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Original Article

A Shrinking Projection Algorithm for Solving Split Equality Problem in Banach Spaces

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Abstract - *In the paper, a new shrinking projection method is proposed for solving split equality problem(SEP) in Banach spaces. For practical purposes, we substitute duality mapping for inner in Banach spaces. Under proper conditions, we give proofs of strong convergence for the SEP with two different choices of the step-size. Finally, we make some extensions and generalization.*

Keywords - *Split equality problem, Strong convergence, Duality mapping, Shrinking projection method, Banach space.*

1. Introduction

In 1994, Censor and Elfving [3] firstly introduced a new problem named split feasibility problem (SFP for short): finding a point x^* in E_1 with the property:

$$x^* \in C \text{ and } Ax^* \in Q, \tag{1.1}$$

where E_1 and E_2 are two real Banach spaces, C and Q are nonempty closed convex subsets of E_1 and E_2, respectively. A is a bounded linear operator.

In 2013, Moudafi [12] introduced the SEP induced by SFP, which can be formulated as:

$$\text{find } x \in C, y \in Q \text{ such that } Ax = By, \tag{1.2}$$

where $H_1,\ H_2,\ H_3$ are three real Hilbert spaces, $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are two bounded linear operators . $C \subset H_1$ and $Q \subset H_2$ are two nonempty closed convex subsets.

When $B = I$, the SEP (1.2) will reduce to the SFP (1.1).

The SEP has received plenty of attention owing to its extraordinary practicality and wide applicability in numerous fields of applied mathematics; for example: decomposition methods for partial differential equations, applications in game theory (see [1]) and intensity-modulated radiation therapy (see [4][5]) and so on. Furthermore, Moudafi proposed the following scheme for dealing with the SEP in Hilbert space:

$$\begin{cases} x_{k+1} = P_{C_k}\big(x_k - \gamma A^*(Ax_k - By_k)\big), \\ y_{k+1} = P_{Q_k}\big(y_k + \gamma B^*(Ax_{k+1} - By_k)\big). \end{cases} \tag{1.3}$$

The author obtained the property of weak convergence of (1.3) under certain proper assumptions and the appropriate parameters.

In 2014, to get the strong convergence result, Shi et al. [16] offered a modification of Moudafi's ACQA algorithms in Hilbert spaces to handle the SEP as follow:

$$w_{n+1} = P_S\{(1 - \alpha_n)[I - \gamma G^*G]w_n\}, \tag{1.4}$$

i.e.,

$$\begin{cases} x_{k+1} = P_C\{(1 - \alpha_k)x_k - \gamma A^*(Ax_k - By_k)\}, n \geq 0, \\ y_{k+1} = P_Q\{(1 - \alpha_k)y_k + \gamma B^*(Ax_k - By_k)\}, n \geq 0. \end{cases} \tag{1.5}$$

Based on the relationship between the SFP (1.1) and the SEP (1.2), we would seek an iterative algorithm to solve SEP (1.2) by the methods for handling the SFP (1.1).

Next, we will recommend a slice of iterative algorithms, which have dealt with the SFP in Banach spaces.

In 2014, Takahashi [23] suggested a new projection method for solving the SFP in Banach spaces:

$$\begin{cases} z_n = n - r_n J_{X^*} A^* J_Y\big(I - P_Q\big)Ax_n, \\ C_n = \{z \in C : \langle z_n - z,\ J_X(x_n - z_n)\rangle \geq 0\}, \\ Q_n = \{z \in C : \langle x_n - z,\ J_X(x_0 - x_n)\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \forall n \in N, \end{cases} \tag{1.6}$$

where r_n is a positive parameter, J_X and J_Y are two duality mappings on X and Y, respectively.

In 2015, by referring to the shrinking projection method, Takahashi [24] proposed the other method:

$$\begin{cases} z_n = x_n - r_n J_{X^*} A^* J_Y\big(I - P_Q\big)Ax_n, \\ Q_{n+1} = \{z \in Q_n : \langle z_n - z,\ J_X(x_n - z_n)\rangle \geq 0\}, \\ x_{n+1} = P_{Q_n}(x_0), \forall n \in N, \end{cases} \tag{1.7}$$

Under the assumption that X is reflexive, smooth and uniformly convex, which is obviously weaker than the condition applied in [13], the author proved that the two methods are strongly convergence to the solution of SFP.
There are more articles solving the SEP and other related problem; see, for instance, [2, 9, 17, 11, 19, 21, 18, 20, 14].

The aim of this paper is to build a new algorithm by modifying the iterative scheme (1.7) in Banach space for solving the SEP and prove the result of strong convergence. The paper will be organized as follow: In Section 2, we firstly recall several necessary lemmas and definitions. In Section 3, we secondly recommended our shrinking projection algorithm and prove the strong convergence property. In Section 4, some extensions will be introduced.

2. Preliminaries

In this section, we introduce the meaning of letters in the article and a few basic lemmas.

Let X, Y, Z be real 2-convex and uniformly smooth and thus reflexive Banach space. So, X^*, Y^*, Z^* are three 2-smooth and uniformly convex dual space of X, Y, Z. In our paper, $<\cdot,\cdot>$ means the duality pairing between the original space and its dual space. The notion "$\rightarrow$" represents the result of strong convergence; "$\rightharpoonup$" means weak convergence; and $\omega_w(z_n)$ is the set of weak cluster points of a sequence $\{z_n\}$. $T^{-1}(0) = \{a \in Z : Ta = 0\}$ shows the null-point set of an operator T which is defined on Z. For $z \in Z$, we let $J_t(I - P_X)u = J_t(u - P_X u)$. We assume that C, Q are nonempty closed convex subsets of X, Y, respectively. $A: X \to Z$, $B: Y \to Z$ are two bounded linear operators. Let $f_2(z)$ is a function under these assumptions

$$f_2(z) := \frac{1}{2}\|z\|^2, \qquad z \in S,$$

which is strictly convex and $Fréchet$ differentiable, where $S := X \times Y$. Its derivative is

$$J_2 := f_2'.$$

By the definition of J_2^*, we let the duality mapping of the dual X^* and Y^* with the gauge function $t \mapsto t$. Among that, J_2, J_2^* are uniformly continuous on bounded sets and bijective with $(J_2)^{-1} = J_2^*$.

Then, the following several lemmas will be applied in the proof of principal theorem.

Lemma 2.1. [22] Let J_Y be the duality mapping on space Y.
(1) J_Y is surjective if and only if Y is reflexive.
(2) J_Y is injective if and only if Y is strictly convex.
(3) J_Y is single-valued if and only if Y is smooth.
(4) If Y is smooth, then J_Y is monotone, that is,

$$\langle x - y, J_Y x - J_Y y\rangle \geq 0, \forall x, y \in Y.$$

Additionally, if Y is a strictly convex space, then J_Y is strictly monotone, that is,

$$\langle x-y, J_Y x - J_Y y\rangle = 0 \Longrightarrow x = y.$$

(5) If Y is smooth, reflexive and strictly convex, we can get that J_Y is one-to-one, single-valued with the property ${J_Y}^{-1} = J_{Y^*}$, where J_{Y^*} is the duality mapping of Y^* .

Lemma 2.2. [13] Let $\{u_n\}$ be a sequence in Z, and $C \subseteq \mathrm{Z}$ be a nonempty closed convex subset. After that, for $u \in Z$, the following inequalities hold.
(1) $\langle z - P_C(u), J_X(u - P_C(u))\rangle \le 0, \forall z \in C$.
(2) $\|u - P_C(u)\|^2 \le \langle u - z, J_X(u - P_C(u))\rangle, \forall z \in C$.
(3) If $x_n \rightharpoonup x$ and $\|x - P_C(x)\| \longrightarrow 0$, then $x \in C$.

Lemma 2.3. [7, 6, 23] Let $N > 0$ and $\{x_n\}, \{y_n\}$ be two sequences in X such that $\|x_n\| = N$, $\|y_n\| = N$ and $\|x_n + y_n\| \longrightarrow 2N$ as $n \to \infty$. If X is uniformly convex, we can get $\lim\limits_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 2.4. [7, 6, 23] Let $\{x_n\}$ be a sequence in X such that $x_n \rightharpoonup x$ as well as $\lim\limits_{n\to\infty} \|x_n\| = \|x\|$. Furthermore, if X is uniformly convex, we can get $\lim\limits_{n\to\infty} x_n = x$.

3. Main Results

To get a perfect algorithm, the following lemmas are essential. By the lemma, the split equality problem can be converted to an equivalent null-point problem, which actually also be equivalent to the fixed-point problem.

Lemma 3.1. Let $\mathrm{T(z)} \coloneqq J_r^*[J_t(I - P_S)z + G^* J_2 G z]$. Then $\Gamma = T^{-1}(0)$, where $\mathrm{z = (x, y)}$, $\mathrm{G = [A, -B]}$, $J_r^* = [J_2^*, J_2^*]^T$, $J_t = [J_2, J_2]^T$.

Proof: Clearly, $\Gamma \subseteq T^{-1}(0)$. Now let $\mathrm{z} \in T^{-1}(0)$. By Lemma 2.2, we have

$$\|z - P_S z\|^2 \le \langle J_t(I - P_S)z, z - u\rangle,$$

and

$$\langle J_2 G z, Gz - Gu\rangle = \|Gz\|^2 = \langle J_2 G z - J_2 G u, Gz - Gu\rangle = \langle G^* J_2 G z, z - u\rangle,$$

where $u \in \Gamma$. By these inequalities, we have

$$\|z - P_S z\|^2 + \langle J_2 G z, Gz - Gu\rangle \le \langle J_t(I - P_S)z\ , z - u\rangle + \langle G^* J_2 G z, z - u\rangle = 0.$$

Then, since

$$\|z - P_S z\|^2 \ge 0 \text{ and } \langle J_2(Gz - Gu), Gz - Gu\rangle = \|Gz\|^2 \ge 0,$$

we can get:

$$\|z - P_S z\|^2 = \|Gz\|^2 = 0.$$

Thus, $z = P_S z$ and $Gz = 0$, that is, $\mathrm{z} \in \Gamma$. Hence $T^{-1}(0) \subseteq \Gamma$.
Altogether, we have $T^{-1}(0) = \Gamma$. This completes our proof.

The next algorithm will be proposed for coping with the split equality problem in Banach spaces. Choose $z_0 = (x_0, y_0) \in \mathrm{X \times Y}$ as well as $\mathrm{S_0 = X \times Y}$. Given z_n, update z_{n+1} by the iteration formula:

$$\begin{cases} w_n = z_n - r_n J_r^*[J_t(I - P_S)z_n + G^* J_2 G z_n], \\ S_{n+1} = \{w \in S_n : \langle w_n - w, J_t(z_n - w_n)\rangle \ge 0\}, \\ z_{n+1} = P_{S_{n+1}}(z_0), \forall n \in \mathbb{N}. \end{cases}$$

where $z_n = (x_n, y_n)$, $w_n = (u_n, v_n)$, $S_n = C_n \times Q_n$, $w = (u, v)$.

Lemma 3.2. Assume that X, Y and Z all are reflexive, smooth and strictly convex Banach spaces. If r_n is chosen so that $0 < a \le r_n \le 1/(1 + \|G\|^2)$, where $a > 0$ is a real number; then for every $n \in N$, the set S_n is nonempty, closed and convex. Consequently, the algorithm that we proposed is well defined.

Proof: Now it is enough to show that S_n is nonempty because S_n is obviously closed and convex. We next show that $\Gamma \subseteq S_n$. Let $m \in \Gamma$, we have

$$\begin{aligned} \langle z_n - m, J_t(z_n - w_n)\rangle &= r_n\langle z_n - m, J_t(I - P_S)z_n + G^* J_2 G z_n\rangle = r_n\langle z_n - m, J_t(I - P_S)z_n\rangle + r_n\langle G z_n - Gm, J_2 G z_n\rangle \\ &\ge r_n\|z_n - P_S z_n\|^2 + r_n\langle G z_n, J_2 G z_n\rangle = r_n(\|z_n - P_S z_n\|^2 + \|G z_n\|^2), \end{aligned} \tag{3.2}$$

which implies

$$\langle w_n - m, J_t(z_n - w_n)\rangle = \langle w_n - z_n, J_t(z_n - w_n)\rangle + \langle z_n - m, J_t(z_n - w_n)\rangle$$
$$= \langle z_n - m, J_t(z_n - w_n)\rangle - \|w_n - z_n\|^2 \geq -\|z_n - w_n\|^2 + r_n\|z_n - P_S z_n\|^2 + \|Gz_n\|^2).$$

On the other hand, according to Young's inequality,

$$\begin{aligned}\|z_n - w_n\|^2 &= {r_n}^2\|J_t(I - P_S)z_n + G^* J_2 G z_n\|^2 \leq {r_n}^2(\|J_t(I - P_S)z_n\| + \|G\|\|J_2 G z_n\|)^2\\ &= {r_n}^2(\|J_t(I - P_S)z_n\|^2 + \|G\|^2\|J_2 G z_n\|^2) + 2{r_n}^2\|J_t(I - P_S)z_n\|\|G\|\|J_2 G z_n\|\\ &\leq {r_n}^2(\|J_t(I - P_S)z_n\|^2 + \|G\|^2\|J_2 G z_n\|^2) + {r_n}^2(\|G\|^2\|J_t(I - P_S)z_n\|^2 + \|J_2 G z_n\|^2)\\ &= {r_n}^2(1 + \|G\|^2)(\|J_t(I - P_S)z_n\|^2 + \|J_2 G z_n\|^2) = {r_n}^2(1 + \|G\|^2)(\|z_n - P_S z_n\|^2 + \|Gz_n\|^2),\end{aligned}$$

then,

$$\langle w_n - m, J_t(z_n - w_n)\rangle \geq r_n(1 - r_n(1 + \|G\|^2))(\|z_n - P_S z_n\|^2 + \|Gz_n\|^2) \geq 0.$$

Hence, $m \in S_n$. Since m is decided in Γ arbitrarily, we get that $\Gamma \in S_n$ for all $n \in N$. Now it is obviously that the set S_n is nonempty, closed and convex. Therefore, the algorithm we proposed is well defined.

Next, we will show the convergence of the recommended algorithm.

Theorem 3.3. We suppose that X and Y are reflexive, smooth and uniformly convex Banach spaces. We also assume that Z is a smooth, reflexive and strictly convex space. If r_n is fixed and satisfies the inequality: $0 < a \leq r_n \leq \frac{1}{1+\|G\|^2}$, the algorithm (3.1) will generate a sequence $\{z_n\}$ which is converges strongly to $\hat{z} \in \Gamma$, where $\hat{z} \in P_\Gamma(z_0)$.

Proof: We initially prove this limit:

$$\lim_{n\to\infty}\|z_n - w_n\| = 0. \tag{3.3}$$

Let $\bar{w} \in \Gamma$, we know that $\bar{w} \in S_n$, $z_{n+1} \in S_{n+1} \subseteq S_n$. Thus, for each $n \in \mathbb{N}$, we have

$$\|z_0 - z_n\| = \left\|z_0 - P_{S_n}(z_0)\right\| \leq \min(\|z_0 - \bar{w}\|, \|z_0 - z_{n+1}\|).$$

The result indicates that $\{\|z_0 - z_n\|\}$ is bounded and nondecreasing; As a consequence, $\lim_{n\to\infty}\|z_0 - z_n\|$ exists. Now we let $\mathrm{M} \coloneqq \lim_{n\to\infty}\|z_0 - z_n\|$. We have

$$\begin{aligned}&\lim_{n\to\infty}\sup\|(z_{n+1} - z_0) + (z_n - z_0)\|\\ &\leq \lim_{n\to\infty}(\|z_{n+1} - z_0\| + \|z_0 - z_n\|) = 2\mathrm{M},\end{aligned}$$

and

$$\begin{aligned}&\lim_{n\to\infty}\inf\|(z_{n+1} - z_0) + (z_n - z_0)\|\\ &= \lim_{n\to\infty}\inf 2\left\|\frac{z_{n+1} + z_n}{2} - z_0\right\| \geq \lim_{n\to\infty}2\|z_0 - z_n\| = 2\mathrm{M},\end{aligned}$$

where $\frac{z_n+z_{n+1}}{2} \in S_n$. So we can get

$$\lim_{n\to\infty}\|(z_{n+1} - z_0) + (z_n - z_0)\| = 2M.$$

Since X , Y are uniformly convex spaces, by the Lemma 2.3 this yields that

$$\lim_{n\to\infty}\|z_{n+1} - z_n\| = \lim_{n\to\infty}\|(z_{n+1} - z_0) - (z_n - z_0)\| = 0.$$

Furthermore, since $z_{n+1} \in S_{n+1}$, by the defifinition of S_{n+1}, then we have:

$$\langle w_n - z_{n+1}, J_t(z_n - w_n)\rangle \geq 0,$$

which implies that

$$\begin{aligned}&\|z_n - w_n\|^2\\ &= \langle z_n - w_n, J_t(z_n - w_n)\rangle\\ &= \langle z_n - z_{n+1}, J_t(z_n - w_n)\rangle + \langle z_{n+1} - w_n, J_t(z_n - w_n)\rangle\\ &\leq \langle z_n - z_{n+1}, J_t(z_n - w_n)\rangle\\ &\leq \|z_n - z_{n+1}\| \cdot \|z_n - w_n\|.\end{aligned}$$

Hence, $\lim_{n\to\infty}\|z_n - w_n\| = 0$.

We next will prove that each weak cluster point of $\{z_n\}$ is a solution of the split equality problem. In order to get the result, we let z be any weak cluster point of $\{z_n\}$ and take a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ converging weakly to z. According to (3.2) and (3.3), we get:

$$\lim_{k\to\infty}\left\|z_{n_k}-P_S z_{n_k}\right\|=\lim_{n\to\infty}\|z_n-P_S z_n\|=0, \tag{3.4}$$

$$\lim_{n\to\infty}\|Gz_n\|=0, \tag{3.5}$$

by the Lemma 2.2, then $z\in S$. And

$$0\le\|Gz\|^2=\langle Gz,J_2Gz\rangle=\langle z,G^*J_2Gz\rangle=\lim_{n\to\infty}\langle z_n,G^*J_2Gz\rangle=\lim_{n\to\infty}\langle Gz_n,J_2Gz\rangle\le\lim_{n\to\infty}\|Gz_n\|\|Gz\|=0.$$

Hence, $z\in\Gamma$. Since z is arbitrary, we get the conclusion we desire.

In the end, we should prove that $\{z_n\}$ converges strongly to $\hat{z}\in P_\Gamma(z_0)$. We can take any $z\in\omega_w(z_n)$. Then, $z\in\Gamma$. There exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ converging weakly to z. Therefore, we have:

$$\|z_0-\hat{z}\|=\|z_0-P_\Gamma(z_0)\|\le\|z_0-z\|\le\lim_{k\to\infty}\left\|z_0-z_{n_k}\right\|=\lim_{k\to\infty}\left\|z_0-P_{S_{n_k}}(z_0)\right\|\le\|z_0-\hat{z}\|.$$

According to the property of projection and the uniqueness of projection, hence,

$$\hat{z}=z,\lim_{k\to\infty}\left\|z_0-z_{n_k}\right\|=\|z_0-\hat{z}\|.$$

Since z is decided arbitrarily, this suggests that $\omega_w(z_n)$ is a single-point set. So, we can get $\{z_n\}$ converges weakly to $\hat{z}$. We can conclude that $z_0-z_{n_k}\rightharpoonup z_0-\hat{z}$. By Lemma 2.4, the uniform convexity implies $\lim_{k\to\infty}z_{n_k}=\hat{z}$. Since $\{z_n\}$ converges weakly, it yields $\lim_{n\to\infty}z_n=\hat{z}$ as desired.

4. Generalization

Reproduced with permission. [Ref.] Copyright Year, Publisher. Let X,Y be real p-uniformly convex and uniformly smooth and thus reflexive Banach spaces. So, there be q-uniformly smooth and uniformly convex dual space X^*,Y^* (see e.g. [8, 10, 6, 15]). The parameters satisfy:

$$p>2,q<2,\frac{1}{p}+\frac{1}{q}=1.$$

Under the above assumptions, the function

$$f_p(z):=\frac{1}{p}\|z\|^p,z\in S,$$

is strictly convex and Fréchet differentiable, where $S:=X\times Y$. Its derivative

$$J_p:=f_p'$$

is a nonlinear mapping from $X\times Y$ to $X^*\times Y^*$ which is called the duality mapping of $X\times Y$ with a gauge function $t\mapsto t^{p-1}$. It is homogenous of degree $p-1$ and we can get

$$\langle J_p(x),x\rangle=\|x\|^p,$$

and

$$\left\|J_p(x)\right\|=\|x\|^{p-1},$$

where we write $\langle x^*,x\rangle=\langle x,x^*\rangle=x^*(x)$ for the application of $x^*\in S^*$ on $x\in S$. By J_q^* we define the duality mapping of the dual S^* with a gauge function $t\mapsto t^{q-1}$. Both J_p and J_q^* are uniformly continuous on bounded sets and bijective with $(J_p)^{-1}=J_q^*$. If we consider the function in Hilbert spaces, then J_2 will be the identity mapping.

Lemma 4.1. Let U be a real Banach space, and $C\subseteq U$ be a nonempty closed convex subset. Then, for $u\in X$, the following inequality hold:

$$\|u-P_Cu\|^p\le\langle -J_p(P_Cu-u),u-z\rangle,\forall z\in C.$$

Proof: We know that the projection can be expressed by a variational inequality (see [13]): the element P_Cu is the metric projection of u onto C iff

$$\langle J_p(P_Cu-u),z-P_Cu\rangle\ge 0,\forall z\in C.$$

i.e.,

$$\langle J_p(P_Cu-u),z-u\rangle+\langle J_p(P_Cu-u),u-P_Cu\rangle\ge 0,\forall z\in C.$$

Then

$$\|u-P_Cu\|^p\le\langle -J_p(P_Cu-u),u-z\rangle,\forall z\in C.$$

Hence, the inequality is valid.

Lemma 4.2. Let $T(z) := J_r^*\left[G^* J_p G z - J_t(P_S x - I)z\right]$. Then $\Gamma = T^{-1}(0)$, where $z = (x, y)$, $G = [A, -B]$, $J_r^* = [J_p^*, J_p^*]^T$, $J_t = [J_p, J_p]^T$.

Proof: Clearly, $\Gamma \subseteq T^{-1}(0)$. Now let $z \in T^{-1}(0)$. By Lemma 4.1, for each $m \in \Gamma$, we have

$$\|z - P_S z\|^p \leq \langle -J_t(P_S x - I)z, z - m\rangle,$$

$$\|Gz\|^p = \langle J_p Gz, Gz\rangle = \langle J_p Gz, Gz - Gm\rangle = \langle J_p Gz, G(z - m)\rangle = \langle G^* J_p Gz, z - m\rangle,$$

By these inequalities, we have

$$\|z - P_S z\|^p + \|Gz\|^p \leq \langle -J_t(P_S x - I)z, z - m\rangle + \langle G^* J_p Gz, z - m\rangle = 0.$$

On the other hand, since

$$\|z - P_S z\|^p \geq 0 \text{ and } \|Gz\|^p \geq 0,$$

thus

$$\|z - P_S z\| = \|Gz\| = 0.$$

So $z = P_S z$ and $Gz = 0$, that is, $z \in \Gamma$. Hence $T^{-1}(0) \subseteq \Gamma$.
Altogether, we have $T^{-1}(0) = \Gamma$.

We provide the following algorithm which can cope with the split equality problem in Banach spaces. Let $z_0 = (x_0, y_0) \in X \times Y$ and $S_0 = X \times Y$. We should calculate and get z_n, then update z_{n+1} by the iteration formula:

$$\begin{cases} w_n = z_n - r_n J_r^*\left[G^* J_p G z_n - J_t(P_S - I)z_n\right], \\ S_{n+1} = \{w \in S_n : \langle w_n - w, J_t(z_n - w_n)\rangle \geq 0\}, \\ z_{n+1} = P_{S_{n+1}}(z_0), \forall n \in \mathbb{N}. \end{cases} \tag{4.1}$$

where $z_n = (x_n, y_n)$, $w_n = (u_n, v_n)$, $S_n = C_n \times Q_n$, $w = (u, v)$.

Lemma 4.3. We suppose that X and Y are two reflexive, smooth and strictly convex spaces. The set S_n is nonempty, convex and closed. The algorithm we proposed is well defined iff r_n is chosen so that

$$r_n^{p-1} = \frac{\|z_n - P_S z_n\|^p + \|Gz_n\|^p}{(\|J_t(P_S x - I)z_n\| + \|G\|\|J_p G z_n\|)^q}.$$

Proof: It is enough to prove that S_n is nonempty because it is obviously closed and convex. Hence, we should reveal that $\Gamma \subseteq S_n$. Let $u \in \Gamma$, we have

$$\begin{aligned} &\langle z_n - u, J_t(z_n - w_n)\rangle = r_n\langle z_n - u, G^* J_p G z_n - J_t(P_S - I)z_n\rangle \\ &= r_n\langle Gz_n - Gu, J_p G z_n\rangle - r_n\langle z_n - u, J_t(P_S - I)z_n\rangle \\ &\geq r_n\langle Gz_n, J_p G z_n\rangle - r_n\langle z_n - P_S z_n, J_t(P_S - I)z_n\rangle \\ &= r_n(\|z_n - P_S z_n\|^p + \|Gz_n\|^p), \end{aligned}$$

which implies

$$\begin{aligned} &\langle w_n - u, J_t(z_n - w_n)\rangle \\ &= \langle w_n - z_n, J_t(z_n - w_n)\rangle + \langle z_n - u, J_t(z_n - w_n)\rangle \\ &= -\|w_n - z_n\|^p + \langle z_n - u, J_t(z_n - w_n)\rangle \\ &\geq r_n(\|z_n - P_S z_n\|^p + \|Gz_n\|^p) - \|z_n - w_n\|^p, \end{aligned}$$

On the other hand,

$$\begin{aligned} &\|z_n - w_n\|^p \\ &= \left\|r_n J_r^*\left[G^* J_p G z_n - J_t(P_S - I)z_n\right]\right\|^p \\ &= r_n{}^p\left\|G^* J_p G z_n - J_t(P_S - I)z_n\right\|^q \\ &\leq r_n{}^p(\|J_t(P_S x - I)z_n\| + \|G\|\left\|J_p G z_n\right\|)^q \\ &= r_n(\|z_n - P_S z_n\|^p + \|Gz_n\|^p). \end{aligned} \tag{4.2}$$

Then

$$\langle w_n - u, J_t(z_n - w_n)\rangle \geq 0.$$

Hence, $m \in S_n$. Since m is decided in Γ arbitrarily, we can get that $\Gamma \subseteq S_n$ for all $n \in \mathbb{N}$. We can know that the set S_n is nonempty, closed and convex. Consequently, the algorithm we proposed is well defined.

We then prove that the sequence $\{z_n\}$ generated by (4.1) converges strongly to $\hat{z} \in \Gamma$. According to the proof of the foregoing theorem, it is enough to confirm that (3.4) and (3.5) still hold. Similarly, we obtain $\lim_{n\to\infty}\|z_n - w_n\| = 0$. By (4.2), we have

$$\lim_{n\to\infty} r_n(\|z_n - P_S z_n\|^p + \|Gz_n\|^p) = 0. \tag{4.3}$$

This together with (4.3) yields (3.4) and (3.5) as desired. Hence, the proof is completed.

Remark 4.4. Our algorithm is new even in Hilbert spaces. In fact, if we consider SEP in Hilbert space, the algorithm we proposed will be simplified as:

$$\begin{cases} w_n = z_n - r_n[G^*Gz_n - (P_S - I)z_n], \\ S_{n+1} = \{w \in S_n: \langle w_n - w, z_n - w_n\rangle \geq 0\}, \\ z_{n+1} = P_{S_{n+1}}(z_0), \forall n \in \mathbb{N}. \end{cases}$$

5. Conclusion

In this paper, we propose a new shrinking projection iterative algorithm. It can cope with the split equality problem (SEP) in Banach spaces. Under several proper conditions, we give proofs of strong convergence for the SEP with two different choices of the step-size. Finally, we make some extensions and generalization.

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