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Original Article

Application of Linear Multiplier Fractional Q-Differintegral Operator on Analytic Functions

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Abstract *- In the present article, by making use of linear multiplier fractional q-differ integral operator we define new subclass of analytic function and discuss some important properties of the class.*

Keywords and phrases - *Harmonic Functions, Univalent functions, Al-OboudiDifferential Operator, q-differintegral.*

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1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic and univalent in the open unit disc U.

Let T denote the subclass of A in U, consisting of analytic functions whose non- zero coefficients from the second terms onwards are negative. That is, an analytic function$f \in T$ if it has a Taylor series expansion of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \quad , \qquad a_k \geq 0 \tag{1.2}$$

are analytic in the open disc U.

A linear multiplier fractional q – differ-integral operator is defined as

$$\begin{aligned}
&\mathcal{D}_{q,\lambda}^{\delta,0} f(z) = f(z) \\
&\mathcal{D}_{q,\lambda}^{\delta,1} f(z) = (1-\lambda)\,\Omega_q^{\delta} f(z) + \lambda z \mathcal{D}_q\, \Omega_q^{\delta} f(z) \\
&\mathcal{D}_{q,\lambda}^{\delta,2} f(z) = \mathcal{D}_{q,\lambda}^{\delta,1}\left(\mathcal{D}_{q,\lambda}^{\delta,1} f(z)\right) \\
&\vdots \\
&\mathcal{D}_{q,\lambda}^{\delta,n} f(z) = \mathcal{D}_{q,\lambda}^{\delta,1}\left(\mathcal{D}_{q,\lambda}^{\delta,n-1} f(z)\right)
\end{aligned} \tag{1.3}$$

We note that if $f \in A$ is given by (1.1), then by (1.3), we have

$$\mathcal{D}_{q,\lambda}^{\delta,n} f(z) = z + \sum_{k=2}^{\infty} \mathcal{B}(k,\delta,\lambda,n,q) a_k z^k \tag{1.4}$$

where,

$$\mathcal{B}(k,\delta,\lambda,n,q) = \left(\frac{\Gamma_q(2-\delta)\Gamma_q(k+1)}{\Gamma_q(k+1-\delta)}\left[([k]_q - 1)\lambda + 1\right]\right)^n \tag{1.5}$$

It can be seen that, by specializing the parameters, the operator $\mathcal{D}_{q,\lambda}^{\delta,n} f(z)$ reduces to many known and new integral and differential operators. In particular, when $\delta = 0$, $q \to 1^-$ the operator $\mathcal{D}_{q,\lambda}^{\delta,n} f(z)$ reduces to the operator introduced by Al-Oboudi [1] and if $\delta = 0$, $\lambda = 1$ and $q \to 1^-$ it reduces to the operator introduced by Sălăgean [4].

Now using the above differ-integral operator, we define the following subclass of .

Let $\mathcal{T}_q^n(b,k,\alpha,\delta,\lambda)$ be the subclass of T consisting of functions which satisfy the condition

$$\Re\left\{\frac{(b-2)\,\mathcal{D}_{q,\lambda}^{\delta,n} f + 2\ \mathcal{D}_{q,\lambda}^{\delta,n+1} f}{(b-2\lambda)\,\mathcal{D}_{q,\lambda}^{\delta,n} f + 2\lambda\,\mathcal{D}_{q,\lambda}^{\delta,n+1} f}\right\} \geq \alpha \tag{1.6}$$

For different parametric values of *q, λ, δ, b* and *n* we get the classes definedby Mostafa [3] and Annapoorna et. all [2].

2. Main Results

Now, we prove the sufficient conditions for the class $\mathcal{T}_q^n(b,k,\alpha,\delta,\lambda)$.

Theorem 2.1. *A function f defined by* (1.2), *is in the class* $\mathcal{T}_q^n(b,k,\alpha,\delta,\lambda)$ *if and only if*

$$\sum_{k=2}^{\infty} \mathcal{B}(k,\delta,\lambda,n,q)a_k\ \{b(1-\alpha)+2(1-\lambda\alpha)[1-\mathcal{B}(k,\delta,\lambda,1,q)]\} \leq b(1-\alpha)$$

where $\mathcal{B}(k,\delta,\lambda,n,q)$ *defined in* (1.5).

Proof:. Let

$$\Re\left\{\frac{(b-2)\,\mathcal{D}_{q,\lambda}^{\delta,n} f + 2\ \mathcal{D}_{q,\lambda}^{\delta,n+1} f}{(b-2\lambda)\,\mathcal{D}_{q,\lambda}^{\delta,n} f + 2\lambda\,\mathcal{D}_{q,\lambda}^{\delta,n+1} f}\right\} \geq \alpha$$

$$\Re\left\{\frac{bz-\sum_{k=2}^{\infty} B(k,\delta,\lambda,n,q)a_k z^k[b+2-2B(k,\delta,\lambda,1,q)]}{bz-\sum_{k=2}^{\infty} B(k,\delta,\lambda,n,q)a_k z^k[b+2\lambda-2\lambda B(k,\delta,\lambda,1,q)]}\right\} \geq \alpha$$

Letting $z \to 1$, we have

$$b-\sum_{k=2}^{\infty} \mathcal{B}(k,\delta,\lambda,n,q)a_k[b+2-2\,\mathcal{B}(k,\delta,\lambda,1,q)] \geq \ \alpha\left\{b-\sum_{k=2}^{\infty} \mathcal{B}(k,\delta,\lambda,n,q)a_k[b+2\lambda-2\,\lambda\,\mathcal{B}(k,\delta,\lambda,1,q)]\right\}$$

Therefore

$$\sum_{k=2}^{\infty} \mathcal{B}(k,\delta,\lambda,n,q)a_k\ \{b(1-\alpha)+2(1-\lambda\alpha)[1-\mathcal{B}(k,\delta,\lambda,1,q)]\} \leq b(1-\alpha).$$

Conversely,

Suppose $f \in \mathcal{T}_q^n(b,k,\alpha,\delta,\lambda)$ satisfies (1.6), equivalently

$$\left|\left\{\frac{(b-2)\,\mathcal{D}_{q,\lambda}^{\delta,n} f + 2\ \mathcal{D}_{q,\lambda}^{\delta,n+1} f}{(b-2\lambda)\,\mathcal{D}_{q,\lambda}^{\delta,n} f + 2\lambda\,\mathcal{D}_{q,\lambda}^{\delta,n+1} f}\right\}-1\right| \leq (1-\alpha)$$

$$\left|\frac{bz-\sum_{k=2}^{\infty} B(k,\delta,\lambda,n,q)a_k z^k[b+2-2B(k,\delta,\lambda,1,q)]}{bz-\sum_{k=2}^{\infty} B(k,\delta,\lambda,n,q)a_k z^k[b+2\lambda-2\lambda B(k,\delta,\lambda,1,q)]}-1\right| \leq (1-\alpha)$$

As $z \to 1$

$$\left|\frac{b-\sum_{k=2}^{\infty} B(k,\delta,\lambda,n,q)a_k[b+2-2B(k,\delta,\lambda,1,q)]}{b-\sum_{k=2}^{\infty} B(k,\delta,\lambda,n,q)a_k[b+2\lambda-2\lambda B(k,\delta,\lambda,1,q)]}-1\right| \leq (1-\alpha)$$

$$\left|\frac{\sum_{k=2}^{\infty} B(k,\delta,\lambda,n,q)a_k[2(\lambda-1)+2(\lambda-1)B(k,\delta,\lambda,1,q)]}{b-\sum_{k=2}^{\infty} B(k,\delta,\lambda,n,q)a_k[b+2\lambda-2\lambda B(k,\delta,\lambda,1,q)]}-1\right| \leq (1-\alpha)$$

This expression is bounded by $(1-\alpha)$, if

$$\sum_{k=2}^{\infty}\mathcal{B}(k,\delta,\lambda,n,q)a_k\, 2(1-\alpha)[\mathcal{B}(k,\delta,\lambda,1,q)-1]$$

$$\leq (1-\alpha)\left\{b-\sum_{k=2}^{\infty}\mathcal{B}(k,\delta,\lambda,n,q)a_k[b-2\,\lambda\,\{\mathcal{B}(k,\delta,\lambda,1,q)-1\}]\right\}$$

which is true by hypothesis. Hence the theorem.

As $q \to 1$ we get the following result.

Corollary 2.2. *A function f defined by* (1.2), *is in the class* $\mathcal{T}^n(b,k,\alpha,\delta,\lambda)$ *if and only if*

$$\sum_{k=2}^{\infty}\mathcal{B}(k,\delta,\lambda,n,1)a_k\ \{b(1-\alpha)+2(1-\lambda\alpha)[1-\mathcal{B}(k,\delta,\lambda,1{,}1)]\}\leq b(1-\alpha).$$

Corollary 2.3. *If* $f\in\mathcal{T}^n(b,k,\alpha,\delta,\lambda)$ *then*

$$|a_k|\leq\frac{b(1-\alpha)}{\mathcal{B}(k,\delta,\lambda,n,q)\,\{b(1-\alpha)+2(1-\lambda\alpha)[1-\mathcal{B}(k,\delta,\lambda,1,q)]\}}$$

Theorem 2.4. *Let* $0\leq\alpha<1$, $0\leq\lambda_1\leq\lambda_2<1$, $n>-1$, *b is any non-zero number, then*

$$\mathcal{T}_q^n(b,k,\alpha,\delta,\lambda_1)\subset\mathcal{T}_q^n(b,k,\alpha,\delta,\lambda_2)$$

Proof: From the Theorem 2.1

$$\sum_{k=2}^{\infty}\mathcal{B}(k,\delta,\lambda_2,n,q)\ \{b(1-\alpha)+2(1-\lambda_2\alpha)[1-\mathcal{B}(k,\delta,\lambda_2,1,q)]\}$$

$$\leq\sum_{k=2}^{\infty}\mathcal{B}(k,\delta,\lambda_1,n,q)\ \{b(1-\alpha)+2(1-\lambda_2\alpha)[1-\mathcal{B}(k,\delta,\lambda_1,1,q)]\}\leq b(1-\alpha).$$

For $f(z)\in\mathcal{T}_q^n(b,k,\alpha,\delta,\lambda_2)$

Hence $f(z)\in\mathcal{T}_q^n(b,k,\alpha,\delta,\lambda_1)$.

Theorem 2.5. Let $f(z)\in\mathcal{T}_q^n(b,k,\alpha,\delta,\lambda)$ define $f_1(z)=z$ and

$$f_k(z)\ =z-\frac{b(1-\alpha)}{\mathcal{B}(k,\delta,\lambda,n,q)\,\{b(1-\alpha)+2(1-\lambda\alpha)[1-\mathcal{B}(k,\delta,\lambda,1,q)]\}}z^k$$

where $k=2,3,4,\ldots$, $z\in U$, $0\leq\alpha<1$, $0\leq\lambda<1$, *b is non-zero real number.*
Then $f(z)\in\mathcal{T}_q^n(b,k,\alpha,\delta,\lambda)$ *if and only if* f *can be expressed as*

$$f(z)=\sum_{k=1}^{\infty}\mu_k f_k(z),$$

$$Where\ \ \mu_k\geq 0\ \ and\ \ \sum_{k=1}^{\infty}\mu_k=1$$

Proof: *If*

$$f(z) = \sum_{k=1}^{\infty} \mu_k f_k(z) \ \ with \ \ \mu_k \geq 0 \ \ and \ \ \sum_{k=1}^{\infty} \mu_k = 1 \ \ then$$

$$\sum_{k=2}^{\infty} \frac{\mathcal{B}(k,\delta,\lambda,n,q)\, b(1-\alpha) + 2(1-\lambda\alpha)[1-\mathcal{B}(k,\delta,\lambda,1,q)]}{\mathcal{B}(k,\delta,\lambda,n,q)\, b(1-\alpha) + 2(1-\lambda\alpha)[1-\mathcal{B}(k,\delta,\lambda,1,q)]}\, b(1-\alpha) = \sum_{k=1}^{\infty} \mu_k (1-\alpha)\, b \qquad = (1-\mu_1) b(1-\alpha)$$

$$\leq b(1-\alpha)$$

Hence $f(z) \in \mathcal{T}_q^n(b,k,\alpha,\delta,\lambda)$.

Conversly,

$$Let \ \ f(z) = z - \sum_{k=2}^{\infty} a_k z^k \in \mathcal{T}^n(b,k,\alpha,\delta,\lambda) \ \ \ define$$

$$\mu_k = \frac{\mathcal{B}(k,\delta,\lambda,n,q)\, b\,(1-\alpha) + 2(1-\lambda\alpha)[1-\mathcal{B}(k,\delta,\lambda,1,q)]|a_k|}{b(1-\alpha)} \qquad k = 2,3,4,\dots$$

Define ,

$$\mu_1 = 1 - \sum_{k=2}^{\infty} \mu_k \,. \quad From \ \ theorem\ 2.1\,, \qquad \sum_{k=2}^{\infty} \mu_k \ \leq 1 \ \ and \ \ so \ \ \mu_1 \geq 0\,.$$

Since $\mu_k f_k(z) = \mu_k f(z) + a_k z^k$,

$$\sum_{k=2}^{\infty} \mu_k f_k(z) = z - \sum_{k=2}^{\infty} a_k z^k = f(z)\ .$$

Hence the theorem.

Theorem 2.6. *The class* $\mathcal{T}_q^n(b,k,\alpha,\delta,\lambda)$ *is closed under convex linear combination.*

Proof: *Let* $f, g \in \mathcal{T}_q^n(b,k,\alpha,\delta,\lambda)$ *and let*

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \ \ , \ \ g(z) = z - \sum_{k=2}^{\infty} b_k z^k$$

For η such that $0 \leq \eta \leq 1$, it is enough to show that the function defined by

$$h(z) = (1-\eta) f(z) + \eta g(z) \in \mathcal{T}_q^n(b,k,\alpha,\delta,\lambda)$$

$$Now, \qquad h(z) = z - \sum_{k=2}^{\infty} [(1-\eta)a_k + \eta b_k]\ z^k$$

Applying Theorem 2.1, to $f, g \in \mathcal{T}_q^n(b,k,\alpha,\delta,\lambda)$ we have,

$$\sum_{k=2}^{\infty} \mathcal{B}(k,\delta,\lambda,n,q)\ \{b(1-\alpha) + 2(1-\lambda\alpha)[1-\mathcal{B}(k,\delta,\lambda,1,q)]\}[(1-\eta)a_k + \eta b_k]$$

$$\leq (1-\eta) \sum_{k=2}^{\infty} \mathcal{B}(k,\delta,\lambda,n,q)\ \{b(1-\alpha) + 2(1-\lambda\alpha)[1-\mathcal{B}(k,\delta,\lambda,1,q)]\} a_k$$

$$+\eta \sum_{k=2}^{\infty} \mathcal{B}(k,\delta,\lambda,n,q)\ \{b(1-\alpha) + 2(1-\lambda\alpha)[1-\mathcal{B}(k,\delta,\lambda,1,q)]\} b_k$$

$$\leq b(1-\alpha)(1-\eta) + \eta b(1-\alpha) = b(1-\alpha).$$

Implies that $h(z) \in \mathcal{T}_q^n(b,k,\alpha,\delta,\lambda)$.

Hence the theorem.

3. Conclusion

Here, in our present investigation, we have successfully introduced a new sub-classes of analytic functions $\mathcal{T}_q^n(b,k,\alpha,\delta,\lambda)$ using the q-differintegral operator. Many properties and characteristics of this newly defined Function have been studied.

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References

[1] F.M. Al-Oboudi, "On Univalent Functions Defined by a Generalized SăLăGean Operator," *International Journal of Mathematics and Mathematical Science,* vol. 2004, pp. 1429-1436, 2004. *Crossref,* https://doi.org/10.1155/S0161171204108090

[2] Annapoorna S, Deepthi K M and C. Goutham, "A Note Q- SăLăGean Type NegativeFunctions," *Gradiva Review Journal,* vol.7, no. 10, pp. 152-159, 2021. *Crossref,* https://doi.org/10.37897.GRJ.2021.V7I9.21.384

[3] A. O. Mostafa, "A Study on Starlike and Convex Properties for HypergeometricFunctions," *Journal of Inequalities in Pure and Applied Mathematics,* vol. 10, no. 3, 2009.

[4] L. Dileep, and S Latha, "A Note on Convolution Conditions," *International Journal of Mathematics Trends and Technology,* vol. 6, no. 2, pp. 189-191, 2014.

[5] L. Dileep, and S Latha, "Neighborhood Properties of Ruscheweyh Type Analytic Functions," *International Journal of Mathematics Trends and Technology,* vol. 9, no. 3, pp. 189-191, 2014.

[6] N. Ravikumar, L. Dileep, and S Latha, "Subclasses of Analytic Functions Associated with Hypergeometric Functions," *International Journal of Mathematics Trends and Technology,* vol. 30, no. 1, pp. 16-22, 2016. *Crossref,* https://doi.org/10.14445/22315373/IJMTT-V30P503

[7] Annapoorna S and Dr Dileep L, Application of linear multiplier fractional q-differintegral operator on analytic functions, *International Journal of Mathematics Trends and Technology (Accpeted for Publucation)*

[8] Grigore Stefan Salagean, "Subclasses of Univalent Functions," *Lecture Notes in Mathematics, Springer-Verlag,* pp. 362-372, 1983. *Crossref,* https://doi.org/10.1007/BFb0066543

[9] N. Ravikumar, L. Dileep, and S Latha, "A Note on Al-Oboudi Type Functions," *The Journal of Rajasthan Academy of Physical Sciences,* vol. 9, no. 2, pp. 155-164, 2010.

[10] Stephan Ruscheweyh, "New Criteria for Univalent Functions," *Proceedings of American Mathematical Society*, vol. 49, no. 1, pp. 109-115, 1975. *Crossref,* https://doi.org/10.2307/2039801

[11] P. L. Duren, *Univalent functions,* Springer-Verlag, 1983.

[12] Dileep L and Mallige Rajeev, "Convolution Operators in Geometric Function Theory," *GIS Science Journal,* vol.7, no.7, pp. 473-483, 2020. *Crossref,* https://doi.org/20.18001.GSJ.2020.V7I7.20.35571

[13] L. Dileep, and S. Latha, "Certain Subclasses of Analytic Functions Involving Salagean- Ruscheweyh Operator," *Vietnam Journal of Mathematics*, vol.38, no. 4, pp. 403-412, 2010.

[14] A.W. Goodman, *Univalent functions*, Marnier Publication Company, 1983.

[15] A.W. Goodman, "On Uniformly Convex Functions, *Annales of Polon Mathematics,* vol. 56, no. 1, pp. 87-92, 1991.

[16] Cho, Nak-Eun, and Kim, Tae-Hwa, "Multiplier Transformation and Strongly Close - To - Convex Functions," *Bullatien of Korean Mathematical Society*, vol. 40, no. 3, pp. 399 - 410, 2003. *Crossref,* https://doi.org/10.4134/BKMS.2003.40.3.399

[17] S.D Bernardi, *Bibliography of Schlicht functions*, Mariner Publication Company, 1983.

[18] Ch. Pommerenke, *Univalent functions,* Vandenhoeck & Ruprecht, Gottingen, 1975.

[19] M. S, Ganesan, "On Certain Classes of Analytic Functions," *Indian Journal of Pure and Applied Mathematics,* vol. 13, no. 1, pp. 47-57, 1982.

[20] Z. Nehari, *Conformal Mapping*, McGraw Hill Book Company, 1952.

[21] K.S, Padmanabhan, "On Certain Classes of Starlike Functions in the Unit Disc," *The Journal of the Indian Mathematical Society,* vol. 32, pp. 89-103, 1968.

[22] G.Lakshma Reddy, and K.S. Padmanabhan, *On Analytic Functions wih Reference to the Bernardi Integral Operator, Bulletin of the Australian Mathematical Society,* vol. 25, no. 3, pp. 387-396, 1982.

[23] M.S, Roberston, "On the Theory of Univalent Functions," *Annals of Math*ematics, vol. 37, pp. 374-406, 1936.

[24] Dorothy Browne Shaffer, "Distortion Theorems for a Special Class of Analytic Functions," *Proceedings of the American Mathematical Society,* vol. 39, no. 2, pp. 281-287, 1973. *Crossref*, https://doi.org/10.2307/2039632

[25] L. Dileep, and S.V. Divya Rashmi, "Certain Subclasses of Analytic Function by SALAGEAN q-Differential Operator," *TWMS Journal of Applied and Engineering Mathematics,* vol. 13, no. 1, pp. 46-52, 2023.