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Original Article

New Arithmetic for Triangular Fuzzy Matrix Inverse

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Abstract - *Numerous authors have provided numerous arithmetic for the triangular fuzzy numbers $\tilde{a} = (a, l_a, r_a)$. Particularly for the operations of addition, subtraction, and scalar multiplication, the author doesn't make much of a distinction. For division and multiplication operations, there are many options available. Finding $\tilde{a}(r) \otimes \frac{1}{\tilde{a}(r)} = \tilde{\imath}(r)$ has proven to be impossible. As a result, we will occasionally create $m_p(\tilde{a})$ of the fuzzy number in this essay. The middle value is used to construct fuzzy division and multiplication. Then, by changing $\tilde{a}(r) = [\underline{a}(r), \overline{a}(r)]$ to give $\tilde{a}(r) \otimes \frac{1}{\tilde{a}(r)} = \tilde{\imath}$. These results will be used to construct the adjoint fuzzy method's inverse of the triangular fuzzy matrix and produce $\tilde{A}(r) \otimes \tilde{A}^{-1}(r) = \tilde{I}_z(r)$.*

Keywords - *Adjoint fuzzy method, Inverse fuzzy, Triangular fuzzy number.*

1. Introduction

Fuzzy was first introduced by L. A. Zadeh in 1965, and numerous authors have since covered it in a variety of scientific works. There are a lot of fuzzy numbers, including trapezoidal, hexagonal, and triangular ones.

In the formula for the triangular fuzzy number, there are three parameters. In accordance with [9,16,17], this is therefore denoted as $\tilde{a} = (a, l_a, r_a)$, where a is the center point, l_a is the left distance to the center point of a and r_a is the right distance to the point center of a. The fuzzy number triangular has three points: a_1, a_2, and a_3, which represent the center, left, and right sides, respectively. $\tilde{a} = (a_1, a_2, a_3)$.is the format used by other authors, such as [7,10], to explain fuzzy numbers. Fuzzy numbers can therefore be expressed as intervals known as fuzzy intervals for instance, $\tilde{a} = (a, l_a, r_a)$ will become $\tilde{a}(r) = [\underline{a}(r), \overline{a}(r)]$.

In this work, the author will show an arithmetic alternative to get $\tilde{A}(r) \otimes \tilde{A}^{-1}(r) = \tilde{I}_z(r)$ for $\tilde{A}(r)$ matrix.

2. Basic and Concept

This section will present several theories to support the findings of the study. Below is a definition of the category of fuzzy numbers mentioned in [7,11,21].

Definition 2.1. If R is any non-empty set, then the set $\mu_{\tilde{a}}: R \to [0,1]$ is represented by the membership function $\tilde{a}$ in x. The definition of $\tilde{a}$

$$\tilde{a} = \{(x, \mu_{\tilde{a}}(x)) \mid x \in R, 0 \leq \mu_{\tilde{a}}(x) \leq 1\}.$$

The development of mathematics has also led to the emergence of fuzzy numbers, such as triangular fuzzy numbers and trapezoidal fuzzy numbers. Triangular fuzzy numbers in the definition of 2.2 are defined as follows by [7,9,10,16].

Definition 2.2. $\tilde{a}$ a is a triangular fuzzy number if $\tilde{a} = (a, l_a, r_a)$, where $\tilde{a}$ is the center point, l_a is the left distance to the center, and r_a is the right distance to the center.

The triangular fuzzy number membership function's formula is

$$\mu_{\tilde{a}}(x) = \begin{cases} 1 - \dfrac{\tilde{a} - x}{l_a} & , \tilde{a} - l_a \leq x \leq \tilde{a}, \\ 1 - \dfrac{x - \tilde{a}}{r_a} & , \tilde{a} \leq x \leq \tilde{a} + r_a, \\ 0 & , \text{etc.} \end{cases}$$

The definition of triangular fuzzy membership as stated in [4,7,9,16] is also as follows:

Definition 2.3. Based on the triangular fuzzy membership function in the definition of 2.2, we are able to produce a triangular fuzzy number equation in the form of a parameter. For instance $\tilde{a} = (a, l_a, r_a)$, is represented by the ordered pair $\tilde{a}(r) = [\underline{a}(r), \overline{a}(r)]$, with

$$\underline{a}(r) = a - (1 - r)l_a \text{ and } \overline{a}(r) = a + (1 - r)r_a, r \in [0,1]. \tag{2.1}$$

There is a parametric form for fuzzy numbers as well, which has been discussed in [3,4,7,9,10,13,19] as in the definition of 2.4.

Definition 2.4. Any fuzzy number in parameter form that satisfies the following conditions is taken into consideration, defined as the ordered pair $\tilde{a}(r) = (a, l_a, r_a)$ of the function $[\underline{a}(r), \overline{a}(r)], 0 \leq r \leq 1$.

(i) $\underline{a}(r)$ monotonically increasing, bounded, and left-continuous at $[0,1]$.
(ii) $\overline{a}(r)$ monotonically decreasing, bounded, and right-continuous at $[0,1]$.
(iii) $\underline{a}(r) \leq \overline{a}(r)$ where $0 \leq r \leq 1$
(iv) Fuzzy number arithmetic has been presented in a variety of ways by numerous authors. Here are some

Examples of calculations using fuzzy numbers, specifically those from [3,4,7,9,10,13,16,19].

Definition 2.5. Consider fuzzy number arithmetic.

$$\tilde{a}(r) = (a, l_a, r_a) = [\underline{a}(r), \overline{a}(r)].$$
$$\tilde{b}(r) = (b, l_b, r_b) = [\underline{b}(r), \overline{b}(r)].$$

The subsequent operations therefore apply:

(i) *Addition*

$$\begin{aligned} \tilde{a}(r) + \tilde{b}(r) &= [\underline{a}(r), \overline{a}(r)] + [\underline{b}(r), \overline{b}(r)] \\ &= [\underline{a}(r) + \underline{b}(r), \overline{a}(r) + \overline{b}(r)]. \end{aligned} \tag{2.2}$$

(ii) *Subtraction*

$$\begin{aligned} \tilde{a}(r) - \tilde{b}(r) &= [\underline{a}(r), \overline{a}(r)] - [\underline{b}(r), \overline{b}(r)] \\ &= [\underline{a}(r) - \overline{b}(r), \overline{a}(r) - \underline{b}(r)]. \end{aligned} \tag{2.3}$$

(iii) *Scalar multiplication*

$$k\tilde{a}(r) = \begin{cases} k[\underline{a}(r), \overline{a}(r)], k \geq 0, \\ k[\overline{a}(r), \underline{a}(r)], k < 0. \end{cases} \tag{2.4}$$

Because multiplication is defined differently by different authors, examples include Multiplication defined by [9,16] as mentioned in Definitions 2.6 and Multiplication defined by [7,10,13,11] as found in Definitions 2.7.

Definition 2.6. For instance, if two arbitrary triangular fuzzy numbers $\tilde{a}(r) = \left[\underline{a}(r), \overline{a}(r)\right]$ and $\left[\underline{b}(r), \overline{b}(r)\right]$, are given, then $\tilde{z}(r) = \tilde{a}(r) \times \tilde{b}(r) = \left[\underline{z}(r), \overline{z}(r)\right]$. These are some case of fuzzy number multiplication.

(a) If $\tilde{a}(r) > 0$ and $\tilde{b}(r) > 0$,

$$\begin{cases} \underline{z}(r) = \underline{a}(r)\underline{b}(1) + \underline{a}(1)\underline{b}(r) - \underline{a}(1)\underline{b}(1), \\ \overline{z}(r) = \overline{a}(r)\overline{b}(1) + \overline{a}(1)\overline{b}(r) - \overline{a}(1)\overline{b}(1). \end{cases}$$

(b) If $\tilde{a}(r) > 0$ and $\tilde{b}(r) < 0$,

$$\begin{cases} \underline{z}(r) = \overline{a}(r)\underline{b}(1) + \overline{a}(1)\overline{b}(r) - \overline{a}(1)\underline{b}(1), \\ \overline{z}(r) = \underline{a}(r))\overline{b}(1) + \underline{a}(1)\underline{b}(r) - \underline{a}(1)\overline{b}(1). \end{cases}$$

(c) If $\tilde{a}(r) < 0$ and $\tilde{b}(r) > 0$,

$$\begin{cases} \underline{z}(r) = \overline{a}(r)\underline{b}(1) + \underline{a}(1)\underline{b}(r) - \underline{a}(1)\overline{b}(1), \\ \overline{z}(r) = \overline{a}(r)\underline{b}(1) + \overline{a}(1)\underline{b}(r) - \overline{a}(1)\underline{b}(1). \end{cases}$$

(d) If $\tilde{a}(r) < 0$ and $\tilde{b}(r) < 0$,

$$\begin{cases} \underline{z}(r) = \overline{a}(r)\overline{b}(1) + \overline{a}(1)\overline{b}(r) - \overline{a}(1)\overline{b}(1), \\ \overline{z}(r) = \underline{a}(r)\underline{b}(1) + \underline{a}(1)\underline{b}(r) - \underline{a}(1)\underline{b}(1). \end{cases}$$

Definition 2.7. For example, an arbitrary fuzzy number $\tilde{a}(r) = \left[\underline{a}(r), \overline{a}(r)\right]$ and $\tilde{b}(r) = \left[\underline{b}(r), \overline{b}(r)\right], 0 \leq r \leq 1$ where $k \in \mathbb{R}$.

(a) *Division*

$$\frac{\tilde{a}(r)}{\tilde{b}(r)} = \left[\frac{\underline{a}(r)}{\underline{b}(r)}, \frac{\overline{a}(r)}{\overline{b}(r)}\right], \text{ where}$$

$$\left(\frac{\underline{a}(r)}{\underline{b}(r)}\right) = \min \left\{\frac{\underline{a}(r)}{\underline{b}(r)}, \frac{\underline{a}(r)}{\overline{b}(r)}, \frac{\overline{a}(r)}{\underline{b}(r)}, \frac{\overline{a}(r)}{\overline{b}(r)}\right\},$$

$$\left(\frac{\overline{a}(r)}{\overline{b}(r)}\right) = \max \left\{\frac{\underline{a}(r)}{\underline{b}(r)}, \frac{\underline{a}(r)}{\overline{b}(r)}, \frac{\overline{a}(r)}{\underline{b}(r)}, \frac{\overline{a}(r)}{\overline{b}(r)}\right\}, r \in [0, 1].$$

(b) *Multiplication*

$$\begin{aligned} \tilde{a}(r) \times \tilde{b}(r) &= \left[\underline{a}(r)\underline{b}(r), \overline{a}(r)\overline{b}(r)\right], \text{ where} \\ \left(\underline{a}(r)\underline{b}(r)\right) &= \min\left\{\underline{a}(r)\underline{b}(r), \underline{a}(r)\overline{b}(r), \overline{a}(r)\underline{b}(r), \overline{a}(r)\overline{b}(r)\right\}, \\ \left(\overline{a}(r)\overline{b}(r)\right) &= \max\left\{\underline{a}(r)\underline{b}(r), \underline{a}(r)\overline{b}(r), \overline{a}(r)\underline{b}(r), \overline{a}(r)\overline{b}(r)\right\}. \end{aligned}$$

3. Triangular Fuzzy Matrix

Before discussing the fuzzy matrix's inverse, we first present the fuzzy matrix. Fuzzy integers are the building blocks of a fuzzy matrix. The entries, or components, of the fuzzy matrix can be written as $\tilde{a}_{ij}$ and the fuzzy matrix itself can be written as $\tilde{A}(r)$. A triangular fuzzy matrix is defined as follows by [6,20,17].

Definition 3.1. The triangular fuzzy matrix of order m x n is defined by $\tilde{A}(r) = \left(\tilde{A}_{ij}\right)_{m\times n}$ where $\tilde{a}_{ij} = \left(a_{ij}, l_{ij}, r_{ij}\right)$ from $\tilde{A}(r)$.

Based on the definition of 3.1, the following is the procedure for creating a triangular fuzzy number matrix of size $m \times n$

$$\tilde{A}(r) = \begin{bmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \cdots & \tilde{a}_{1n} \\ \tilde{a}_{21} & \tilde{a}_{22} & \cdots & \tilde{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{m1} & \tilde{a}_{m2} & \cdots & \tilde{a}_{mn} \end{bmatrix}. \quad (3.1)$$

The definition of triangular fuzzy numbers is as follows, and it is taken from [6,15]:

Definition 3.2. The fuzzy triangular number It is a pure zero triangular fuzzy according to the definition of $\tilde{0} = (0,0,0)$, which is represented by $\tilde{0}$. A pure triangular fuzzy number is the triangular fuzzy number $\tilde{0} = (0, l_a, r_a)$, which is denoted by the symbol $\tilde{0}_z$. Another pure triangular fuzzy number is the triangular fuzzy number $\tilde{\imath} = (1,0,0)$ which is denoted by the symbol. The triangular fuzzy number $\tilde{\imath} = (1, l_a, r_a)$, which is also known as the triangular fuzzy identity

Following the definition of triangular fuzzy integers, we will discuss triangular fuzzy matrices, which are covered in [17,20].

Definition 3.3. If all of the entries in a triangular fuzzy number matrix are zero, or if all of the elements are (0,0,0), then the matrix is said to be triangular fuzzy pure zero.

To make it possible to express the triangular fuzzy pure zero matrix in the following form:

$$\tilde{O}(r) = \begin{bmatrix} [0,0] & \cdots & [0,0] \\ \vdots & \ddots & \vdots \\ [0,0] & \cdots & [0,0] \end{bmatrix}. \tag{3.2}$$

Definition 3.4. A triangular fuzzy matrix of the form $(0, l_a, r_a)$is said to be a zero fuzzy matrix if all of its elements zero fuzzy

The triangular fuzzy zero matrix can thus be represented using the following form:

$$\tilde{O}_z(r) = \begin{bmatrix} [-l_{11} + l_{11}r, r_{11} - r_{11}r] & \cdots & [-l_{n1} + l_{n1}r, r_{n1} - r_{n1}r] \\ \vdots & \ddots & \vdots \\ [-l_{m1} + l_{m1}r, r_{m1} - r_{m1}r] & \cdots & [-l_{mn} + l_{mn}r, r_{mn} - r_{mn}r] \end{bmatrix}. \tag{3.3}$$

Definition 3.5. The triangular fuzzy matrix is referred to as a pure triangular fuzzy identity matrix if $\tilde{a}_{ii} = (1,0,0)$ and $\tilde{a}_{ij} = (0,0,0), i \neq j$, for every $i \neq j$.

Consequently, the following is how the fuzzy triangular pure identity matrix is expressed:

$$\tilde{I}(r) = \begin{bmatrix} [1,1] & \cdots & [0,0] \\ \vdots & \ddots & \vdots \\ [0,0] & \cdots & [1,1] \end{bmatrix}. \tag{3.4}$$

Definition 3.6. The triangular fuzzy matrix is referred to as a fuzzy identity if $\tilde{a}_{ii} = (1, l_a, r_a)$ and $\tilde{a}_{ij} = (0, l_a, r_a)$.

As a result, the triangular fuzzy identity matrix can be written as follows:

$$\tilde{I}_z(r) = \begin{bmatrix} [1 - l_{11} + l_{11}r, 1 + r_{11} - r_{11}r] & \cdots & [-l_{n1} + l_{n1}r, r_{n1} - r_{n1}r] \\ \vdots & \ddots & \vdots \\ [-l_{m1} + l_{m1}r, r_{m1} - r_{m1}r] & \cdots & [1 - l_{mn} + l_{mn}r, 1 + r_{mn} - r_{mn}r] \end{bmatrix}. \tag{3.5}$$

4. Result and Discussion

To construct the arithmetic of multiplication, division, and inverse triangular fuzzy numbers, the middle value of the fuzzy number was previously defined as follows:

Definition 4.1. The middle value of the triangular fuzzy number $\tilde{a}(r) = (a, l_a, r_a)$ is employed in the parameter form $\tilde{a}(r) = [\underline{a}(r), \overline{a}(r)]$. The symbol is as follows:

$$m_p(\tilde{a}) = \frac{[\underline{a}(r) \oplus \overline{a}(r)]}{2}. \tag{4.1}$$

The following multiplication arithmetic can be created using the definition of 4.1:

$$\tilde{a}(r) \otimes \tilde{b}(r) = \begin{array}{l}[\underline{a}(r)m_p(\tilde{b}) + \underline{b}(r)m_p(\tilde{a}) - m_p(\tilde{a})m_p(\tilde{b}), \\ \overline{a}(r)m_p(\tilde{b}) + \overline{b}(r)m_p(\tilde{a}) - m_p(\tilde{a})m_p(\tilde{b})].\end{array} \quad (4.2)$$

Additionally, any triangular fuzzy number can be represented by the equation (4.2), which can be derived inversely from the definition of 4.1 and is denoted as $\frac{1}{\tilde{a}(r)}$.

$$\tilde{a}(r) \otimes \tilde{x}(r) = \tilde{\iota}(r) = [1,1]. \quad (4.3)$$

Then, proof theorem 4 is presented as an example of the (4.3):

Theorem 4.2. For any triangular fuzzy number $\tilde{a}(r) = [\underline{a}(r), \overline{a}(r)]$ where $m_p(\tilde{a}) \neq 0$, from $\tilde{x}(r) = \frac{1}{\tilde{a}(r)} = \left[\frac{2m_p(\tilde{a}) - \underline{a}(r)}{(m_p(\tilde{a}))^2}, \frac{2m_p(\tilde{a}) - \overline{a}(r)}{(m_p(\tilde{a}))^2}\right]$, then, by multiplying, does the formula (4.2) hold

$$\tilde{a}(r) \otimes \tilde{x}(r) = \tilde{\iota}(r) = [1,1].$$

Proof: *The value of is first calculated* $m_p(\tilde{x})$, *i.e*

$$\begin{aligned} m_p(\tilde{x}) &= \frac{\left(\left(\frac{2m_p(\tilde{a}) - \underline{a}(r)}{(m_p(\tilde{a}))^2}\right) \oplus \left(\frac{2m_p(\tilde{a}) - \overline{a}(r)}{(m_p(\tilde{a}))^2}\right)\right)}{2}, \\ &= \frac{\left(\frac{4m_p(\tilde{a}) - (\underline{a}(r) + \overline{a}(r))}{(m_p(\tilde{a}))^2}\right)}{2}, \\ &= \frac{\left(\frac{2}{m_p(\tilde{a})}\right)}{2}, \end{aligned}$$

$$m_p(\tilde{x}) = \frac{1}{m_p(\tilde{a})}. \quad (4.4)$$

Furthermore, it can be concluded using the (4.4) equation $\tilde{a}(r) \otimes \tilde{x}(r)$

$$\begin{aligned} \tilde{a}(r) \otimes \tilde{x}(r) &= [\underline{a}(r)m_p(\tilde{x}) + \underline{x}(r)m_p(\tilde{a}) - m_p(\tilde{a})m_p(\tilde{x}), \\ &\quad \overline{a}(r)m_p(\tilde{x}) + \overline{x}(r)m_p(\tilde{a}) - m_p(\tilde{a})m_p(\tilde{x})], \\ &= \left[\underline{a}(r)\frac{1}{m_p(\tilde{a})} + \frac{2m_p(\tilde{a}) - \underline{a}(r)}{(m_p(\tilde{a}))^2} m_p(\tilde{a}) - m_p(\tilde{a})\left(\frac{1}{m_p(\tilde{a})}\right),\right. \\ &\quad \left.\overline{a}(r)\frac{1}{m_p(\tilde{a})} + \frac{2m_p(\tilde{a}) - \overline{a}(r)}{(m_p(\tilde{a}))^2} m_p(\tilde{a}) - m_p(\tilde{a})\left(\frac{1}{m_p(\tilde{a})}\right)\right], \\ &= \left[\underline{a}(r)\frac{1}{m_p(\tilde{a})} + \frac{2m_p(\tilde{a}) - \underline{a}(r)}{m_p(\tilde{a})} - 1,\right. \\ &\quad \left.\overline{a}(r)\frac{1}{m_p(\tilde{a})} + \frac{2m_p(\tilde{a}) - \overline{a}(r)}{m_p(\tilde{a})} - 1\right] \\ &= [1,1] \end{aligned}$$

$$\tilde{a}(r) \otimes \tilde{x}(r) = \tilde{\imath}(r).$$

It is proved that $\tilde{a}(r) \otimes \tilde{x}(r) = \tilde{\imath}(r)$.

Similarly $\tilde{b}(r) = [\underline{b}(r), \overline{b}(r)]$, so

$$\left(\frac{1}{\tilde{b}(r)}\right) = \left[\frac{2m_p(\tilde{b}) - \underline{b}(r)}{\left(m_p(\tilde{b})\right)^2}, \frac{2m_p(\tilde{b}) - \overline{b}(r)}{\left(m_p(\tilde{b})\right)^2}\right] = \frac{1}{m_p(\tilde{b})}.$$

Corollary 4.3. Any triangular fuzzy number has the form $\tilde{a}(r) = [\underline{a}(r), \overline{a}(r)]$ and $\tilde{b}(r) = [\underline{b}(r), \overline{b}(r)]$, In to show $\frac{\tilde{a}(r)}{\tilde{b}(r)}$

Proof:

$$\frac{\tilde{a}(r)}{\tilde{b}(r)} = \tilde{a}(r) \otimes \frac{1}{\tilde{b}(r)}$$

$$= [\underline{a}(r), \overline{a}(r)] \otimes \left[\frac{2m_p(\tilde{b}) - \underline{b}(r)}{\left(m_p(\tilde{b})\right)^2}, \frac{2m_p(\tilde{b}) - \overline{b}(r)}{\left(m_p(\tilde{b})\right)^2}\right]$$

$$= \left[\underline{a}(r)\frac{1}{m_p(\tilde{b})} + \frac{2m_p(\tilde{b}) - \underline{b}(r)}{\left(m_p(\tilde{b})\right)^2} m_p(\tilde{a}) - \frac{m_p(\tilde{a})}{m_p(\tilde{b})}, \right.$$

$$\left. \overline{a}(r)\frac{1}{m_p(\tilde{b})} + \frac{2m_p(\tilde{b}) - \overline{b}(r)}{\left(m_p(\tilde{b})\right)^2} m_p(\tilde{a}) - \frac{m_p(\tilde{a})}{m_p(\tilde{b})}\right]$$

$$= \left[\frac{\underline{a}(r)m_p(\tilde{b}) + 2m_p(\tilde{b})m_p(\tilde{a}) - \underline{b}(r)m_p(\tilde{a}) - m_p(\tilde{a})m_p(\tilde{b})}{\left(m_p(\tilde{b})\right)^2}, \right.$$

$$\left. \frac{\overline{a}(r)m_p(\tilde{b}) + 2m_p(\tilde{b})m_p(\tilde{a}) - \overline{b}(r)m_p(\tilde{a}) - m_p(\tilde{a})m_p(\tilde{b})}{\left(m_p(\tilde{b})\right)^2}\right]$$

$$= \left[\frac{\underline{a}(r)m_p(\tilde{b}) - \underline{b}(r)m_p(\tilde{a}) + m_p(\tilde{a})m_p(\tilde{b})}{\left(m_p(\tilde{b})\right)^2}, \right.$$

$$\left. \frac{\overline{a}(r)m_p(\tilde{b}) - \overline{b}(r)m_p(\tilde{a}) + m_p(\tilde{a})m_p(\tilde{b})}{\left(m_p(\tilde{b})\right)^2}\right].$$

Theorem 4.4. As an illustration, if a triangular fuzzy number $\tilde{a}(r), \tilde{b}(r), \tilde{c}(r)$ is present in the interval, the following characteristics are true:

a. $\tilde{a}(r) \otimes \tilde{0}(r) = \tilde{0}(r)$.
b. $\tilde{a}(r) \otimes \tilde{\imath}(r) = \tilde{a}(r)$.
c. $\tilde{a}(r) \otimes \tilde{b}(r) = \tilde{b}(r) \otimes \tilde{a}(r)$.
d. $\left(\tilde{a}(r) \otimes \tilde{b}(r)\right) \otimes \tilde{c}(r) = \tilde{a}(r) \otimes \left(\tilde{b}(r) \otimes \tilde{c}(r)\right)$.
e. $\left(\tilde{a}(r) \oplus \tilde{b}(r)\right) \otimes \tilde{c}(r) = \tilde{a}(r) \otimes \left(\tilde{c}(r) \oplus \tilde{b}(r) \otimes \tilde{c}(r)\right)$.
f. If $\tilde{a}(r) \otimes \tilde{x}(r) = \tilde{b}(r)$where $\tilde{a}(r) \neq \tilde{0}(r)$, so $\tilde{x}(r) = \frac{\tilde{b}(r)}{\tilde{a}(r)}$.

g. If $\tilde{a}(r) \otimes \tilde{b}(r) = \tilde{0}(r)$, so $\tilde{a}(r) = \tilde{0}(r)$ or $\tilde{b}(r) = \tilde{0}(r)$.
h. If $\tilde{a}(r) \otimes \tilde{b}(r) = \tilde{a}(r) \otimes \tilde{b}(r)$ where $\tilde{a}(r) \neq \tilde{0}(r)$, so $\tilde{b}(r) = \tilde{c}(r)$.
i. If $\tilde{a}(r) \neq \tilde{0}(r)$, so $\frac{1}{\tilde{a}(r)} \neq \tilde{0}(r)$ and $\frac{1}{\frac{1}{\tilde{a}(r)}} = \tilde{a}(r)$.
j. $\tilde{a}(r) \neq \tilde{0}(r)$ and $\tilde{b}(r) \neq \tilde{0}(r)$, so $\frac{\frac{1}{\tilde{a}(r)}}{\tilde{a}(r)\otimes\tilde{b}(r)} = \frac{1}{\tilde{a}(r)} \otimes \frac{1}{\tilde{b}(r)}$.

Proof: *clearly*.

5. Adjoint Matrix Method for Triangular Fuzzy Matrix Inverse

The adjoint method can be used to find the inverse of the $\tilde{A}$ matrix if $\tilde{A}\tilde{B} = \tilde{B}\tilde{A} = \tilde{I}$,

(1) A triangular fuzzy number matrix with $m = n$ is given.
(2) The triangular fuzzy number matrix is then changed into an interval matrix form, which looks like the equation (3.1).
(3) A determinant value for the matrix $\tilde{A}(r)$ should be found.
(4) Next, we will determine the value of $\frac{1}{\det(\tilde{A})}$.
(5) Find the adjoint of the $\tilde{A}(r)$ matrix.
(6) Then, we will find the inverse of the fuzzy $\tilde{A}(r)$ matrix by the following method:

$$\tilde{A}^{-1}(r) = \frac{1}{\det(\tilde{A})} \otimes \text{ajd}(\tilde{A}). \tag{5.1}$$

(7) The value of $\tilde{A}^{-1}(r) \otimes \tilde{A}(r)$ will then be identified.

The inverse triangular fuzzy matrix can be found using the ajdoint method, as proven by the example that follows.

EXAMPLE 5.1. The triangular fuzzy number matrix shown below is fuzzy:

$$\tilde{A} = \begin{bmatrix} (3,2,2) & (3,3,3) \\ (3,1,1) & (1,2,2) \end{bmatrix}. \tag{5.2}$$

The (5.2) equation will be changed into a fuzzy matrix as follows:

$$\tilde{A}(r) = \begin{bmatrix} [1+2r, 5-2r] & [3r, 6-3r] \\ [2+r, 4-r] & [-1+2r, 3-2r] \end{bmatrix}. \tag{5.3}$$

The $\tilde{A}(r)$ matrix's determinant will be established as follows:

$$\begin{aligned} \det(\tilde{A}) &= ([1+2r, 5-2r] \otimes [-1+2r, 3-2r]) - ([2+r, 4-r] \otimes [3r, 6-3r]) \\ &= \left[\frac{7}{18} - \frac{5}{9}r, -\frac{13}{18} + \frac{5}{9}r\right]. \end{aligned}$$

The value of $\frac{1}{\det(\tilde{A})}$ will then be determined.

$$\frac{1}{\left[\frac{7}{18} - \frac{5}{9}r, -\frac{13}{18} + \frac{5}{9}r\right]} = \left[\frac{\frac{13}{18} + \frac{5}{9}r}{\frac{25}{81}}, \frac{\frac{23}{18} + \frac{5}{9}r}{\frac{25}{81}}\right].$$

Next, it will be calculated what $\tilde{A}(r)$ should be.

$$\begin{bmatrix}[-1+2r,3-2r] & [-6+3r,-3r]\\ [-4+r,-2-r] & [1+2r,5-2r]\end{bmatrix}.$$

The inverse of $\tilde{A}(r)$ can then be obtained by multiplying $\frac{1}{\det(\tilde{A})}$ by the adjoint matrix $\tilde{A}(r)$

$$\tilde{A}^{-1}(r)=\left[\frac{\frac{13}{18}+\frac{5}{9}r}{\frac{25}{81}},\frac{\frac{23}{18}+\frac{5}{9}r}{\frac{25}{81}}\right]\cdot\begin{bmatrix}[-1+2r,3-2r] & [-6+3r,-3r]\\ [-4+r,-2-r] & [1+2r,5-2r]\end{bmatrix}. \quad (5.4)$$

Using the calculations given in equation (5.4), the following value for $\tilde{A}^{-1}(r)$ is determined:

$$\tilde{A}^{-1}(r)=\begin{pmatrix}\left[\frac{13}{18}-\frac{8}{9}r,-\frac{19}{18}+\frac{8}{9}r\right] & \left[-\frac{2}{3}+\frac{7}{6}r,\frac{5}{3}-\frac{7}{6}r\right]\\ \left[-1+\frac{3}{2}r,2-\frac{3}{2}r\right] & \left[\frac{3}{2}-2r,-\frac{5}{2}+2r\right]\end{pmatrix}. \quad (5.5)$$

Then, in order to prove the accuracy of the $\tilde{A}^{-1}(r)$ obtained, $\tilde{A}^{-1}(r)\otimes\tilde{A}(r)=\tilde{I}_z(r)$ will be calculated.

$$\begin{pmatrix}\left[\frac{13}{18}-\frac{8}{9}r,-\frac{19}{18}+\frac{8}{9}r\right] & \left[-\frac{2}{3}+\frac{7}{6}r,\frac{5}{3}-\frac{7}{6}r\right]\\ \left[-1+\frac{3}{2}r,2-\frac{3}{2}r\right] & \left[\frac{3}{2}-2r,-\frac{5}{2}+2r\right]\end{pmatrix}\otimes\begin{bmatrix}[1+2r,5-2r] & [3r,6-3r]\\ [2+r,4-r] & [-1+2r,3-2r]\end{bmatrix}.$$

The formula for $\tilde{A}^{-1}(r)\otimes\tilde{A}(r)=\tilde{I}_z(r)$ is displayed in the following.

1. For $\tilde{\iota}_{z11}$

$$\left(\left[\frac{13}{18}-\frac{8}{9}r,-\frac{19}{18}+\frac{8}{9}r\right]\otimes[1+2r,5-2r]\right)\oplus\left[-\frac{2}{3}+\frac{7}{6}r,\frac{5}{3}-\frac{7}{6}r\right]\otimes[2+r,4-r]\right)$$
$$=\left[\frac{5}{2}-3r,-\frac{7}{2}+3r\right]\oplus\left[-\frac{5}{2}+4r,\frac{11}{2}-4r\right]$$
$$=[r,2-r]. \quad (5.6)$$

(2) For $\tilde{0}_{z12}$

$$\left(\left[\frac{13}{18}-\frac{8}{9}r,-\frac{19}{18}+\frac{8}{9}r\right]\otimes[3r,6-3r]\right)\oplus\left(\left[-\frac{2}{3}+\frac{7}{6}r,\frac{5}{3}-\frac{7}{6}r\right]\otimes[-1+2r,3-2r]\right)$$
$$=\left[\frac{8}{3}-\frac{19}{6}r,-\frac{11}{3}+\frac{19}{6}r\right]+\left[-\frac{5}{3}+\frac{13}{6}r,\frac{8}{3}-\frac{13}{6}r\right]$$
$$=[1-r,-1+r]. \quad (5.7)$$

(3) For $\tilde{0}_{z21}$

$$\left(\left[-1+\frac{3}{2}r,2-\frac{3}{2}r\right]\otimes[1+2r,5-2r]\right)\oplus\left(\left[\frac{3}{2}-2r,-\frac{5}{2}+2r\right]\otimes[[2+r,4-r]]\right)$$
$$=\left[-4+\frac{11}{2}r,-8+\frac{13}{2}r\right]\oplus\left[5-\frac{13}{2}r,-8+\frac{13}{2}r\right]$$
$$=[1-r,-1+r]. \quad (5.8)$$

(4) For $\tilde{\iota}_{z22}$

$$\left(\left[-1+\frac{3}{2}r,2-\frac{3}{2}r\right]\otimes[3r,6-3r]\right)\oplus\left(\left[\frac{3}{2}-2r,-\frac{5}{2}+2r\right]\otimes[-1+2r,3-2r]\right)$$
$$=\left[-\frac{9}{2}+6r,\frac{15}{2}-6r\right]\oplus\left[\frac{5}{2}-3r,-\frac{7}{2}+3r\right]$$

$$= [-2 + 3r, 4 - 3r]. \tag{5.9}$$

According to the calculation results provided in equations (5.6), (5.7), (5.9), and (5.8), a fuzzy identity matrix satisfies the requirements. It must be able to be expressed specifically as follows in the matrix form outlined in the definition of (3.6).

$$\tilde{\boldsymbol{I}}_z = \begin{bmatrix} [r, 2-r] & [1-r, -1+r] \\ [1-r, -1+r] & [-2+3r, 4-3r] \end{bmatrix}. \tag{5.10}$$

Therefore, it can be said that $\tilde{A}^{-1}(r) \otimes \tilde{A}(r) = \tilde{I}_z(r)$ is proven..

6. Conclusion

(1) Using multiplication arithmetic, such as the (4.2) equation, we can find the inverse of fuzzy numbers, such that $\tilde{a}(r) = [\underline{a}(r), \overline{a}(r)]$ contains $\tilde{a}^{-1}(r) = \tilde{I}_z = \left[\frac{2m_p(\tilde{a}) - \underline{a}(r)}{\left(m_p(\tilde{a})\right)^2}, \frac{2m_p(\tilde{a}) - \overline{a}(r)}{\left(m_p(\tilde{a})\right)^2}\right]$ Consequently $\tilde{a} \otimes \tilde{a}^{-1}(r) = \tilde{\imath}(r)$.

(2) Additionally, the triangular fuzzy matrix inverse with adjoint is calculated to derive the fuzzy identity value $\tilde{A}(r) \otimes \tilde{A}^{-1}(r) = \tilde{I}_z(r)$.

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