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On a solution of a Delay Differential Equation, via a New Iteration Scheme

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Abstract

In this paper, results on stability and data dependency for a new iteration scheme under contractive-like mappings are discussed. In the framework of uniformly convex Banach spaces, we also provide weak and strong convergence results for mappings satisfying the condition $\beta_{\gamma,\mu}$. To validate our proofs, numerical example are also provided which are supported by graphs and tables. Finally, we exhibit the applicability of our three-step iteration process in delay differential equations.

Keywords: Contractive-like mappings, $B_{\gamma,\mu}$ condition, stability, data dependence, fixed points, iterative algorithm, weak and strong convergence, delay differential equation.

MSC(2010) : 47H09, 47H10, 54H25.

1 Introduction

Let $\mathbb{N}$, $\mathbb{R}$ and $F(\mathcal{T})$ denote the set of positive integers, set of real numbers and set of fixed points respectively and $\mathcal{T}$ be any self mapping defined on a subset $\mathcal{C}$ of a Banach space B.

In the last few years, iteration schemes have been considered an easy tool to calculate the desired fixed point and, due to this, a number of interesting iterative processes have been introduced to obtain the fixed point of various kinds of mappings in different types of domains. Some well-known iterations are Mann iteration [17], Ishikawa iteration [16], Noor [19], S-iteration [5], Abbas et al. [1], Thakur et al. ([35], [36]) K iteration [15], M^*iteration [37], M iteration [38], K^* iteration [39], Picard-S iteration process [13].

Piri et al. [27] introduced the following iteration process:

$$\begin{cases} p_1 = p \in \mathcal{C} \\ p_{n+1} = (1-c_n)\mathcal{T}r_n + c_n\mathcal{T}q_n, \\ q_n = \mathcal{T}r_n, \\ r_n = \mathcal{T}((1-d_n)p_n + d_n\mathcal{T}p_n), \qquad n \in \mathbb{N} \end{cases} \tag{1.1}$$

where $\{c_n\}, \{d_n\}$ are in $(0,1)$. In [27], Piri et al. proved that the iteration process (1.1) converges faster than above mentioned leading iterations for contractive mappings when $(1-c_n) < c_n$ and $1-d_n < d_n$ for all $n \in \mathbb{N}$. With the help of graphs and tables they concluded that their new iteration scheme is more stable than Thakur [35], Abbas [1] and Agarwal [5] iteration processes with respect to selection of initial points and different sets parameters.

In 2020, Chanchal et al. [11] have given chanchal iteration process:

$$\begin{cases} x_1 \in C \\ p_{n+1} = \mathcal{T} q_n, \\ q_n = \mathcal{T}((1-d_n)\mathcal{T} p_n + d_n \mathcal{T} r_n), \\ r_n = \mathcal{T}((1-c_n)p_n + c_n \mathcal{T} p_n), \qquad n \in \mathbb{N} \end{cases} \tag{1.2}$$

where $\{c_n\}, \{d_n\}$ are in $(0, 1)$.

They proved that their iteration scheme (1.2) converges faster than abovementioned iteration algorithms for contractive-like operators.

Very recently, Hussain et al. [15] introduced the D iteration process as followed:

$$\begin{cases} p_1 \in \mathcal{C} \\ p_{n+1} = \mathcal{T} q_n, \\ q_n = \mathcal{T}((1-d_n)\mathcal{T} p_n + d_n \mathcal{T} r_n), \\ r_n = \mathcal{T}((1-c_n)p_n + c_n \mathcal{T} p_n), \qquad n \in \mathbb{N} \end{cases} \tag{1.3}$$

where $\{c_n\}, \{d_n\}$ are in $(0, 1)$.

In 2022 in [18], we introduced a new iteration scheme to study stability, data dependency and fixed point of generalized nonexpansive mappings.

$$\begin{cases} p_1 \in \mathcal{C} \\ p_{n+1} = \mathcal{T} q_n, \\ q_n = \mathcal{T} r_n, \\ r_n = \mathcal{T}((1-c_n)p_n + c_n \mathcal{T} p_n), \qquad n \in \mathbb{N} \end{cases} \tag{1.4}$$

where $\{c_n\} \in (0, 1)$.

Definition 1.1. A mappings $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ is contraction, if $\exists\ k \in (0, 1)$, such that $\|\mathcal{T}u - \mathcal{T}v\| \le k\|u - v\|$ for all $u, v \in \mathcal{C}$.

Definition 1.2. $\mathcal{T}$ is quasi-nonexpansive, if $F(\mathcal{T}) \neq \phi$ and $\|\mathcal{T}u - q\| \le \|u - q\|\ \ \forall\ u \in \mathcal{C}$ and $q \in F(\mathcal{T})$.

In 2008, Suzuki [30] defined a new class of mappings in Banach space.

$$\frac{1}{2}\|x - Tx\| \le \|x - y\| \Rightarrow \|Tx - Ty\| \le \|x - y\| \qquad \forall x, y \in K.$$

Here, T is named as Condition (C), which is also referred to as generalized nonexpansive mapping. Suzuki [30] proved that the above-defined mapping is stronger than quasinonexpansive mappings and weaker than nonexpansive mappings.

It is obvious that every mapping satisfying condition (C) with a fixed point is a quasi nonexpansive mapping (see [30]). Following this, numerous results have been obtained for the class of generalized nonexpansive mappings in various spaces (e.g. [10], [35], [38] and references therein).

Definition 1.3 ([10])**.** Let $(\mathbb{B}, \|\cdot\|)$ be a Banach space and and $\mathcal{C}$ a nonempty subset of $\mathbb{B}$. A mapping $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ satisfies the (E_μ) condition on the set $\mathcal{C}$ if there can be found a real number $\mu \ge 1$ so that

$$\|x - \mathcal{T}y\| \le \mu\|x - \mathcal{T}x\| + \|x - y\|,$$

for all x, $y \in \mathcal{C}$.

Moreover, it is said that $\mathcal{T}$ accomplishes the condition (E) if there exists $\mu \ge 1$ such that $\mathcal{T}$ fulfills the condition (E_μ).

In 2017, Pant and Shukla [25] defined a new class of mapping, known as generalized α-nonexpansive mapping which is larger than the mappings satisfying in condition (C). A self mapping T defined on nonempty subset K of a Banach space, is said to be generalized α-nonexpansive if there exists $0 \leq \alpha < 1$ such that

$$\frac{1}{2}\|x - \mathcal{T}x\| \leq \|x - y\| \Rightarrow \|\mathcal{T}x - \mathcal{T}y\| \leq \alpha\|\mathcal{T}x - y\| + \alpha\|\mathcal{T}y - x\| + (1 - 2\alpha)\|x - y\| \ \ \forall x, y \in \mathcal{C}. \tag{1.5}$$

To approximate the fixed point of generalized $\alpha-$nonexpansive mappings, in 2019 Piri et al. [27] introduced a new iterative process and proved that their iteration process converges faster than some leading iterations, for instance Picard, Mann [17], Ishikawa [16], Noor[19], Agarwal [5], Abbas[1] and Thakur iteration processes [35] for contractive mappings.

In 2018, Patir et al. [26] generalized the notion of (C) condition as follows and presented some fixed point results for this class of operators.

Definition 1.4. Let $\mathcal{C}$ be a nonempty subset of a Banach space $\mathbb{B}$. Let $\gamma \in [0, 1]$ and $\mu \in [0, \frac{1}{2}]$ such that $2\mu \leq \gamma$. A mapping $T : \mathcal{C} \to \mathcal{C}$ is said to satisfy the condition $\beta_{\gamma,\mu}$ on $\mathcal{C}$ if, for all x, y in $\mathcal{C}$,

$$\gamma\|x - \mathcal{T}x\| \leq \|x - y\| + \mu\|y - \mathcal{T}y\|$$

implies

$$\|\mathcal{T}x - \mathcal{T}y\| \leq (1 - \gamma)\|x - y\| + \mu(\|x - \mathcal{T}y\| + \|y - \mathcal{T}x\|)$$

Recently, fixed point theorems for mapping satisfying condition $B_{\gamma,\mu}$ have been studied by a number of authors (see [2], [3], [4]). It is clear that, this class of operators includes the class of nonexpansive mappings (for $\gamma = \mu = 0$).

It is noteworthy that nonexpansive mappings are continuous in their domains, but Suzuki-type generalized nonexpansive mappings, α-nonexpansive mappings, generalized α-nonexpansive mappings, nonexpansive mappings satisfying condition (E) and nonexpansive mappings satisfying condition $B_{\gamma,\mu}$ are not necessarily continuous (see [30], [25], [27], [10], [26]).

In this paper, we used a novel three step iterative scheme to approximate fixed point within a fewer number of steps and shown result of stability, data dependency and convergence behavior of the new iteration process (1.4) under contractive-type mappings and proved some weak and strong convergence theorems in Banach spaces thereby extending the classes of mappings. Moreover, in final section we exhibited the applicability of a special case of our iteration process in delay differential equation.

2 Preliminaries

A Banach space B is uniformly convex if for each $\epsilon \in \mathbb{R}_+$, there is a $\delta \in \mathbb{R}_+$ such that $\|u\| \leq 1, \|v\| \leq 1$ and $\|u - v\| > \epsilon$ implies $\frac{\|u+v\|}{2} \leq (1 - \delta)$ for $u, v \in B$.

A Banach space B is said to have the Opial property [22] if for each weakly convergent sequence $\{u_n\}$ in B, converging weakly to $u \in B$, we have

$$\limsup_{n\to\infty} \|u_n - u\| < \limsup_{n\to\infty} \|u_n - v\| \text{ , for all } v \in B \text{ such that } v \neq u.$$

Assume that $\{u_n\}$ be a bounded sequence in Banach space B. For $u \in B$, we set

$$r(u, \{u_n\}) = \limsup_{n\to\infty} \|u - u_n\|.$$

The asymptotic radius of $\{u_n\}$ relative to a nonempty closed and convex subset $\mathcal{C}$ of Banach space B is given by

$$r(\mathcal{C},\{u_n\}) = \inf\{r(u,\{u_n\}) : u \in \mathcal{C}\}.$$

The asymptotic center of $\{u_n\}$ relative to $\mathcal{C}$ is the set

$$A(\mathcal{C},\{u_n\}) = \{u \in \mathcal{C} : r(u,\{u_n\}) = r(\mathcal{C},\{u_n\}\}.$$

It is noteworthy that $A(\mathcal{C},\{u_n\})$ has exactly one point if B is uniformly convex. Also, $A(\mathcal{C},\{u_n\})$ is nonempty and convex when $\mathcal{C}$ is weakly compact and convex (for more details, see [31]).

Definition 2.1. [8] Let $\mathcal{T} : B \to B$ be any mapping. Suppose $p_0 \in B$ and $p_{n+1} = f(\mathcal{T}, p_n)$ defines an iterative scheme which produces a sequence of points $p_n \in B$. Suppose p_n converges to the fixed point p^* of $\mathcal{T}$. Assume that $\{s_n\}$ be a sequence in B and $\epsilon \in [0,\infty)$ given by $\epsilon = \|s_{n+1} - s_n\|$. Then the iterative scheme define by $p_{n+1} = f(\mathcal{T}, p_n)$ is called stable with respect to $\mathcal{T}$ if $\lim_{n\to\infty} \epsilon_n = 0$ iff $\lim_{n\to\infty} s_n = p^*$.

Definition 2.2. [27] Let $\{p_n\}$ and $\{q_n\}$ be two iteration processes such that both converging to the same fixed point p^* and $\|p_n - p^*\| \le t_n$ and $\|q_n - p^*\| \le w_n \quad \forall n \in \mathbb{N}$.

If $\{t_n\}$ and $\{w_n\}$ be two real number sequences converging to t and w, respectively and $\frac{|t_n - t|}{|w_n - w|} = 0$. This implies that $\{p_n\}$ converges faster than $\{q_n\}$.

Definition 2.3. [23] A mapping $\mathcal{T}$ defined on a Banach space B is known as contractive mapping on B if there exist a nonnegative constant L, $b \in [0,1)$ such that for all $u, v \in B$

$$\|\mathcal{T}u - \mathcal{T}v\| \le L\|u - \mathcal{T}u\| + b\|u - v\| \tag{2.1}$$

Definition 2.4. [8] The operator $\mathcal{T}$ is called contractive-like operator if there exists a constant $b \in (0,1)$ a continuous and strictly increasing function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that for each $u, v \in B$,

$$\|\mathcal{T}u - \mathcal{T}v\| \le g(\|u - \mathcal{T}u\|) + b\|u - v\| \tag{2.2}$$

Osilike [23] worked on many stability results as a generalizations of the works done by Rhoades [28] and Harder et al. [14].

Proposition 2.1. *[10] Let $\mathcal{T}$ be an arbitrary self mapping defined on bounded subset $\mathcal{C}$ of a Banach space B. If*

- *There exists an almost fixed point sequence $\{p_n\}$ for $\mathcal{T}$ in $\mathcal{C}$ such that $p_n \rightharpoonup p$,*
- *$\mathcal{T}$ satisfies condition (E) on $\mathcal{C}$, and*
- *$(B, \|\|)$ satisfies the Opial condition.*

Then, $\mathcal{T}p = p$.

Lemma 2.2. *[26] Let $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ be a mapping which satisfies condition $B_{\gamma,\mu}$ on $\mathcal{C}$. If $\mathcal{T}$ has some fixed point, then T is quasi-nonexpansive. The converse is not true.*

From Lemma 2.2 Abdeljawad et al. [3] obtained the following lemma.

Lemma 2.3. *[3] Let $\mathcal{C}$ be a nonempty subset of a Banach space $\mathbb{B}$ and $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ satisfies condition $B_{\gamma,\mu}$. Then, the set $F(\mathcal{T})$ is closed. Moreover, if $\mathbb{B}$ is strictly convex and $\mathcal{C}$ is convex, then $F(\mathcal{T})$ is also convex.*

Theorem 2.4. *[26] Let $\mathcal{C}$ be a nonempty subset of a Banach space $\mathbb{B}$ having Opial property. Let $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ satisfies $B_{\gamma,\mu}$ condition. If $\{x_n\} \subseteq \mathcal{C}$ be a sequence such that*

1. *$\{x_n\}$ converges weakly to p^**
2. *$\lim_{n\to\infty} \|x_n - \mathcal{T}x_n\| = 0$.*

Then $\mathcal{T}p^* = p^*$.

Following are some important properties of generalized nonexpansive mappings that satisfies the condition $B_{\gamma,\mu}$ on $\mathcal{C}$.

Proposition 2.5. *[26] Let $\mathcal{C} \neq \phi$ be a subset of a Banach space $\mathbb{B}$. Let a self map $\mathcal{T}$ satisfies the condition $B_{\gamma,\mu}$ on $\mathcal{C}$. Then, $\forall x, y \in \mathcal{C}$ and for $\lambda \in [0,1]$,*

1. $\|\mathcal{T}x - \mathcal{T}^2x\| \leq \|x - \mathcal{T}x\|$.
2. *At least one of the following conditions ((a) or (b)) holds:*
   (a) $(\frac{\lambda}{2})\|x - \mathcal{T}x\| \leq \|x - y\|$.
   (b) $(\frac{\lambda}{2})\|\mathcal{T}x - \mathcal{T}^2x\| \leq \|\mathcal{T}x - y\|$.

   The condition (a) implies $\|\mathcal{T}x - \mathcal{T}y\| \leq (1 - \frac{\lambda}{2})\|x - y\| + \mu(\|x - \mathcal{T}y\| + \|y - \mathcal{T}x\|)$.
   The condition (b) implies $\|\mathcal{T}^2x - \mathcal{T}y\| \leq (1 - \frac{\lambda}{2})\|\mathcal{T}x - y\| + \mu(\|\mathcal{T}x - \mathcal{T}y\| + \|y - \mathcal{T}^2x\|)$.
3. $\|x - \mathcal{T}y\| \leq (3 - \lambda)\|x - \mathcal{T}x\| + (1 - \frac{\lambda}{2})\|x - y\| + \mu(2\|x - \mathcal{T}x\| + \|x - \mathcal{T}y\| + \|y - \mathcal{T}x\| + 2\|\mathcal{T}x - \mathcal{T}^2x\|)$.

Theorem 2.6. *[31] Let $\mathbb{B}$ be a UCBS and $0 < \alpha \leq t_n \leq \beta < 1$ for all positive integers n. Let $\{x_n\}$ and $\{y_n\}$ be two sequences such that $\limsup_{n\to\infty} \|x_n\| \leq r, \limsup_{n\to\infty} \|y_n\| \leq r$ and $\lim_{n\to\infty} \|t_nx_n + (1 - t_n)y_n\| = r$ hold for some $r \geq 0$. Then,*

$$\lim_{n\to\infty} \|x_n - y_n\| = 0.$$

Theorem 2.7. *[31] Let $0 < \alpha \leq t_n \leq \beta < 1$ for all positive integers n. Let $\{p_n\}$ and $\{q_n\}$ be two sequences in a uniformly convex Banach space B such that $\limsup_{n\to\infty} \|p_n\|, \limsup_{n\to\infty} \|q_n\| \leq \lambda$ and $\lim_{n\to\infty} \|t_np_n + (1-t_n)q_n\| = \lambda$ hold for some $\lambda \geq 0$. Then,*

$$\lim_{n\to\infty} \|p_n - q_n\| = 0.$$

3 Stability

In 1967, Ostrowski [24] obtained the following classical stability result on metric spaces.

Theorem 3.1. *Let $\mathcal{T} : X \to X$ be a Banach contraction with contraction constant $\mu \in [0, 1)$, where (X, ρ) is a complete metric space. Let p^* be the fixed point of $\mathcal{T}$. Let $p_0 \in X$ and $p_{n+1} = \mathcal{T}p_n$ for $n = 0, 1, 2, ...$ Suppose that $\{q_n\}$ is another sequence in X such that $\epsilon_n = \rho(\mathcal{T}q_n, q_{n+1})$. Then*

$$\rho(p^*, q_{n+1}) \leq \rho(p^*, p_{n+1}) + \mu^{n+1}\rho(p_0, q_0) + \sum_{i=0}^{n} \mu^{n-i}\epsilon_i.$$

In addition, $\lim_{n\to\infty} q_n = p^*$ if and only if $\lim_{n\to\infty} \epsilon_n = 0$.

Later, Harder and Hicks [14] , Osilike [23], Rhoades [28] and Zhou [40] extended above-important result. The convergence and stability of iteration schemes are studied in [[15], [38]] for K iteration and K^* iteration respectively.

Very recently, Hussain et al. [15] studied data dependency and stability for iteration (1.3).

Remark : From classical analysis we have $(1 - \eta) \leq e^{-\eta}, \quad \eta \in [0, 1]$. If $\{b_n\}$ be a sequence defined on $\mathbb{R}^+ \cup \{0\}$ such that $b_n \in (0, 1]$ for all $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty} b_n = \infty$, then $\prod_{n=1}^{\infty} (1 - b_n) = 0$.

Theorem 3.2. *Let $\mathcal{T}$ be a self-contractive-like mappings defined on a nonempty convex and closed subset $\mathcal{C}$ of a Banach space B with $F(\mathcal{T}) \neq \emptyset$ and $\{p_n\}$ be the sequence defined by the new iteration scheme (1.4), where $\{c_n\}$ is a sequence in $[0, 1]$ for all $n \in \mathbb{N}$ such that $\lim_{n\to\infty} \sum_{i=0}^{n} c_i = \infty$. Then $\{p_n\}$ converges strongly to a unique fixed point of $\mathcal{T}$.*

Proof. Assume that p^* is a fixed point of $\mathcal{T}$. By the definition of new iteration (1.4) and contradictive-like mapping

$$\begin{aligned}\|p_{n+1}-p^*\| &= \|\mathcal{T}q_n - \mathcal{T}p^*\| \\ &\leq b\|q_n - p^*\| + f(\|\mathcal{T}p^* - p^*\|) \\ &= b\|q_n - p^*\| \end{aligned} \tag{3.1}$$

$$\begin{aligned}\|q_n-p^*\| &= \|\mathcal{T}r_n - \mathcal{T}p^*\| \\ &\leq b\|r_n - p^*\| + f(\|\mathcal{T}p^* - p^*\|) \\ &= b\|r_n - p^*\| \end{aligned} \tag{3.2}$$

and

$$\begin{aligned}\|r_n-p^*\| &= \|\mathcal{T}((1-c_n)p_n + c_n\mathcal{T}p_n) - \mathcal{T}p^*\| \\ &= b[\|(1-c_n)p_n + c_n\mathcal{T}p_n - p^*\|] + f(\|\mathcal{T}p^* - p^*\|) \\ &\leq b[(1-c_n)\|p_n - p^*\| + c_n\|\mathcal{T}p_n - p^*\|] \\ &\leq b[(1-c_n)\|p_n - p^*\| + bc_n\|p_n - p^*\| + c_n f(\|\mathcal{T}p^* - p^*\|)] \\ &= b[(1-c_n)\|p_n - p^*\| + bc_n\|p_n - p^*\|] \\ &= b[1-c_n(1-b)]\|p_n - p^*\|. \end{aligned} \tag{3.3}$$

By equations (3.1), (3.2) and (3.3), we have

$$\|p_{n+1}-p^*\| \leq b^3[1-c_n(1-b)]\|p_n - p^*\|.$$

Inductively we have

$$\|p_{n+1}-p^*\| \leq b^{3(n+1)} \prod_{i=0}^{i=n} [(1-c_i(1-b)]\|p_0 - p^*\|.$$

Since $b < 1$ so $1 - b > 0$ and $c_i \in [0, 1]$ for all $i \in \mathbb{N}$, we obtain $[1 - c_i(1 - b)] < 1$. It is clear that $(1 - n) \leq e^{-n}$ for all $n \in [0, 1]$. Thus, we see that $(1 - c_i(1 - b) \leq e^{-c_i(1-b)}$. Therefore,

$$\|p_{n+1}-p^*\| \leq b^{3(n+1)} e^{-(1-b)\sum_{i=0}^{i=n} c_i}\|p_0 - p^*\|$$
$$\lim_{n\to\infty}\|p_{n+1}-p^*\| = \lim_{n\to\infty} b^{3(n+1)} e^{-(1-b)\sum_{i=0}^{i=n} \alpha_i}\|p_0 - p^*\|.$$

This yields $\lim_{n\to\infty}\|p_n - p^*\| = 0$.
For the uniqueness of fixed points, suppose that p^* and q^* are any two fixed points of $\mathcal{T}$. Then

$$\|p^* - q^*\| = \|\mathcal{T}p^* - \mathcal{T}q^*\| \leq b\|p^* - q^*\| + f(\|\mathcal{T}p^* - p^*\|) = b\|p^* - q^*\|.$$

Thus, we have $(1 - b)\|p^* - q^*\| = 0$. This implies that $\|p^* - q^*\| = 0$, i.e. $p^* = q^*$. □

Lemma 3.3. *[6] For a real number $\sigma \in [0, 1)$ and a sequence of positive numbers $\{\epsilon_n\}$ such that $\epsilon_n \to 0$ as $n \to \infty$, then for any sequence of positive numbers t_n satisfying*

$$t_{n+1} = \sigma\ t_n + \epsilon_n$$

for $n = 1, 2, 3, ...,$ we have

$$\lim_{n\to\infty} t_n = 0.$$

Theorem 3.4. *Let $\mathcal{T}$ be a self-contractive-like mapping defined on nonempty closed convex subset $\mathcal{C}$ of Banach space B with $F(\mathcal{T}) \neq \emptyset$ and $\{p_n\}$ be the sequence satisfying (1.4), where $\{c_n\} \in [0,1]$ for all positive integers n. Then the new iteration scheme (1.4) is $\mathcal{T}$-stable.*

Proof. Let $\{w_n\}$ be an arbitrary sequence in B and $p_{n+1} = f(\mathcal{T}, p_n)$ be defined by (1.4) which converges to the unique fixed point p^* of $\mathcal{T}$ and $\epsilon_n = \|w_{n+1} - f(\mathcal{T}, w_n)\|$. We need to show that $\lim_{n\to\infty} \epsilon_n = 0$ if and only if $\lim_{n\to\infty} w_n = p^*$.
First we assume that $\lim_{n\to\infty} \epsilon_n = 0$ and

$$\begin{aligned}
\|w_{n+1} - p^*\| &= \|w_{n+1} - f(\mathcal{T}, w_n) + f(\mathcal{T}, w_n) - p^*\| \\
&\leq \|w_{n+1} - f(\mathcal{T}, w_n)\| + \|f(\mathcal{T}, w_n) - p^*\| \\
&= \epsilon_n + \|\mathcal{T}v_n - p^*\| \\
&\leq \epsilon_n + b\|v_n - p^*\| + f(\|\mathcal{T}p^* - p^*\|) \\
&= \epsilon_n + b\|\mathcal{T}u_n - p^*\| \\
&\leq \epsilon_n + b[b\|u_n - p^*\| + f(\|\mathcal{T}p^* - p^*\|)] \\
&= \epsilon_n + b^2\|\mathcal{T}((1-c_n)w_n + c_n\mathcal{T}w_n) - p^*\| \\
&= \epsilon_n + b^2\|\mathcal{T}((1-c_n)w_n + c_n\mathcal{T}w_n) - \mathcal{T}p^*\| \\
&= \epsilon_n + b^2[b[((1-c_n)w_n + c_n\mathcal{T}w_n) - p^*)] + f(\|\mathcal{T}p^* - p^*\|)] \\
&\leq \epsilon_n + b^3[(1-c_n)\|w_n - p\| + c_n\|\mathcal{T}w_n - p^*\|] \\
&\leq \epsilon_n + b^3[(1-c_n)\|w_n - p^*\| + c_n[b\|w_n - p^*\| + f(\|\mathcal{T}p^* - p^*\|)]] \\
&= \epsilon_n + b^3[1 - c_n(1-b)\|w_n - p^*\|]
\end{aligned}$$

Since $(1-b) \leq 1$ and $c_n \in [0,1]$, so $b^3[1 - c_n(1-b)] \leq 1$. By the virtue of Lemma 3.3, we have $\lim_{n\to\infty} \|w_n - p^*\| = 0$ i.e. $\lim_{n\to\infty} w_n = p^*$.

Conversely, suppose that $\lim_{n\to\infty} w_n = p^*$. Then

$$\begin{aligned}
\epsilon_n &= \|w_{n+1} - f(\mathcal{T}, w_n)\| \\
&\leq \|w_{n+1} - p^*\| + \|p^* - f(\mathcal{T}, w_n)\| \\
&\leq \|w_{n+1} - p^*\| + \|\mathcal{T}v_n - p^*\| \\
&\leq \|w_{n+1} - p^*\| + b[\|v_n - p^*\|] + f(\|p^* - \mathcal{T}p^*\|) \\
&= \|w_{n+1} - p^*\| + b[\|\mathcal{T}u_n - p^*\|] \\
&\leq \|w_{n+1} - p^*\| + b[b[\|u_n - p^*\|] + f(\|\mathcal{T}p^* - p^*\|)] \\
&= \|w_{n+1} - p^*\| + b^2[\|u_n - p^*\|] \\
&= \|w_{n+1} - p^*\| + b^2\|\mathcal{T}((1-c_n)w_n + c_n\mathcal{T}w_n) - \mathcal{T}p^*\| \\
&= \|w_{n+1} - p^*\| + b^2\|[b[(1-c_n)w_n + c_n\mathcal{T}w_n - p^*) + f(\|\mathcal{T}p^* - p^*\|)]\| \\
&= \|w_{n+1} - p^*\| + b^3\|(1-c_n)w_n + c_n\mathcal{T}w_n - p^*)\| \\
&\leq \|w_{n+1} - p^*\| + b^3[(1-c_n)\|w_n - p^*\| + c_n\|\mathcal{T}w_n - p^*)\|] \\
&\leq \|w_{n+1} - p^*\| + b^3[(1-c_n)\|w_n - p^*\| + c_n[b\|w_n - p^*\| + f(\|\mathcal{T}p^* - p^*\|)]] \\
&= \|w_{n+1} - p^*\| + b^3[(1 - c_n(1 - bb_n)]\|w_n - p^*\| \qquad (3.4)
\end{aligned}$$

Taking limit as $n \to \infty$ in (3.4) we get $\lim_{n\to\infty} \epsilon_n = 0$. As a result, the new iteration scheme (1.4) is $\mathcal{T}-$stable.

□

4 Data Dependency Theorem

The data-dependence result concerning Mann-Ishikawa iteration is in [32], where the data-dependence of Ishikawa iteration was proved for contraction mappings. Şoltuz et al. in [34] proved data-dependence results for Ishikawa iteration for the contractive-like operators.

We consider the new iteration process (1.4) for the operator S as follows.

$$\begin{cases} w_1 \in C \\ w_{n+1} = Sv_n, \\ v_n = Su_n, \\ u_n = S((1-c_n)w_n + c_n Sw_n), \qquad n \in \mathbb{N} \end{cases} \tag{4.1}$$

where $\{c_n\} \in (0,1)$.

Definition 4.1. [7] Let $\mathcal{T}, S : B \to B$ be two operators. S is said to be approximate operator for $\mathcal{T}$ if for some $\epsilon > 0$ we have $\|\mathcal{T}u - Su\| \leq \epsilon$ for all $u \in B$.

Lemma 4.1. *[34] Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of nonnegative real number for which there exists $m \in \mathbb{N}$ such that for all $n \geq m$ satisfying the relation.*

$$a_{n+1} \leq (1-\lambda_n)a_n + \lambda_n \sigma_n, \tag{4.2}$$

where $\lambda_n \in (0,1)$ for all $n \in \mathbb{N}$, $\sum_{n=0}^{\infty} \lambda_n = \infty$. and $\sigma_n \geq 0$ for all $n \in \mathbb{N}$, Then $0 \leq \limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} \sigma_n$.

Theorem 4.2. *Let $\mathcal{T}$ and S be defined on a nonempty subset $\mathcal{C}$ such that $\mathcal{T}$ is a contractive-like operator with a fixed point p^* and S is an approximate operator for $\mathcal{T}$ with $Sq^* = q^*$. Let $\{p_n\}_{n=0}^{\infty}$ be an iterative sequence generated by (1.4) and iterative sequence $\{w_n\}_{n=0}^{\infty}$ is generated by (4.1) with the assumption $(1-c_n) < c_n$ and $\sum_{n=1}^{\infty} c_n = \infty$. If $\lim_{n\to\infty} w_n = q^*$, then we have $\|p^* - q^*\| \leq \frac{7\epsilon}{1-b}$, where $\epsilon > 0$ is a fixed number.*

Proof. By using equations (1.4) and (4.1), we obtain

$$\begin{aligned}
|r_n - u_n\| &= \|\mathcal{T}((1-c_n)p_n + c_n\mathcal{T}p_n) - S((1-c_n)w_n + c_nSw_n)\| \\
&\leq \|\mathcal{T}((1-c_n)p_n + c_n\mathcal{T}p_n) - \mathcal{T}((1-c_n)w_n + c_nSw_n)\| \\
&\quad + \|\mathcal{T}((1-c_n)w_n + c_nSw_n) - S((1-c_n)w_n + c_nSw_n) \\
&\leq b\|(1-c_n)p_n + c_n\mathcal{T}p_n - ((1-c_n)w_n + c_nSw_n)\| \\
&\quad + f(\|\mathcal{T}((1-c_n)p_n + c_n\mathcal{T}p_n) - (1-c_n)p_n + c_n\mathcal{T}p_n\|) + \epsilon \\
&\leq b[(1-c_n)\|p_n - w_n\| + c_n\|\mathcal{T}p_n - Sw_n\|] \\
&\quad + f(\|\mathcal{T}((1-c_n)p_n + c_n\mathcal{T}p_n) - (1-c_n)p_n + c_n\mathcal{T}p_n\|) + \epsilon \\
&= b[(1-c_n)\|p_n - w_n\| + c_n\|\mathcal{T}p_n - \mathcal{T}w_n + \mathcal{T}w_n - Sw_n\|] \\
&\quad + f(\|\mathcal{T}((1-c_n)p_n + c_n\mathcal{T}p_n) - (1-c_n)p_n + c_n\mathcal{T}p_n\|) + \epsilon \\
&\leq b[(1-c_n)\|p_n - w_n\| + c_n[\|\mathcal{T}p_n - \mathcal{T}w_n\| + \|\mathcal{T}w_n - Sw_n\|]] \\
&\quad + f(\|\mathcal{T}((1-c_n)p_n + c_n\mathcal{T}p_n) - (1-c_n)p_n + c_n\mathcal{T}p_n\|) + \epsilon \\
&\leq b[(1-c_n)\|p_n - w_n\| + c_n[b\|p_n - w_n\| + f(\|\mathcal{T}p_n - p_n\|) + \epsilon]] \\
&\quad + f(\|\mathcal{T}((1-c_n)p_n + c_n\mathcal{T}p_n) - (1-c_n)p_n + c_n\mathcal{T}p_n\|) + \epsilon \\
&\leq b[(1-c_n)\|p_n - w_n\| + c_nb\|p_n - w_n\| + c_nf(\|\mathcal{T}p_n - p_n\|) + c_n\epsilon] \\
&\quad + f(\|\mathcal{T}((1-c_n)p_n + c_n\mathcal{T}p_n) - (1-c_n)p_n + c_n\mathcal{T}p_n\|) + \epsilon \\
&\leq b[(1-c_n + c_nb)\|p_n - w_n\|] + bc_nf(\|\mathcal{T}p_n - p_n\|) + bc_n\epsilon \\
&\quad + f(\|\mathcal{T}((1-c_n)p_n + c_n\mathcal{T}p_n) - (1-c_n)p_n + c_n\mathcal{T}p_n\|) + \epsilon \\
&\leq b[(1-c_n(1-b)]\|p_n - w_n\| + bc_nf(\|\mathcal{T}p_n - p_n\|) + bc_n\epsilon \\
&\quad + f(\|\mathcal{T}((1-c_n)p_n + c_n\mathcal{T}p_n) - (1-c_n)p_n + c_n\mathcal{T}p_n\|) + \epsilon
\end{aligned} \tag{4a}$$

Now
$$\begin{aligned}\|q_n - v_n\| &= \|\mathcal{T}r_n - Su_n\|\\ &\le \|\mathcal{T}r_n - \mathcal{T}u_n\| + \|\mathcal{T}u_n - Su_n\|\\ &\le b\|r_n - u_n\| + f(\|\mathcal{T}r_n - r_n\|) + \|\mathcal{T}u_n - Su_n\|\\ &\le b\|r_n - u_n\| + f(\|\mathcal{T}r_n - r_n\|) + \epsilon \end{aligned} \tag{4b}$$

Then by using equations $(4a)$ and $(4b)$, we obtain

$$\begin{aligned}\|p_{n+1} - w_{n+1}\| &= \|\mathcal{T}q_n - Sv_n\|\\ &\le \|\mathcal{T}q_n - \mathcal{T}v_n\| + \|\mathcal{T}v_n - Sv_n\|\\ &\le b\|q_n - v_n\| + f(\|\mathcal{T}q_n - q_n\|) + \epsilon\\ &\le b[b\|r_n - u_n\| + f(\|\mathcal{T}r_n - r_n\|) + \epsilon] + f(\|\mathcal{T}q_n - q_n\|) + \epsilon\\ &\le b^2\|r_n - u_n\| + bf(\|\mathcal{T}r_n - r_n\|) + b\epsilon + f(\|\mathcal{T}q_n - q_n\|) + \epsilon\\ &\le b^2[b[(1-c_n(1-b)]\|p_n - w_n\| + bc_nf(\|\mathcal{T}p_n - p_n\|) + bc_n\epsilon\\ &+ f(\|\mathcal{T}((1-c_n)p_n + c_n\mathcal{T}p_n) - (1-c_n)p_n + c_n\mathcal{T}p_n\|) + \epsilon]\\ &+ bf(\|\mathcal{T}r_n - r_n\|) + b\epsilon + f(\|\mathcal{T}q_n - q_n\|) + \epsilon\\ &\le b^3[(1-c_n(1-b)]\|p_n - w_n\| + c_nf(\|\mathcal{T}p_n - p_n\|) + c_n\epsilon]\\ &+ b^2f(\|\mathcal{T}((1-c_n)p_n + c_n\mathcal{T}p_n) - (1-c_n)p_n + c_n\mathcal{T}p_n\|) + b^2\epsilon\\ &+ bf(\|\mathcal{T}r_n - r_n\|) + b\epsilon + f(\|\mathcal{T}q_n - q_n\|) + \epsilon.\end{aligned}$$

Since $b \in (0,1)$ so

$$\begin{aligned}\|p_{n+1} - w_{n+1}\| &\le (1-c_n(1-b))\|p_n - w_n\| + c_nf(\|\mathcal{T}p_n - p_n\|) + c_n\epsilon\\ &+ f(\|\mathcal{T}((1-c_n)p_n + c_n\mathcal{T}p_n) - (1-c_n)p_n + c_n\mathcal{T}p_n\|) + \epsilon\\ &+ f(\|\mathcal{T}r_n - r_n\|) + \epsilon + f(\|\mathcal{T}q_n - q_n\|) + \epsilon.\end{aligned}$$

By the assumption $(1-c_n) < c_n$, i.e. $1 < 2c_n$ and taking $(1-c_n)p_n + c_nTp_n) = a_n$, we obtain

$$\begin{aligned}&\|p_{n+1} - w_{n+1}\|\\ &\le (1-c_n(1-b))\|p_n - w_n\| + c_nf(\|\mathcal{T}p_n - p_n\|) + c_n\epsilon\\ &\quad + (1-c_n+c_n)f(\|\mathcal{T}(a_n) - (a_n)\|) + (1-c_n+c_n)\epsilon\\ &\quad + (1-c_n+c_n)f(\|\mathcal{T}r_n - r_n\|) + (1-c_n+c_n)\epsilon + (1-c_n+c_n)f(\|\mathcal{T}q_n - q_n\|)\\ &\quad + (1-c_n+c_n)\epsilon\\ &\le (1-c_n(1-b))\|p_n - w_n\| + c_nf(\|\mathcal{T}p_n - p_n\|) + c_n\epsilon\\ &\quad + (2c_n-c_n+c_n)f(\|\mathcal{T}(a_n) - (a_n)\|) + (2c_n-c_n+c_n)\epsilon\\ &\quad + (2c_n-c_n+c_n)f(\|\mathcal{T}r_n - r_n\|) + (2c_n-c_n+c_n)\epsilon + (2c_n-c_n+c_n)f(\|\mathcal{T}q_n - q_n\|)\\ &\quad + (2c_n-c_n+c_n)\epsilon\\ &\le (1-c_n(1-b))\|p_n - w_n\| + c_nf(\|\mathcal{T}p_n - p_n\|) + c_n\epsilon\\ &\quad + (2c_n)f(\|\mathcal{T}(a_n) - (a_n)\|) + (2c_n)\epsilon\\ &\quad + (2c_n)f(\|\mathcal{T}r_n - r_n\|) + (2c_n)\epsilon + (2c_n)f(\|\mathcal{T}q_n - q_n\|) + (2c_n)\epsilon\\ &= (1-c_n(1-b))\|p_n - w_n\| + c_n[f(\|\mathcal{T}p_n - p_n\|) + \epsilon\\ &\quad + 2f(\|\mathcal{T}(a_n) - (a_n)\|) + 2\epsilon + 2f(\|\mathcal{T}r_n - r_n\|) + 2\epsilon + 2f(\|\mathcal{T}q_n - q_n\|) + 2\epsilon]\\ &= (1-c_n(1-b))\|p_n - w_n\| + c_n[f(\|\mathcal{T}p_n - p_n\|) + 2f(\|\mathcal{T}(a_n) - (a_n)\|)\\ &\quad + 2f(\|\mathcal{T}r_n - r_n\|) + 2f(\|\mathcal{T}q_n - q_n\|) + 7\epsilon]\\ &= (1-c_n(1-b))\|p_n - w_n\| + c_n(1-b)\cdot\\ &\quad \frac{[f(\|\mathcal{T}p_n - p_n\|) + 2f(\|\mathcal{T}(a_n) - (a_n)\|) + 2f(\|\mathcal{T}r_n - r_n\|) + 2f(\|\mathcal{T}q_n - q_n\|) + 7\epsilon]}{(1-b)}\end{aligned}$$

Take $a_n = \|p_n - w_n\|, \lambda_n = c_n(1-b)$, and

$$\sigma_n = \frac{[f(\|\mathcal{T}p_n - p_n\|) + 2f(\|\mathcal{T}(a_n) - (a_n)\|) + 2f(\|\mathcal{T}r_n - r_n\|) + 2f(\|\mathcal{T}q_n - q_n\|) + 7\epsilon]}{(1-b)}.$$

Since f is a continuous strictly increasing mapping and $\{p_n\}, \{q_n\}, \{r_n\}$ are sequences converge to the fixed point of $\mathcal{T}$, then

$$\lim_{n\to\infty} f(\|\mathcal{T}p_n - p_n\|) = \lim_{n\to\infty} f(\|\mathcal{T}(a_n) - (a_n)\|) = \lim_{n\to\infty} f(\|\mathcal{T}r_n - r_n\|) = \lim_{n\to\infty} f(\|\mathcal{T}q_n - q_n\|) = 0.$$

By using Lemma 4.1, we have

$$\begin{aligned}
&\limsup_{n\to\infty} \|p_n - w_n\| \\
&\leq \limsup_{n\to\infty} \frac{[f(\|\mathcal{T}p_n - p_n\|) + 2f(\|\mathcal{T}(a_n) - (a_n)\|) + 2f(\|\mathcal{T}r_n - r_n\|) + 2f(\|\mathcal{T}q_n - q_n\|) + 7\epsilon]}{(1-b)} \\
&\leq \limsup_{n\to\infty} \frac{7\epsilon}{(1-b)}
\end{aligned}$$

By using Theorem 3.2 and the assumption $\lim_{n\to\infty} w_n = q^*$, we have

$$\|p^* - q^*\| \leq \frac{7\epsilon}{(1-b)}$$

This completes the proof. □

5 Rate of Convergence

In this section, we will prove that the new iteration process (1.4) converges faster than the iteration process (1.1) and has the same rate of convergence as that of iteration (1.2) for contrtactive-like mappings.

Theorem 5.1. *Let $\mathcal{C}, B, \mathcal{T}$ and $\{p_n\}$ be as in Theorem 3.4. If $p^* \in F(\mathcal{T})$ then the iteration process (1.4) converges faster than iteration (1.1) and have same rate of convergence as that of iteration process (1.2).*

Proof. For the given iteration process (1.4), we have

$$\begin{aligned}
\|r_n - p^*\| &= \|\mathcal{T}((1-c_n)p_n + c_n\mathcal{T}p_n) - \mathcal{T}p^*\| \\
&\leq b[\|(1-c_n)p_n + c_n\mathcal{T}p_n - p^*\|] + f(\|\mathcal{T}p^* - p^*\|) \\
&\leq b[(1-c_n)\|p_n - p^*\| + c_n\|\mathcal{T}p_n - p^*\|] \\
&\leq b[(1-c_n)\|p_n - p^*\| + c_n\|\mathcal{T}p_n - \mathcal{T}p^*\|] \\
&\leq b[(1-c_n)\|p_n - p^*\| + c_n[b\|p_n - p^*\| + f(\|\mathcal{T}p^* - p^*\|)] \\
&= b[1 - (1-b)c_n]\|p_n - p^*\|.
\end{aligned}$$

Since $b \in [0,1), c_n \in (0,1)$ so we have $c_n(1-b) < 1$. This implies that $[1 - c_n(1-b)] < 1$.

$$\text{Thus, we get} \qquad \|r_n - p^*\| \leq b\|p_n - p^*\| \tag{5.1}$$

$$\begin{aligned}
\|q_n - p^*\| &= \|\mathcal{T}r_n - \mathcal{T}p^*\| \\
&\leq b\|r_n - p^*\| + f(\|\mathcal{T}p^* - p^*\|) \\
&\leq b^2\|p_n - p^*\|.
\end{aligned} \tag{5.2}$$

$$\text{Therefore}\qquad \begin{aligned}\|p_{n+1}-p^*\| &= \|\mathcal{T}q_n-\mathcal{T}p^*\|\\ &\le b\|q_n-p^*\|+f(\|\mathcal{T}p^*-p^*\|)\\ &\le b^3\|p_n-p^*\|\\ &\vdots\\ &\le b^{3n}\|p_1-p^*\| \end{aligned}\tag{5.3}$$

Similarly, from equation (1.1), we obtain

$$\begin{aligned}\|p_n-p^*\| &= \|\mathcal{T}((1-d_n)p_n+d_n\mathcal{T}p_n)-\mathcal{T}p^*\|\\ &\le b[\|(1-d_n)p_n+d_n\mathcal{T}p_n-p^*\|]+f(\|\mathcal{T}p^*-p^*\|)\\ &\le b[(1-d_n)\|p_n-p^*\|+d_n\|\mathcal{T}p_n-p^*\|]\\ &\le b[(1-d_n)\|p_n-p^*\|+d_n[b\|p_n-p^*\|+f(\|\mathcal{T}p^*-p^*\|)]]\\ &= b[1-(1-b)d_n]\|p_n-p^*\|. \end{aligned}$$

Since $b\in[0,1), d_n\in(0,1)$ so $d_n(1-b)<1$. This implies that $[1-d_n(1-b)]<1$. Hence

$$\|r_n-p^*\|\le b\|p_n-p^*\| \tag{5.4}$$

$$\text{and}\qquad \begin{aligned}\|q_n-p^*\| &= \|\mathcal{T}r_n-\mathcal{T}p^*\|\\ &\le b\|r_n-p^*\|+f(\|\mathcal{T}p^*-p^*\|)\\ &\le b^2\|p_n-p^*\|. \end{aligned}\tag{5.5}$$

Then

$$\begin{aligned}\|p_{n+1}-p^*\| &= \|(1-c_n)\mathcal{T}r_n+c_n\mathcal{T}r_n-p^*\|\\ &\le (1-c_n)\|\mathcal{T}r_n-p^*\|+c_n\|\mathcal{T}q_n-p^*\|\\ &\le (1-c_n)[b\|r_n-p^*\|+f(\|\mathcal{T}p^*-p^*\|)]+c_n[b\|q_n-p^*\|+f(\|\mathcal{T}p^*-p^*\|)]\\ &= (1-c_n)b\|r_n-p^*\|+c_nb\|q_n-p^*\|\\ &\le b(1-c_n)\|r_n-p^*\|+c_nb^2\|r_n-p^*\|\\ &\le b^2[1-c_n(1-b)]\|p_n-p^*\| \end{aligned}$$

Since $b\in[0,1), c_n\in(0,1)$ so $c_n(1-b)<1$. This implies that $[1-c_n(1-b)]<1$. Thus, we have

$$\begin{aligned}\|p_{n+1}-p^*\| &\le b^2\|p_n-p^*\|\\ &\vdots\\ &\le b^{2n}\|p_1-p^*\|. \end{aligned}$$

Let $\zeta_n=b^{3n}\|p_1-p^*\|$ and $\delta_n=b^{2n}\|p_1-p^*\|$. Then

$$\lim_{n\to\infty}\frac{\zeta_n}{\delta_n}=\lim_{n\to\infty}\frac{b^{3n}\|p_1-p^*\|}{b^{2n}\|p_1-p^*\|}=0.$$

This proves that the new iteration (1.4) converges faster than the iteration (1.1).

Similarly, under a contractive-like condition, we get the following result using iteration procedure (1.2).

$$\|p_{n+1} - p^*\| \leq b^{3n}\|p_1 - p^*\|. \tag{5.6}$$

For iterations (1.4) and (1.2), let $\zeta_n = b^{3n}\|p_1 - p^*\|$ and $\vartheta_n = b^{3n}\|p_1 - p^*\|$. Then

$$\lim_{n\to\infty} \frac{\zeta_n}{\vartheta_n} = \lim_{n\to\infty} \frac{b^{3n}\|p_1 - p^*\|}{b^{3n}\|p_1 - p^*\|} = 1.$$

This proves that the iteration (1.4) converges at the same rate as that of iteration (1.2) for contractive-like mappings.

□

Example 5.1. Let $B = \mathbb{R}$, $C = [0,3]$, and let $\mathcal{T} : C \to C$ be defined by

$$\mathcal{T}(u) = \begin{cases} \frac{u}{4}, & \text{if } u \in [0,1) \\ \frac{u}{8}, & \text{if } u \in [1,3]. \end{cases}$$

It is clear that $0 \in F(\mathcal{T})$. The discontinuity of $\mathcal{T}$ at 1 shows that $\mathcal{T}$ is neither contraction mapping nor nonexpansive mapping. Next, we will prove that $\mathcal{T}$ is contractive-like mapping. We define $f : [0,\infty) \to [0,\infty)$ as a strictly increasing continuous function, by

$$f(u) = \begin{cases} \frac{u}{3}, & \text{if } u \in (0,\infty) \\ 0, & \text{if } u = 0. \end{cases}$$

If $u \in [0,1)$, then

$$\|u - \mathcal{T}u\| = \|u - \frac{u}{4}\| = \frac{3u}{4} \text{ and } f(\frac{3u}{4}) = \frac{u}{4}.$$

If $u \in [1,3]$, then

$$\|u - \mathcal{T}u\| = \|u - \frac{u}{8}\| = \frac{7u}{8} \text{ and } f(\frac{7u}{8}) = \frac{7u}{24}.$$

Now we consider the following four cases.

Case 1 : If $u, v \in [0,1)$, then

$$\begin{aligned} \|\mathcal{T}u - \mathcal{T}v\| &= \|\frac{u}{4} - \frac{v}{4}\| = \frac{1}{4}\|u - v\| \leq \frac{1}{4}\|u - v\| + \|\frac{u}{4}\| \\ &= \frac{1}{4}\|u - v\| + f(\|\frac{3u}{4}\|) = \frac{1}{4}\|u - v\| + f(\|u - \mathcal{T}u\|). \end{aligned}$$

Case 2 : If $u \in [0,1), v \in [1,3]$, then

$$\begin{aligned} \|\mathcal{T}u - \mathcal{T}v\| &= \|\frac{u}{4} - \frac{v}{8}\| = \|\frac{u}{8} + \frac{u}{8} - \frac{v}{8}\| \leq \frac{1}{8}\|u - v\| + \|\frac{u}{8}\| \\ &\leq \frac{1}{4}\|u - v\| + \|\frac{u}{4}\| \leq \frac{1}{4}\|u - v\| + f(\|\frac{3u}{4}\|) \\ &= \frac{1}{4}\|u - v\| + f(\|u - \mathcal{T}u\|). \end{aligned}$$

Case 3 : If $u \in [1,3], v \in [0,1)$, then clearly **Case 3** is similar to **Case 2**.

Case 4 : If $u, v \in [1,3]$, then

$$\begin{aligned} \|\mathcal{T}u - \mathcal{T}v\| &= \|\frac{u}{8} - \frac{v}{8}\| \leq \frac{1}{8}\|u - v\| + \|\frac{7u}{24}\| \\ &\leq \frac{1}{4}\|u - v\| + f(\|\frac{7u}{8}\|) = \frac{1}{4}\|u - v\| + f(\|u - \mathcal{T}u\|). \end{aligned}$$

Case 1, 2, 3 and **Case 4** show that $\mathcal{T}$ is a contractive-like mapping for $b = \frac{1}{4}$.

Steps	Mann	Ishikawa	Noor	PicardS	Thakur	Piri	Chanchal	New iteration
0	2.000000000000	2.000000000000	2.000000000000	2.000000000000	2.000000000000	2.000000000000	2.000000000000	2.000000000000
1	0.750000000000	0.750000000000	0.750000000000	0.031250000000	0.250000000000	0.011718750000	0.003906250000	0.001464843750
2	0.398437500000	0.392578125000	0.392567952474	0.001892089844	0.060231526693	0.000369644165	0.000057506561	0.000012159348
3	0.232421875000	0.226732042101	0.226722236537	0.000115518217	0.014632973648	0.000012915085	0.000000855901	0.000000110827
4	0.145263671875	0.140809380019	0.140801898689	0.000007105489	0.003583541457	0.000000488508	0.000000012897	0.000000001082
5	0.095741965554	0.092406155638	0.092400697310	0.000000439047	0.000882059973	0.000000019620	0.000000000196	0.000000000011
6	0.065822601318	0.063331985255	0.063328003540	0.000000027206	0.000217831526	0.000000000826	0.000000000003	**0.000000000000**
7	0.046835312477	0.044957746190	0.044954804198	0.000000001689	0.000053917551	0.000000000036	**0.000000000000**	0.000000000000
8	0.034290139492	0.032855453021	0.032853243432	0.000000000105	0.000013367479	0.000000000002	0.000000000000	0.000000000000
9	0.025717604619	0.024605564050	0.024603876666	0.000000000007	0.000003318175	**0.000000000000**	0.000000000000	0.000000000000

Steps	S iteration	Abbas	K^* iteration	Thakur new	M^* iteration	M iteration	New iteration
0	2.000000000000	2.000000000000	2.000000000000	2.000000000000	2.000000000000	2.000000000000	2.000000000000
1	0.250000000000	0.250000000000	0.011718750000	0.031250000000	0.011718750000	0.046875000000	0.001464843750
2	0.060546875000	0.060231526693	0.000369644165	0.001892089844	0.000383377075	0.001556396484	0.000012159348
3	0.014786331742	0.014632973648	0.000012915085	0.000115518217	0.000013838626	0.000056743622	0.000000110827
4	0.003638010319	0.003583541457	0.000000488508	0.000007105489	0.000000537145	0.000002216548	0.000000001082
5	0.000899167323	0.000882059973	0.000000019620	0.000000439047	0.000000022031	0.000000091307	0.000000000011
6	0.000222872502	0.000217831526	0.000000000826	0.000000027206	0.000000000944	0.000000003923	**0.000000000000**
7	0.000055347221	0.000053917551	0.000000000036	0.000000001689	0.000000000042	0.000000000174	0.000000000000
8	0.000013762891	0.000013367479	0.000000000002	0.000000000105	0.000000000002	0.000000000008	0.000000000000
9	0.000003425632	0.000003318175	**0.000000000000**	0.000000000007	**0.000000000000**	**0.000000000000**	0.000000000000

By using example 5.1, we tried to show that the rate of convergence of the iteration process (1.4) is better than some known iteration processes for contractive-like mapping. Parameters are

$$a_n = \frac{5}{n+7}, b_n = \frac{n}{7n+8}, c_n = \frac{n}{(5n+7)^2}, \quad \text{for all} \quad n \in \mathbb{N}.$$

Clearly $p^* = 0$ is a fixed point of contractive-like mapping. $\mathcal{T}$. Table 1 and Table 2 below show the behaviour of some iteration processes to the fixed point of $\mathcal{T}$ for an initial value of $x_0 = 2$.

Table 1: Comparison Table

Convergence behaviour of the iterative schemes of Mann, Ishikawa, Noor, Agarwal, PicardS, Thakur et al., Piri et al., Chanchal et al. and the new iteration (1.4) for the function given in Example 5.1 when initial guess $x_0 = 2$.

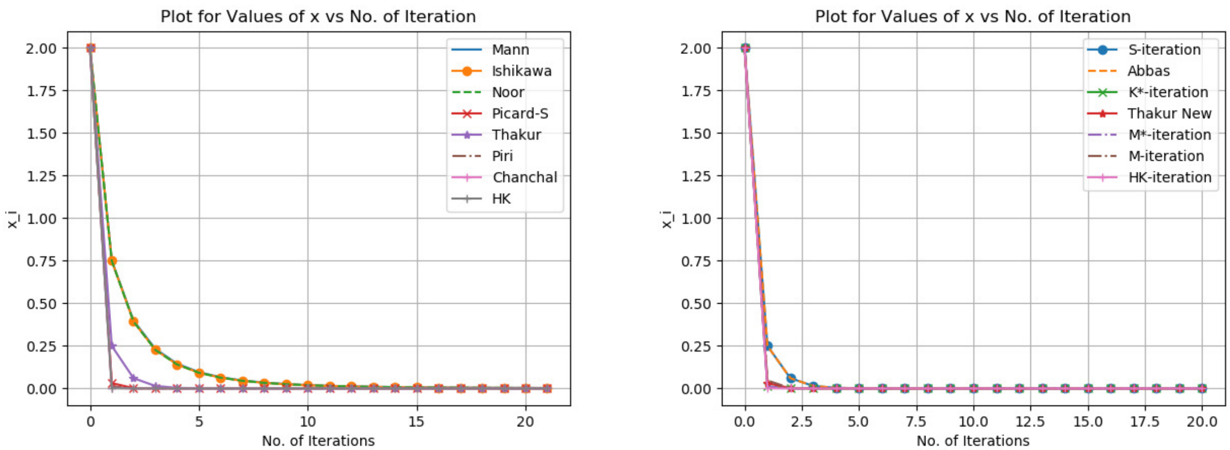
Table 2: Comparison Table

Convergence behaviour of iterative schemes of S-iteration, Abbas, K^* iteration, Thakur new iteration, M^* iteration, M iteration and the new iteration (1.4) for the function given in Example 5.1 when initial guess $x_0 = 2$.

6 Convergence Theorem

Now, we introduce the convergence theorem.

Theorem 6.1. *Let $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ be a mapping satisfying the condition $B_{\gamma,\mu}$ defined on a nonempty convex and closed subset $\mathcal{C}$ of a UCBS B with $F(\mathcal{T}) \neq \emptyset$ and $\{p_n\}$ be the sequence defined by iteration scheme (1.4), where sequences $\{c_n\} \in [0,1]$ for all $n \in \mathbb{N}$. Then $F(\mathcal{T}) \neq \emptyset$ if and only if $\{p_n\}$ is bounded and $\lim_{n\to\infty} \|\mathcal{T}p_n - p_n\| = 0$.*



(a) Graph corresponding to Table 1.

(b) Graph corresponding to Table 2.

Figure 1: Comparison graph of various iteration processes

Proof. Since $F(\mathcal{T})$ is nonempty, by Lemma 2.2 $\mathcal{T}$ is quasi-nonexpansive. If $p^* \in F(\mathcal{T})$, then from the iteration (1.4)

$$\begin{aligned} \|r_n - p^*\| &= \|\mathcal{T}((1-c_n)p_n + c_n\mathcal{T}r_n) - p^*\| \\ &\leq \|(1-c_n)p_n + c_n\mathcal{T}p_n - p^*\| \\ &\leq (1-c_n)\|p_n - p^*\| + c_n\|\mathcal{T}p_n - p^*\| \\ &\leq (1-c_n)\|p_n - p^*\| + c_n\|p_n - p^*\| \leq \|p_n - p^*\| \end{aligned} \tag{6.1}$$

$$\text{and} \qquad \|q_n - p^*\| = \|\mathcal{T}r_n - p^*\| \leq \|r_n - p^*\| \leq \|p_n - p^*\|. \tag{6.2}$$

from equations (6.1) and (6.2), we have

$$\|p_{n+1} - p^*\| = \|\mathcal{T}q_n - p^*\| \leq \|q_n - p^*\| \leq \|p_n - p^*\|. \tag{6.3}$$

Thus $\|p_n - p^*\|$ is bounded below and nonincreasing. Hence $\lim_{n\to\infty} \|p_n - p^*\|$ exists. Assume that

$$\lim_{n\to\infty} \|p_n - p^*\| = \varepsilon. \tag{6.4}$$

Again by (6.2), we have

$$\limsup \|q_n - p^*\| \leq \limsup \|p_n - p^*\| = \varepsilon.$$

Therefore,

$$\limsup \|q_n - p^*\| \leq \varepsilon. \tag{6.5}$$

Since $\mathcal{T}$ is a mapping satisfying the condition $B_{\gamma,\mu}$ with a fixed point, it implies that $\mathcal{T}$ is quasi-nonexpansive mapping. Hence,

$$\limsup \|\mathcal{T}p_n - p^*\| \leq \limsup \|p_n - p^*\| = \varepsilon. \tag{6.6}$$

Again by equation (6.1), we obtain

$$\|r_n - p^*\| \leq \|p_n - p^*\|$$

$$\text{Thus,} \qquad \|p_{n+1} - p^*\| = \|\mathcal{T}q_n - p^*\| \leq \|q_n - p^*\|$$

which implies that

$$\liminf \|p_{n+1} - p^*\| \leq \liminf \|q_n - p^*\|.$$

As a result, we have

$$\varepsilon \leq \liminf \|q_n - p^*\|. \tag{6.7}$$

From equations (6.5) and (6.7), we have

$$\lim_{n\to\infty} \|q_n - p^*\| = \varepsilon. \tag{6.8}$$

By equations (6.4) and (6.6), we have

$$\begin{aligned}
\varepsilon = \lim_{n\to\infty} \|q_n - p^*\| &= \lim_{n\to\infty} \|\mathcal{T}r_n - p^*\| \\
&= \lim_{n\to\infty} \|\mathcal{T}(\mathcal{T}((1-c_n)p_n + c_n\mathcal{T}p_n)) - p^*\| \\
&\leq \lim_{n\to\infty} \|(\mathcal{T}((1-c_n)p_n + c_n\mathcal{T}p_n)) - p^*\| \\
&\leq \lim_{n\to\infty} \|((1-c_n)p_n + c_n\mathcal{T}p_n) - p^*\| \\
&\leq \lim_{n\to\infty} ((1-c_n)\|p_n - p^*\| + c_n\|\mathcal{T}p_n - p^*\|) \\
&\leq \lim_{n\to\infty} ((1-c_n)\|p_n - p^*\| + c_n\|p_n - p^*\|) \\
&\leq \lim_{n\to\infty} ((1-c_n)\varepsilon + c_n\varepsilon) = \varepsilon.
\end{aligned}$$

This implies that

$$\lim_{n\to\infty} ((1-c_n)\|p_n - p^*\| + c_n\|\mathcal{T}p_n - p^*\| = \varepsilon. \tag{6.9}$$

Now, using equations (6.4), (6.6) and Theorem 2.6, we conclude that $\lim_{n\to\infty} \|p_n - \mathcal{T}p_n\| = 0$.

Conversely, suppose that $\{p_n\}$ is bounded and $\lim_{n\to\infty} \|p_n - \mathcal{T}p_n\| = 0$. Let $p^* \in A(\mathcal{C}, \{p_n\})$. By Proposition 2.5 (3), for $\gamma = \frac{\lambda}{2}, \lambda \in [0,1]$ we have

$$\begin{aligned}
\|p_n - \mathcal{T}p^*\| \leq{}& (3-\lambda)\|p_n - \mathcal{T}p_n\| + (1 - \frac{\lambda}{2})\|p_n - p^*\| + \mu(2\|p_n - \mathcal{T}p_n\| + \|p_n - \mathcal{T}p^*\| + \|p^* - \mathcal{T}p_n\| \\
&+ 2\|\mathcal{T}p_n - \mathcal{T}^2p_n\|). \\
&\leq (3-\lambda)\|p_n - \mathcal{T}p_n\| + (1 - \frac{\lambda}{2})\|p_n - p^*\| + \mu(2\|p_n - \mathcal{T}p_n\| + \|p_n - \mathcal{T}p^*\| + \|p_n - p^*\| \qquad + \|p_n - \mathcal{T}p_n\| + 2\|p_n
\end{aligned}$$

Since $\lim_{n\to\infty} \|p_n - \mathcal{T}p_n\| = 0$ and by proposition 2.5 (1), we have

$$(1-\mu)\|p_n - \mathcal{T}p^*\| \leq (1 - \frac{\lambda}{2} + \mu)\|p_n - p^*\|.$$

Taking lim sup in both sides, this yields

$$(1-\mu)\limsup_{n\to\infty}\|p_n - \mathcal{T}p^*\| \le (1-\frac{\lambda}{2}+\mu)\limsup_{n\to\infty}\|p_n - p^*\|$$
$$\limsup_{n\to\infty}\|p_n - \mathcal{T}p^*\| \le \frac{(1-\frac{\lambda}{2}+\mu)}{(1-\mu)}\limsup_{n\to\infty}\|p_n - p^*\|$$

Since $2\mu \le \gamma = \frac{\lambda}{2}$ so $\frac{(1-\frac{\lambda}{2}+\mu)}{(1-\mu)} \le 1$

Hence, the conclusion is that $r(\mathcal{T}p^*, \{p_n\}) \le r(p^*, \{p_n\})$. So $\mathcal{T}p^* \in A(\mathcal{C}, \{p_n\}$ Since $\mathbb{B}$ is uniformly convex, so it consists only one member. Thus we have $\mathcal{T}p^* = p^*$.

□

It is obvious that if $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ is nonexpansive, then it satisfies condition B_γ, μ, for $\gamma = \mu = 0$. Thus, we obtain the following corollary.

Corollary 6.2. *Let $B, \mathcal{C}$ and $\{p_n\}$ be as in Theorem 6.1 and $\mathcal{T}$ be a nonexpansive self mapping defined on a nonempty closed and convex subset $\mathcal{C}$ of a UCBS B, sequence $\{p_n\}$ defined by (1.4), where $\{c_n\}$ is sequences in $[0,1]$ for all $n \in \mathbb{N}$. Then $F(\mathcal{T}) \neq \emptyset$ iff $\{p_n\}$ is bounded and $\lim_{n\to\infty}\|p_n - \mathcal{T}p_n\| = 0$.*

Theorem 6.1 plays an important role in proving the following weak convergence theorem.

Theorem 6.3. *$\mathcal{T}$ be a generalized nonexpansive self mapping defined on a nonempty closed convex subset $\mathcal{C}$ of a uniformly convex Banach space B satisfying condition $B_{\gamma,\mu}, \{p_n\}$ defined by (1.4), where $\{c_n\}$ is sequences in $[0,1]$ for all $n \in \mathbb{N}$. Suppose that B have Opial's property and $F(\mathcal{T}) \neq \phi$. Then the sequence $\{p_n\}$ converges weakly to an element of $F(\mathcal{T})$.*

Proof. In Theorem 6.1, it is proved that $\{p_n\}$ is bounded sequence, $\lim_{n\to\infty}\|p_n - \mathcal{T}p_n\| = 0$ and $\lim_{n\to\infty}\|p_n - p^*\|$ exists. Since B is uniformly convex, it is reflexive. Therefore, there are a subsequence $\{p_{n_i}\}$ of $\{p_n\}$ such that $\{p_{n_i}\} \rightharpoonup p_1 \in C$. By Proposition 2.1, p_1 is an element of $F(\mathcal{T})$. It is sufficient to prove that $\{p_n\}$ converges weakly to p_1 . If we assume that $\{p_n\}$ does not converge weakly to p_1. Then, there exists a weakly convergent subsequence $\{p_{n_k}\}$ of $\{p_n\}$ which converges weakly to $p_2 \in \mathcal{C}$ and $p_1 \neq p_2$.

Again, by Proposition 2.1, $p_2 \in F(\mathcal{T})$. By Theorem 6.1, $\lim_{n\to\infty}\|p_n - p^*\|$ exists for all fixed points $p^* \in F(\mathcal{T})$. By the Opial's property

$$\begin{aligned}\lim_{n\to\infty}\|p_n - p_1\| &= \lim_{j\to\infty}\|p_{n_i} - p_1\| < \lim_{j\to\infty}\|p_{n_i} - p_2\| \\ &= \lim_{n\to\infty}\|p_n - p_2\| = \lim_{k\to\infty}\|p_{n_k} - p_2\| \\ &< \lim_{k\to\infty}\|p_{n_k} - p_1\| = \lim_{n\to\infty}\|p_n - p_1\|,\end{aligned}$$

which is a contradiction. Thus, $p_1 = p_2$. This proves that $\{p_n\} \rightharpoonup p_1 \in F(\mathcal{T})$. □

We now a prove strong convergence theorem for the mapping satisfying the condition $B_{\gamma,\mu}$.

Theorem 6.4. *Let $\mathcal{T}, B, \mathcal{C}$ and $\{p_n\}$ be as in Theorem 6.1. Suppose that $p^* \in F(\mathcal{T}) \neq \emptyset$ and $\liminf_{n\to\infty}\rho(p_n, F(\mathcal{T})) = 0$ (where $\rho(p, F(\mathcal{T})) = \inf_{p\in F(\mathcal{T})}\|p - p^*\|$). Then, $\{p_n\}$ converges strongly to an element of $F(\mathcal{T})$.*

Proof. In Theorem 6.1, $\lim_{n\to\infty}\|p_n - p^*\|$ exists for all $p \in F(\mathcal{T})$. So $\lim_{n\to\infty}\rho(p_n, F(\mathcal{T}))$ exists. Thus

$$\lim_{n\to\infty}\rho(p_n, F(\mathcal{T})) = 0.$$

Therefore, there exists a sequence $\{q_j\}$ in $F(\mathcal{T})$ and $\{p_n\}$ has a subsequence $\{p_{n_j}\}$ which satisfies following inequality $\|p_{n_j} - q_j\| \le \dfrac{1}{2^j}$ for all $j \in \mathbb{N}$. In Theorem 6.1, it is proved that $\{p_n\}$ is nonincreasing, so

$$\|p_{n_{j+1}} - q_j\| \le \|p_{n_j} - q_j\| \le \frac{1}{2^j}.$$

$$\text{Therefore} \qquad \|q_{n_{j+1}} - q_j\| \leq \|q_{n_{j+1}} - p_{n_{j+1}}\| + \|p_{n_{j+1}} - q_j\|$$
$$\leq \frac{1}{2^{j+1}} + \frac{1}{2^j} \leq \frac{1}{2^{j-1}} \to 0 \text{ as } j \to \infty.$$

Th above argument shows that $\{q_j\}$ is a Cauchy sequence in $F(\mathcal{T})$, so $\{q_j\}$ converges to some $p^* \in F(\mathcal{T})$. Now, apply triangle inequality

$$\|p_{n_j} - p^*\| \leq \|p_{n_j} - q_j\| + \|q_j - p^*\|.$$

$$\lim_{j\to\infty} \|p_{n_j} - p^*\| \leq \lim_{j\to\infty} \|p_{n_j} - z_j\| + \lim_{j\to\infty} \|q_j - p^*\|.$$

Above argument completes that $\{p_{n_j}\}$ converges strongly to p^*. Since by Theorem 6.1, $\lim_{n\to\infty} \|p_n - p^*\|$ exists, hence $\{p_n\}$ converges strongly to $p^* \in F(\mathcal{T})$.

□

In 1974, Senter et al. [29] introduced the condition (A) as follows.

A mapping $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ satisfies the condition (A) if there exists a nondecreasing function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0, g(a) > 0$ for all $a \in (0, \infty)$ such that $\rho(x, \mathcal{T}x) \geq g(\rho(x, F(\mathcal{T})))$ for all $x \in \mathcal{C}$.

Theorem 6.5. *Let $\mathcal{T}, B, \mathcal{C}$ and $\{p_n\}$ be as in Theorem 6.1 such that $F(\mathcal{T}) \neq \emptyset$. If $\{p_n\}$ is a sequence defined by iteration process (1.4) and $\mathcal{T}$ satisfy the condition (A), then $\{p_n\}$ strongly converges to an element of $F(\mathcal{T})$.*

Proof. By Theorem 6.1, $\lim_{n\to\infty} \|p_n - p^*\|$ exists for all p^* of $\mathcal{T}$ and

$$\|p_{n+1} - p^*\| \leq \|p_n - p^*\|$$

Taking $\inf_{p^* \in F(\mathcal{T})}$ on both sides

$$\inf_{p^* \in F(\mathcal{T})} \|p_{n+1} - p^*\| \leq \inf_{p^* \in F(\mathcal{T})} \|p_n - p^*\|$$

which yields

$$\|p_{n+1} - F(\mathcal{T})\| \leq \|p_n - F(\mathcal{T})\|.$$

From the above inequality, it is obvious that the sequence $\{\|p_n - F(\mathcal{T})\|\}$ is bounded below and non-increasing. Therefore, by Theorem 6.1 $\lim_{n\to\infty} \|p_n - F(\mathcal{T})\|$ exists.

Also, by Theorem 6.1, we have

$$\lim_{n\to\infty} \|p_n - \mathcal{T}p_n\| = 0.$$

By the condition (A),

$$\lim_{n\to\infty} g(\rho(p_n, F(\mathcal{T})) \leq \lim_{n\to\infty} \rho(p_n, \mathcal{T}p_n) = 0.$$

Thus, we have

$$\lim_{n\to\infty} g(\rho(p_n, F(\mathcal{T})) = 0.$$

Becuase g is a non-decreasing function satisfying $g(0) = 0$ and $g(a) > 0$ for all points $a \in (0, \infty)$.

It is unavoidable that $\lim_{n\to\infty} \rho(p_n, F(\mathcal{T})) = 0$. All the relevant conditions for Theorem 6.4 are satisfied. Thus, the sequence $\{p_n\}$ converges strongly to a fixed point of $\mathcal{T}$.

□

In Theorem 6.5, if we assume that T is a generalized nonexpansive mapping satisfying condition B_γ, μ, for $\gamma = \mu = 0$, Then T becomes a nonexpansive mapping and thus we obtain the following corollary.

Corollary 6.6. *Let $\mathcal{T}$ be a nonexpansive self mapping defined on a nonempty, closed and convex subset $\mathcal{C}$ of a uniformly convex Banach space B such that $F(\mathcal{T}) \neq \emptyset$. If $\{p_n\}$ is a sequence defined by the iteration process (1.4) and $\mathcal{T}$ satisfies the condition (A), then $\{p_n\}$ converges strongly to an element of $F(\mathcal{T})$.*

Now, we give an example for generalized nonexpansive mapping which satisfies the condition $B_{\gamma,\mu}$.

Example 6.1. Let $\mathcal{C} = [1, 5]$ which is a closed, and convex subset of the Banach space space $B = \mathbb{R}$, endowed with the usual norm. $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ is defined by

$$\mathcal{T}(u) = \begin{cases} \frac{u+3}{2}, & \text{if } \ u \in [0,5) \\ 3, & \text{if } \ u = 5. \end{cases}$$

It is obvious that $3 \in F(\mathcal{T})$ and $\mathcal{T}$ fulfills the $B_{1,\frac{1}{2}}$ condition.

It is obvious that for $u = \frac{45}{10}$ and $v = 5$, $\mathcal{T}$ doesn't satisfy condition (C).

Case 1: If $u, v \in [0, 5)$, then

$$\begin{aligned}(1-\gamma)\|u-v\| + \mu\big(\|u-\mathcal{T}v\| + \|v-\mathcal{T}u\|\big) &= \frac{1}{2}(\|u - \frac{v+3}{2}\| + \|v - \frac{u+3}{2}\|) \\ &= \frac{1}{2}(\|u - \frac{v}{2} - \frac{3}{2}\| + \|v - \frac{u}{2} - \frac{3}{2}\|) \\ &= \frac{1}{2}(\|u - \frac{v}{2} - \frac{3}{2}\| + \|\frac{u}{2} + \frac{3}{2} - v\|) \\ &\geq \frac{1}{2}(\|u - \frac{v}{2} - \frac{3}{2} + \frac{u}{2} + \frac{3}{2} - v\|) \\ &= \frac{1}{2}\|\frac{3}{2}u - \frac{3}{2}v\| = \frac{3}{4}\|u-v\|\end{aligned}$$

and

$$\|\mathcal{T}u - \mathcal{T}v\| = \|\frac{u+3}{2} - \frac{v+3}{2}\| = \frac{1}{2}\|u-v\|.$$

Thus

$$\|\mathcal{T}u - \mathcal{T}v\| \leq (1-\gamma)\|u-v\| + \mu\big(\|u-\mathcal{T}v\| + \|v-\mathcal{T}u\|\big).$$

Case 2: If $u \in [0, 5)$ and $v = 5$, then

$$\begin{aligned}(1-\gamma)\|u-v\| + \mu(\|u-\mathcal{T}v\| + \|v-\mathcal{T}u\|) &= \frac{1}{2}(\|u-3\| + \|v - \frac{u+3}{2}\|) \\ &\geq \frac{1}{2}\|u-3\|\end{aligned}$$

and

$$\|\mathcal{T}u - \mathcal{T}v\| = \|\frac{u+3}{2} - 3\| = \frac{1}{2}\|u-3\|.$$

Thus

$$\|\mathcal{T}u - \mathcal{T}v\| \leq (1-\gamma)\|u-v\| + \mu\big(\|u-\mathcal{T}v\| + \|v-\mathcal{T}u\|\big).$$

Case 1 and **Case 2** show that $\mathcal{T}$ is a generalised nonexpansive mapping that meets the condition $B_{\gamma,\mu}$ for $\gamma = 1$ and $\mu = \frac{1}{2}$.

7 Applications to a Delay Differential Equation

Delay differential equations are used in many physical phenomena of interest in biology, medicine, chemistry, physics, engineering, economics and among others (for example, see ([9], [12], [20], [21], and the references therein). Our purpose in this section is to exhibit the applicability of our three-step iteration process (1.4).

The following lemma will play a vital role in the furthur theorem.

Lemma 7.1. *[33] Let $\{a_n\}$ be a sequence of positive real numbers which satisfies:*

$$a_{n+1} \leq (1-\theta_n)a_n$$

If $\theta_n \in (0,1)$ and $\sum_{n=1}^{\infty} \theta_n = \infty$, then $\lim_{n\to\infty} s_n = 0$.

Let the space $C([a,b])$ of all continuous real-valued functions on a closed interval $[a,b]$ and $\|\|_\infty$ is a Chebyshev norm $\|u-v\|_\infty = \max_{t\in[a,b]} |u(t)-v(t)|$.

Our interest now is to consider the following delay differential equation

$$x'(t) = f(t, x(t), x(t-\tau)), \qquad t \in [t_0, b] \tag{7.1}$$

with initial condition

$$x(t) = \psi(t) \qquad t \in [t_0-\tau, t_0] \tag{7.2}$$

Now, we will show that the sequence generated by our iteration scheme (1.4) converges to the solution of the delay differential equations (7.1)and (7.2).

We assume that the following conditions are satisfied.

1. $t_0, b \in \mathbb{R}, \tau > 0$;
2. $f \in C([t_0, b] \times \mathbb{R}^2, \mathbb{R})$;
3. $\psi \in C([t_0-\tau, b], \mathbb{R})$;
4. There exists $L_f > 0$ such that
$|f(t,u_1,u_2) - f(t,v_1,v_2)| \leq L_f(|u_1-v_1| + |u_2-v_2|), \qquad \forall u_1, u_2, v_1, v_2 \in \mathbb{R}, t \in [t_0, b]$;
5. $2L_f(b-t_0) < 1$.

The problems (7.1) and (7.2) can be reformulated in the following integral equation:

$$x(t) = \begin{cases} \psi(t), & t \in [t_0-\tau, t_0] \\ \psi(t_0) + \int_{t_0}^{t} f(s, x(s), x(s-\tau))ds, & t \in [t_0, b]. \end{cases}$$

Here $x \in C([t_0-\tau, b], \mathbb{R}) \cap C^1([t_0, b], \mathbb{R})$.

Coman et al. [9] obtained the following results.

Theorem 7.2. *If conditions 1 to 5 are satisfied. Then the problems (7.1)- (7.2) has a unique solution, $q \in C([t_0-\tau, b], \mathbb{R}) \cap C^1([t_0, b], \mathbb{R})$ and $q = \lim_{n\to\infty} T^n x$ for any $x \in C([t_0-\tau, b], \mathbb{R})$.*

Now, we are ready to prove the strong convergence of (1.4) to the unique solution of the delay differential equation.

Theorem 7.3. *Suppose that conditions 1 to 5 are satisfied. Then the iterative sequence $\{x_n\}$ generated by iteration process (1.4) converges strongly to the unique solution of problem (7.1) - (7.2), say $q \in C([t_0-\tau, b], \mathbb{R}) \cap C^1([t_0, b], \mathbb{R})$.*

Proof. Let $\{p_n\}$ be an iterative sequence generated by the iteration process (1.4) for an operator defined by

$$\mathcal{T}x(t) = \begin{cases} \psi(t), & t \in [t_0-\tau, t_0] \\ \psi(t_0) + \int_{t_0}^{t} f(s, x(s), x(s-\tau))ds, & t \in [t_0, b], \end{cases}$$

where $c_n \in (0,1),\ n \in \mathbb{N}$ such that $\sum_{n=0}^{\infty} c_n = \infty$. Let $p^* \in F(T)$. We will prove that $p_n \to p^*$ as $n \to \infty$. Apparently, it is easy to see that $p_n \to p^*$ as $n \to \infty$, for $t \in [t_0 - \tau, t_0]$. For $t \in [t_0 - \tau, b]$ we have

$$\begin{aligned}
\|r_n - p^*\|_\infty &= \|\mathcal{T}((1-c_n)p_n + c_n\mathcal{T}p_n) - \mathcal{T}p^*\|_\infty \\
&= \max_{t\in[t_0-\tau,b]} |\mathcal{T}((1-c_n)p_n + c_n\mathcal{T}p_n)(t) - \mathcal{T}p^*(t)| \\
&= \max_{t\in[t_0-\tau,b]} |\psi(t_0) + \int_{t_0}^{t} f(s, ((1-c_n)p_n + c_n\mathcal{T}p_n)(s), ((1-c_n)p_n + c_n\mathcal{T}p_n)(s-\tau))ds - \psi(t_0) \\
&- \int_{t_0}^{t} f(s, p^*(s), p^*(s-\tau))ds| \\
&\le \max_{t\in[t_0-\tau,b]} \int_{t_0}^{t} |f(s, ((1-c_n)p_n + c_n\mathcal{T}p_n)(s), ((1-c_n)p_n + c_n\mathcal{T}p_n)(s-\tau)) - f(s, p^*(s), p^*(s-\tau))|ds \\
&\le \max_{t\in[t_0-\tau,b]} \int_{t_0}^{t} L_f(|((1-c_n)p_n + c_n\mathcal{T}p_n)(s) - p^*(s)| + |((1-c_n)p_n + c_n\mathcal{T}p_n)(s-\tau)) - p^*(s-\tau)|) \\
&\le \int_{t_0}^{t} L_f(\max_{t\in[t_0-\tau,b]} |((1-c_n)p_n + c_n\mathcal{T}p_n)(s) - p^*(s)| \\
&+ \max_{t\in[t_0-\tau,b]} |((1-c_n)x_n + c_n\mathcal{T}p_n)(s-\tau)) - p^*(s-\tau)|)ds \\
&\le \int_{t_0}^{t} L_f(\|((1-c_n)p_n + c_n\mathcal{T}p_n) - p^*\|_\infty + \|((1-c_n)p_n + c_n\mathcal{T}p_n) - p^*\|_\infty)ds \\
&\le 2L_f(b-t_0)\|((1-\alpha_n)p_n + c_n\mathcal{T}p_n) - p^*\|_\infty
\end{aligned} \tag{7.3}$$

And

$$\begin{aligned}
\|((1-c_n)p_n + c_n\mathcal{T}p_n) - p^*\|_\infty &= \|((1-c_n)p_n + c_n\mathcal{T}p_n) - \mathcal{T}p^*\|_\infty \\
&\le (1-c_n)\|p_n - p^*\|_\infty + c_n\|\mathcal{T}p_n - \mathcal{T}p^*\|_\infty \\
&= (1-c_n)\|p_n - p^*\|_\infty + c_n \max_{t\in[t_0-\tau,b]} |\psi(t_0) + \int_{t_0}^{t} f(s, p_n(s), p_n(s-\tau))ds \\
&- \psi(t_0) - \int_{t_0}^{t} f(s, p^*(s), p^*(s-\tau))ds| \\
&\le (1-c_n)\|p_n - p^*\|_\infty + c_n \max_{t\in[t_0-\tau,b]} \int_{t_0}^{t} |f(s, p_n(s), p_n(s-\tau))ds - f(s, p^*(s), p^*(s-\tau))|ds \\
&\le (1-c_n)\|p_n - p^*\|_\infty + c_n \max_{t\in[t_0-\tau,b]} \int_{t_0}^{t} L_f(|p_n(s) - p^*(s)| + |p_n(s-\tau) - p^*(s-\tau)|)ds \\
&\le (1-c_n)\|p_n - p^*\|_\infty + c_n \int_{t_0}^{t} L_f(\|p_n - p^*\|_\infty + \|p_n - p^*\|_\infty) \\
&\le (1-\alpha_n)\|p_n - p^*\|_\infty + 2\alpha_n L_f(b-t_0)\|p_n - p^*\|_\infty \\
&\le [1 - \alpha_n(1 - 2L_f(b-t_0))]])\|p_n - p^*\|_\infty
\end{aligned} \tag{7.4}$$

Similarly

$$\begin{aligned}
\|q_n - p^*\|_\infty &= \max_{t\in[t_0-\tau,b]} \|\mathcal{T}r_n(t) - \mathcal{T}p^*(t)\| \\
&= \max_{t\in[t_0-\tau,b]} |\psi(t_0) + \int_{t_0}^{t} f(s, r_n(s), r_n(s-\tau))ds - \psi(t_0) - \int_{t_0}^{t} f(s, p^*(s), p^*(s-\tau))ds| \\
&\leq \max_{t\in[t_0-\tau,b]} \int_{t_0}^{t} |f(s, r_n(s), r_n(s-\tau)) - f(s, p^*(s), p^*(s-\tau))|ds \\
&\leq \max_{t\in[t_0-\tau,b]} \int_{t_0}^{t} L_f(|r_n(s) - p^*(s)| + |r_n(s-\tau) - p^*(s-\tau)|)ds \\
&\leq \int_{t_0}^{t} L_f(\max_{t\in[t_0-\tau,b]} |r_n(s) - p^*(s)| + \max_{t\in[t_0-\tau,b]} |r_n(s-\tau) - p^*(s-\tau)|)ds \\
&\leq \int_{t_0}^{t} L_f(\|r_n - p^*\|_\infty + \|r_n - p^*\|_\infty)ds \\
&\leq 2L_f(b - t_0)\|r_n - p^*\|_\infty.
\end{aligned} \tag{7.5}$$

Finally

$$\begin{aligned}
\|p_{n+1} - p^*\|_\infty &= \max_{t\in[t_0-\tau,b]} |\mathcal{T}q_n(t) - \mathcal{T}p^*(t)| \\
&= \max_{t\in[t_0-\tau,b]} |\psi(t_0) + \int_{t_0}^{t} f(s, q_n(s), q_n(s-\tau))ds - \psi(t_0) - \int_{t_0}^{t} f(s, p^*(s), p^*(s-\tau))ds| \\
&= \max_{t\in[t_0-\tau,b]} |\int_{t_0}^{t} [f(s, q_n(s), q_n(s-\tau)]ds - f(s, p^*(s), p^*(s-\tau))|ds \\
&\leq \max_{t\in[t_0-\tau,b]} |\int_{t_0}^{t} L_f(|q_n(s) - p^*(s)| + |q_n(s-\tau) - p^*(s-\tau)|)ds \\
&\leq 2L_f(b - t_0)\|q_n - p^*\|_\infty
\end{aligned} \tag{7.6}$$

Using equations (7.3), (7.4), (7.5) and (7.6), we obtained

$$\|p_{n+1} - p^*\|_\infty = [2L_f(b - t_0)]^3[1 - c_n(1 - 2L_f(b - t_0))]\|p_n - p^*\|_\infty \tag{7.7}$$

Since, $2L_f(b - t_0) < 1$, so equation (7.7) becomes

$$\|p_{n+1} - p^*\|_\infty \leq [1 - c_n(1 - 2L_f(b - t_0))]\|p_n - p^*\|_\infty$$

Hence, by induction we get

$$\|p_{n+1} - p^*\|_\infty \leq \prod_{i=0}^{n} [1 - c_i(1 - 2L_f(b - t_0))]\|p_0 - p^*\|_\infty$$

Since, $0 \leq c_n \leq 1$, for all $n \in \mathbb{N}$, we have $1 - c_i(1 - 2L_f(b - t_0)) < 1$ and if we consider $c_i(1 - 2L_f(b - t_0)) = \theta_n$, then all the conditions of Lemma 7.1 are satisfied. Thus, $\lim_{n\to\infty} \|p_n - p^*\|_\infty = 0$. This completes the proof. □

8 Conclusion

We proved that the iteration scheme (1.4) is $\mathcal{T}$-stable and the data dependence results for the iteration process (1.4) under contractive-like conditions. We also used scheme (1.4) to speed the approximation of a fixed point and compared this with some leading iterations such as Mann iteration [17], Ishikawa iteration

[16], Noor [19], S-iteration [5], Abbas et al. [1], Thakur et al. ([35], [36]) K iteration [15], M^*iteration [37], M iteration [38], K^* iteration [39], Picard-S iteration process [13] to show the efficiency and effectiveness of new scheme. Our assertion is supported by a numerical example with Table-1 and Table-2 and graphs. Next section investigates some convergence theorems for generalised nonexpansive mappings with property $B_{\gamma,\mu}$ in the context of uniformly convex Banach spaces. In final section an application that raised from delay differential equation is given to show the applicability of our scheme.

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