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Degree Sum Energy in Context of Some Graph Operations on Regular Graph

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Abstract

Let G be a graph with order p and size q. The degree sum energy of a graph G is defined as the sum of the absolute values of the eigenvalues of the degree sum matrix of G. In this paper, we obtain the degree sum energy of $m-$splitting graph of a regular graph and central graph of regular graph. We define a new operation $m-$semishadow graph and obtain degree sum energy of $m-$semishadow graph of a regular graph.

Keywords: Degree sum energy, $m-$ Splitting graph, $m-$Semishadow graph, Central graph, Regular graph.
AMS Subject Classification (2020): 05C50.

1 Introduction

Let G be a simple graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \dots, e_n\}$. The number of edges incident with vertex v_i is called degree of v_i; denoted by d_i. The adjacency matrix $A(G)$ of the graph G is a square matrix of order n whose (i, j) - entry is equal to 1 if the vertices v_i and v_j are adjacent, and is equal to zero otherwise.
The absolute sum of eigen values of adjacency matrix of graph G is called Energy of graph G. Energy of graph G was first introduced by Ivan Gutman[5] in 1978 and he briefly outlined the connection between the energy of a graph and the total $\pi-$electron energy of organic molecules. He also presented some fundamental results on graph energy, the relation between energy $E(G)$ of the graph G and the characteristic polynomial of G. H. S. Ramane and I. Gutman [6] gave list of research papers on Graph Energy published in 2019, which became resourceful during our study.
H. S. Ramane, D. S. Revankar, and J. B. Patil defined degree sum energy of a graph as follows.

Definition 1.1. *[4] Let G be a simple graph with n vertices $v_1, v_2, \ldots, v_n$ and let d_i be the degree of $v_i, i = 1, 2, \ldots, n$. Then the degree sum matrix of a graph G is $DS(G) = [d_{ij}]$, where*

$$d_{ij} = \begin{cases} d_i + d_j; & v_iv_j \in E(G). \\ 0; & otherwise \end{cases}$$

The characteristic polynomial of degree sum matrix of G is $|\lambda I_n - DS(G)|$, where I_n is the identity matrix of order n. The roots of the characteristic equation $|\lambda I_n - DS(G)| = 0$ are called the degree sum eigenvalues of G. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the degree sum eigenvalues of G then the degree sum energy of a graph G is defined as

$$E_{DS}(G) = \sum_{i=1}^{n} |\lambda_i|.$$

We begin with simple, finite, connected and undirected graph $G = (V(G), E(G))$ with order p and size q, which is denoted as $G(p, q)$. We refer to Bondy and Murty [8] for the standard terminology and notations related to graph theory.

Lemma 1.2. *[2] If a and b are scalars then*

$$\begin{vmatrix} a & b & b & \ldots & b \\ b & a & b & \ldots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \ldots & a \end{vmatrix} = (a-b)^{n-1}[a + (n-1)b]$$

Lemma 1.3. *[2] Let A and C be square matrices then*

$$\begin{vmatrix} A & B \\ 0 & C \end{vmatrix} = |A| \times |C|.$$

2 Results

Definition 2.1. *[11] The splitting graph $S'(G)$ of a graph G is obtained by adding to each vertex v a new vertex v' such that v' is adjacent to every vertex that is adjacent to v in G.*

Definition 2.2. *[11] The $m-$splitting graph $Spl_m(G)$ of a graph G is obtained by adding to each vertex v of G new m vertices, say $v^1, v^2, v^3, \ldots, v^m$ such that $v^i, 1 \le i \le m$ is adjacent to each vertex that is adjacent to v in G.*

Theorem 2.3. *Let G be $r-$ regular graph then* $E_{DS}(Spl_m(G)) = 2r\Big(2mn - m + n - 2 + \sqrt{(m-n)^2 + mn^2(m+2)^2}\Big)$.

Proof. Let $v_1, v_2, \ldots, v_n$ be n vertices of $r-$ regular graph G. So the degree sum matrix of $r-$ regular graph G is of order n and matrix is

$$DS(G) = \begin{bmatrix} 0 & 2r & 2r & \ldots & 2r \\ 2r & 0 & 2r & \ldots & 2r \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2r & 2r & 2r & \ldots & 0 \end{bmatrix}$$

Let $v_i^1, v_i^2, \ldots, v_i^m$ be the vertices corresponding to each vertex v_i, where $1 \leq i \leq n$, which are added in G to construct $Spl_m(G)$ such that $N(v_i^1) = N(v_i^2) = N(v_i^3) = \ldots = N(v_i^m) = N(v_i)$.
The degree sum matrix of $m-$splitting graph of $r-$ regular graph G is of order $mn + n$ and matrix is

$$DS(Spl_m(G)) = \left[\begin{array}{cccc|cccc} 0 & 2r(m+1) & \ldots & 2r(m+1) & r(m+2) & r(m+2) & \ldots & r(m+2) \\ 2r(m+1) & 0 & \ldots & 2r(m+1) & r(m+2) & r(m+2) & \ldots & r(m+2) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2r(m+1) & 2r(m+1) & \ldots & 0 & r(m+2) & r(m+2) & \ldots & r(m+2) \\ \hline r(m+2) & r(m+2) & \ldots & r(m+2) & 0 & 2r & \ldots & 2r \\ r(m+2) & r(m+2) & \ldots & r(m+2) & 2r & 0 & \ldots & 2r \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ r(m+2) & r(m+2) & \ldots & r(m+2) & 2r & 2r & \ldots & 0 \end{array}\right]$$

Here upper left block is of order n, upper right block is of order $n \times mn$, lower left block is of order $mn \times n$ and lower right block if of order mn.
So, the characteristic polynomial of $DS(Spl_m(G))$ is $|\lambda I_n - DS(Spl_m(G))|$

$$= \left|\begin{array}{cccc|cccc} \lambda & -2r(m+1) & \ldots & -2r(m+1) & -r(m+2) & -r(m+2) & \ldots & -r(m+2) \\ -2r(m+1) & \lambda & \ldots & -2r(m+1) & -r(m+2) & -r(m+2) & \ldots & -r(m+2) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -2r(m+1) & -2r(m+1) & \ldots & \lambda & -r(m+2) & -r(m+2) & \ldots & -r(m+2) \\ \hline -r(m+2) & -r(m+2) & \ldots & -r(m+2) & \lambda & -2r & \ldots & -2r \\ -r(m+2) & -r(m+2) & \ldots & -r(m+2) & -2r & \lambda & \ldots & -2r \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -r(m+2) & -r(m+2) & \ldots & -r(m+2) & -2r & -2r & \ldots & \lambda \end{array}\right|$$

$$= \left|\begin{array}{c|c} (\lambda + 2r(m+1))I_n - 2r(m+1)J_n & -r(m+2)J_{n\times mn} \\ \hline -r(m+2)J_{mn\times n} & (\lambda+2r)I_{mn} - 2rJ_{mn} \end{array}\right|$$

Where I_n is the identity matrix of order n and J_n is the matrix of ones of order n.
To reduce the lower left block containing all "$-r(m+2)$" to zero in above determinant, we perform
$R_i + \dfrac{r(m+2)}{\lambda - 2r(m+1)(n-1)} \sum_{j=1}^{n} R_j$, where $i = n+1, \ldots, mn+n$, Then above determinant is

$$\left|\begin{array}{c|c} (\lambda + 2r(m+1))I_n - 2r(m+1)J_n & -r(m+2)J_{n\times mn} \\ \hline 0 & \beta J_{mn} - (\beta-\alpha)I_{mn} \end{array}\right|$$

Where $\alpha = \lambda - \dfrac{r^2(m+2)^2 n}{\lambda - 2r(m+1)(n-1)}$ and $\beta = -2r - \dfrac{r^2(m+2)^2 n}{\lambda - 2r(m+1)(n-1)}$.
On applying Lemma 1.3 we get,

$$\left| \ \lambda + 2r(m+1))I_n - 2r(m+1)J_n \ \right| \times \left| \ \beta J_{mn} - (\beta-\alpha)I_{mn} \ \right|$$

On applying Lemma 1.2 we get,

$$|\lambda I_n - DS(Spl_m(G))| = (\lambda + 2r(m+1))^{n-1}(\lambda+2r)^{mn-1}\left\{\lambda - r\left(2mn - m + n - 2 \pm \sqrt{(m-n)^2 + mn^2(m+2)^2}\right)\right\}$$

Thus, spectrum of $DS(Spl_m(G))$ is

$$\begin{pmatrix} -2r(m+1) & -2r & r\Big(2mn-m+n-2 \pm \sqrt{(m-n)^2+mn^2(m+2)^2}\Big) \\ n-1 & mn-1 & 1 \end{pmatrix}$$

Calculating absolute sum of eigen values, we get

$$\begin{aligned} E_{DS}(Spl_m(G)) &= 2r(m+1)(n-1)+2r(mn-1) \\ &\quad +r\Big(\sqrt{(m-n)^2+mn^2(m+2)^2}+2mn-m+n-2\Big) \\ &\quad +r\Big(\sqrt{(m-n)^2+mn^2(m+2)^2}-2mn+m-n+2\Big). \\ &= 2r\Big(2mn-m+n-2+\sqrt{(m-n)^2+mn^2(m+2)^2}\Big). \end{aligned}$$

□

Example 2.4. *The Degree sum matrix of* $2-$ *splitting graph of complete graph* K_4 *(*$3-$*regular graph of* 4 *vertices) is*

$$\left[\begin{array}{cccc|cccccccc} 0 & 18 & 18 & 18 & 12 & 12 & 12 & 12 & 12 & 12 & 12 & 12 \\ 18 & 0 & 18 & 18 & 12 & 12 & 12 & 12 & 12 & 12 & 12 & 12 \\ 18 & 18 & 0 & 18 & 12 & 12 & 12 & 12 & 12 & 12 & 12 & 12 \\ 18 & 18 & 18 & 0 & 12 & 12 & 12 & 12 & 12 & 12 & 12 & 12 \\ \hline 12 & 12 & 12 & 12 & 0 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\ 12 & 12 & 12 & 12 & 6 & 0 & 6 & 6 & 6 & 6 & 6 & 6 \\ 12 & 12 & 12 & 12 & 6 & 6 & 0 & 6 & 6 & 6 & 6 & 6 \\ 12 & 12 & 12 & 12 & 6 & 6 & 6 & 0 & 6 & 6 & 6 & 6 \\ 12 & 12 & 12 & 12 & 6 & 6 & 6 & 6 & 0 & 6 & 6 & 6 \\ 12 & 12 & 12 & 12 & 6 & 6 & 6 & 6 & 6 & 0 & 6 & 6 \\ 12 & 12 & 12 & 12 & 6 & 6 & 6 & 6 & 6 & 6 & 0 & 6 \\ 12 & 12 & 12 & 12 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 0 \end{array}\right]$$

So, the characteristic polynomial of $DS(Spl_2(K_4)$ *is*

$$(\lambda+18)^3(\lambda+6)^7(\lambda-48\pm 6\sqrt{129}).$$

Thus, spectrum of $DS(Spl_3(K_4))$ *is*

$$\begin{pmatrix} -18 & -6 & 48+6\sqrt{129} & 48-6\sqrt{129} \\ 3 & 7 & 1 & 1 \end{pmatrix}$$

Calculating absolute sum of eigen values, we get

$$E_{DS}(Spl_2(K_4)) = 12(8+\sqrt{129}).$$

.

We define $m-$ semi shadow graph as under:

Definition 2.5. *Let G be a graph with vertices $v_1, v_2, \ldots, v_n$. The $m-$semi shadow graph $SD_m(G)$ of graph G is constructed by taking graph G and m copies of G say $G^1, G^2, \ldots, G^m$ with vertices $v_i^1, v_i^2, \ldots, v_i^m$, where $1 \leq i \leq n$, and joining each vertex $v_i^{(j)}$ to all vertices which are adjacent to v_i in G, where $1 \leq i \leq n$ and $1 \leq j \leq m$.*

Let G be a graph with n vertices and e edges then the number of vertices and the number of edges in $SD_m(G)$ is respectively $(m+1)n$ and $e(3m+1)$.

Remark 2.6. *The $1-$ semi shadow graph of graph G is same as Shadow graph $D_2(G)$ of graph G.*

Theorem 2.7. *Let G be $r-$ regular graph then*
$E_{DS}(SD_m(G)) = 2r\Big(3mn - m + n - 3 + \sqrt{(m-1)^2(n+1)^2 + (m+3)^2mn^2}\Big)$.

Proof. Let $v_1, v_2, \ldots, v_n$ be n vertices of $r-$ regular graph G. So the n^{th} order degree sum matrix of G is same as in theorem 2.3.
Consider $m-$ copies of graph G, say $G^1, G^2, \ldots, G^m$ with vertices $v_i^1, v_i^2, \ldots, v_i^m$, where $1 \leq i \leq n$. To construct $SD_m(G)$, we join each vertex $v_i^{(j)}$ to all vertices which are adjacent to v_i in G, where $1 \leq i \leq n$ and $1 \leq j \leq m$.
The degree sum matrix of $m-$semi shadow graph of $r-$ regular graph G is of order $mn+n$ and matrix is

$$DS(SD_m(G)) = \left[\begin{array}{cccc|cccc} 0 & 2r(m+1) & \ldots & 2r(m+1) & r(m+3) & r(m+3) & \ldots & r(m+3) \\ 2r(m+1) & 0 & \ldots & 2r(m+1) & r(m+3) & r(m+3) & \ldots & r(m+3) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2r(m+1) & 2r(m+1) & \ldots & 0 & r(m+3) & r(m+3) & \ldots & r(m+3) \\ \hline r(m+3) & r(m+3) & \ldots & r(m+3) & 0 & 4r & \ldots & 4r \\ r(m+3) & r(m+3) & \ldots & r(m+3) & 4r & 0 & \ldots & 4r \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ r(m+3) & r(m+3) & \ldots & r(m+3) & 4r & 4r & \ldots & 0 \end{array}\right]$$

Here upper left block is of order n, upper right block is of order $n \times mn$, lower left block is of order $mn \times n$ and lower right block if of order mn.
So, the characteristic polynomial of $DS(SD_m(G))$ is $|\lambda I_n - DS(SD_m(G))|$

$$= \left|\begin{array}{cccc|cccc} \lambda & -2r(m+1) & \ldots & -2r(m+1) & -r(m+3) & -r(m+3) & \ldots & -r(m+3) \\ -2r(m+1) & \lambda & \ldots & -2r(m+1) & -r(m+3) & -r(m+3) & \ldots & -r(m+3) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -2r(m+1) & -2r(m+1) & \ldots & \lambda & -r(m+3) & -r(m+3) & \ldots & -r(m+3) \\ \hline -r(m+3) & -r(m+3) & \ldots & -r(m+3) & \lambda & -4r & \ldots & -4r \\ -r(m+3) & -r(m+3) & \ldots & -r(m+3) & -4r & \lambda & \ldots & -4r \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -r(m+3) & -r(m+3) & \ldots & -r(m+3) & -4r & -4r & \ldots & \lambda \end{array}\right|$$

$$= \left|\begin{array}{c|c} (\lambda + 2r(m+1))I_n - 2r(m+1)J_n & -r(m+3)J_{n\times mn} \\ \hline -r(m+3)J_{mn\times n} & (\lambda+4r)I_{mn} - 4rJ_{mn} \end{array}\right|$$

Where I_n is the identity matrix of order n and J_n is the matrix of ones of order n.
To reduce the lower left block containing all "$-r(m+3)$" to zero in above determinant, we perform
$R_i + \dfrac{r(m+3)}{\lambda - 2r(m+1)(n-1)} \sum_{j=1}^{n} R_j$, where $i = n+1, \ldots, mn+n$, Then above determinant is

$$\left|\begin{array}{c|c} (\lambda + 2r(m+1))I_n - 2r(m+1)J_n & -r(m+3)J_{n\times mn} \\ \hline 0 & \beta J_{mn} - (\beta - \alpha)I_{mn} \end{array}\right|$$

Where $\alpha = \lambda - \dfrac{r^2(m+3)^2 n}{\lambda - 2r(m+1)(n-1)}$ and $\beta = -4r - \dfrac{r^2(m+3)^2 n}{\lambda - 2r(m+1)(n-1)}$.
On applying Lemma 1.3 we get,

$$\left| \lambda + 2r(m+1))I_n - 2r(m+1)J_n \right| \times \left| \beta J_{mn} - (\beta - \alpha)I_{mn} \right|$$

On applying Lemma 1.2 we get, $|\lambda I_n - DS(SD_m(G))|$

$$= (\lambda+2r(m+1))^{n-1}(\lambda+4r)^{mn-1}\Big\{\lambda - r\Big(3mn - m + n - 3 \pm \sqrt{(m-1)^2(n+1)^2 + (m+3)^2mn^2}\Big)\Big\}$$

Thus, spectrum of $DS(SD_m(G))$ is

$$\begin{pmatrix} -2r(m+1) & -4r & r\Big(3mn - m + n - 3 \pm \sqrt{(m-1)^2(n+1)^2 + (m+3)^2mn^2}\Big) \\ n-1 & mn-1 & 1 \end{pmatrix}$$

Calculating absolute sum of eigen values, we get

$$\begin{aligned} E_{DS}(SD_m(G)) = {} & 2r(m+1)(n-1) + 4r(mn-1) \\ & + r\Big(\sqrt{(m-1)^2(n+1)^2 + (m+3)^2mn^2} + 3mn - m + n - 3\Big) \\ & + r\Big(\sqrt{(m-1)^2(n+1)^2 + (m+3)^2mn^2} - 3mn + m - n + 3\Big). \\ = {} & 2r\Big(3mn - m + n - 3 + \sqrt{(m-1)^2(n+1)^2 + (m+3)^2mn^2}\Big). \end{aligned}$$

□

Example 2.8. *The Degree sum matrix of $2-$ semi shadow graph of cycle C_4 ($2-$regular graph of 4 vertices) is*

$$DS(SD_2(C_4) = \left[\begin{array}{cccc|cccccccc} 0 & 12 & 12 & 12 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 \\ 12 & 0 & 12 & 12 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 \\ 12 & 12 & 0 & 12 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 \\ 12 & 12 & 12 & 0 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 \\ \hline 10 & 10 & 10 & 10 & 0 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\ 10 & 10 & 10 & 10 & 8 & 0 & 8 & 8 & 8 & 8 & 8 & 8 \\ 10 & 10 & 10 & 10 & 8 & 8 & 0 & 8 & 8 & 8 & 8 & 8 \\ 10 & 10 & 10 & 10 & 8 & 8 & 8 & 0 & 8 & 8 & 8 & 8 \\ 10 & 10 & 10 & 10 & 8 & 8 & 8 & 8 & 0 & 8 & 8 & 8 \\ 10 & 10 & 10 & 10 & 8 & 8 & 8 & 8 & 8 & 0 & 8 & 8 \\ 10 & 10 & 10 & 10 & 8 & 8 & 8 & 8 & 8 & 8 & 0 & 8 \\ 10 & 10 & 10 & 10 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 0 \end{array}\right]$$

So, the characteristic polynomial of $DS(SD_2(C_4)$ is

$$(\lambda+12)^3(\lambda+8)^7(\lambda-46\pm 10\sqrt{33}).$$

Thus, spectrum of $DS(SD_2(C_4))$ is

$$\begin{pmatrix} -12 & -8 & 46+10\sqrt{33} & 46-10\sqrt{33} \\ 3 & 7 & 1 & 1 \end{pmatrix}$$

Calculating absolute sum of eigen values, we get

$$E_{DS}(SD_2(K_4)) = 4(23+5\sqrt{33}).$$

.

Definition 2.9. *[1] The Central graph of a simple, connected and undirected graph G, denoted by $C(G)$ is obtained by subdividing each edge of G exactly once and joining all the non-adjacent vertices of G in $C(G)$.*

Let G be a simple $r-$ regular graph with n vertices then the number of vertices and the number edges in $C(G)$ is $n+\frac{nr}{2}$ and $\frac{n(n+r-1)}{2}$ respectively.

Theorem 2.10. *Let G be $r-$ regular graph then*
$E_{DS}(C(G)) = 2(n^2+nr-2n-1)+2\sqrt{\left(nr-2-(n-1)^2\right)^2+\frac{n^2r(n+1)^2}{2}}$.

Proof. Let $v_1, v_2, \ldots, v_n$ be n vertices of $r-$ regular graph G. The degree sum matrix of $C(G)$ of $r-$ regular graph G is of order $n+\frac{nr}{2}$ and matrix is

$$DS(C(G)) = \left[\begin{array}{cccc|cccc} 0 & 2(n-1) & \ldots & 2(n-1) & n+1 & n+1 & \ldots & n+1 \\ 2(n-1) & 0 & \ldots & 2(n-1) & n+1 & n+1 & \ldots & n+1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2(n-1) & 2(n-1) & \ldots & 0 & n+1 & n+1 & \ldots & n+1 \\ \hline n+1 & n+1 & \ldots & n+1 & 0 & 4 & \ldots & 4 \\ n+1 & n+1 & \ldots & n+1 & 4 & 0 & \ldots & 4 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ n+1 & n+1 & \ldots & n+1 & 4 & 4 & \ldots & 0 \end{array}\right]$$

Here upper left block is of order n, upper right block is of order $n\times\frac{nr}{2}$, lower left block is of order $\frac{nr}{2}\times n$ and lower right block if of order $\frac{nr}{2}$.
So, the characteristic polynomial of $DS(C(G))$ is $|\lambda I_n - DS(C(G))|$

$$= \left|\begin{array}{cccc|cccc} \lambda & -2(n-1) & \ldots & -2(n-1) & -(n+1) & -(n+1) & \ldots & -(n+1) \\ -2(n-1) & 0 & \ldots & -2(n-1) & -(n+1) & -(n+1) & \ldots & -(n+1) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -2(n-1) & -2(n-1) & \ldots & \lambda & -(n+1) & -(n+1) & \ldots & -(n+1) \\ \hline -(n+1) & -(n+1) & \ldots & -(n+1) & \lambda & -4 & \ldots & -4 \\ -(n+1) & -(n+1) & \ldots & -(n+1) & -4 & \lambda & \ldots & -4 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -(n+1) & -(n+1) & \ldots & -(n+1) & -4 & -4 & \ldots & \lambda \end{array}\right|$$

$$= \left| \begin{array}{c|c} (\lambda + 2(n-1))I_n - 2(n-1)J_n & -(n+1)J_{n\times \frac{nr}{2}} \\ \hline -(n+1)J_{\frac{nr}{2}\times n} & (\lambda+4)I_{\frac{nr}{2}} - 4J_{\frac{nr}{2}} \end{array} \right|$$

Where I_n is the identity matrix of order n and J_n is the matrix of ones of order n.
To reduce the lower left block containing all " $-(n+1)$" to zero in above determinant, we perform
$R_i + \dfrac{(n+1)}{\lambda - 2(n-1)^2} \sum_{j=1}^{n} R_j$, where $i = n+1, \ldots, n + \frac{nr}{2}$, Then above determinant is

$$\left| \begin{array}{c|c} (\lambda + 2(n-1))I_n - 2(n-1)J_n & -(n+1)J_{n\times \frac{nr}{2}} \\ \hline 0 & \beta J_{\frac{nr}{2}} - (\beta-\alpha)I_{\frac{nr}{2}} \end{array} \right|$$

Where $\alpha = \lambda - \dfrac{n(n+1)^2}{\lambda - 2(n-1)^2}$ and $\beta = -4 - \dfrac{n(n+1)^2}{\lambda - 2(n-1)^2}$.
On applying Lemma 1.3 we get,

$$\left| \ (\lambda + 2(n-1))I_n - 2(n-1)J_n \ \right| \times \left| \ \beta J_{\frac{nr}{2}} - (\beta-\alpha)I_{\frac{nr}{2}} \ \right|$$

On applying Lemma 1.2 we get, $|\lambda I_n - DS(C(G))|$

$$= (\lambda+2(n-1))^{n-1}(\lambda+4)^{\frac{nr}{2}-1}\left\{\lambda - \left(nr-2+(n-1)^2 \pm \sqrt{\left(nr-2-(n-1)^2\right)^2 + \frac{n^2r(n+1)^2}{2}}\right)\right\}$$

Thus, spectrum of $DS(SD_m(G))$ is

$$\begin{pmatrix} -2(n-1) & -4 & nr-2+(n-1)^2 \pm \sqrt{\left(nr-2-(n-1)^2\right)^2 + \frac{n^2r(n+1)^2}{2}} \\ n-1 & \frac{nr}{2}-1 & 1 \end{pmatrix}$$

Calculating absolute sum of eigen values, we get

$$\begin{aligned} E_{DS}(C(G)) = {} & 2(n-1)^2 + 4(\frac{nr}{2} - 1) \\ & + \sqrt{\left(nr-2-(n-1)^2\right)^2 + \frac{n^2r(n+1)^2}{2}} + nr - 2 + (n-1)^2 \\ & + \sqrt{\left(nr-2-(n-1)^2\right)^2 + \frac{n^2r(n+1)^2}{2}} - nr + 2 - (n-1)^2. \\ = {} & 2(n^2 + nr - 2n - 1) + 2\sqrt{\left(nr-2-(n-1)^2\right)^2 + \frac{n^2r(n+1)^2}{2}}. \end{aligned}$$

□

Example 2.11. *The Degree sum matrix of central graph of cycle C_5 (2−regular graph of* 5 *vertices) is*

$$DS(C(C_5)) = \left[\begin{array}{ccccc|ccccc} 0 & 8 & 8 & 8 & 8 & 6 & 6 & 6 & 6 & 6 \\ 8 & 0 & 8 & 8 & 8 & 6 & 6 & 6 & 6 & 6 \\ 8 & 8 & 0 & 8 & 8 & 6 & 6 & 6 & 6 & 6 \\ 8 & 8 & 8 & 0 & 8 & 6 & 6 & 6 & 6 & 6 \\ 8 & 8 & 8 & 8 & 0 & 6 & 6 & 6 & 6 & 6 \\ \hline 6 & 6 & 6 & 6 & 6 & 0 & 4 & 4 & 4 & 4 \\ 6 & 6 & 6 & 6 & 6 & 4 & 0 & 4 & 4 & 4 \\ 6 & 6 & 6 & 6 & 6 & 4 & 4 & 0 & 4 & 4 \\ 6 & 6 & 6 & 6 & 6 & 4 & 4 & 4 & 0 & 4 \\ 6 & 6 & 6 & 6 & 6 & 4 & 4 & 4 & 4 & 0 \end{array}\right]$$

So, the characteristic polynomial of $DS(C(C_5)$ is

$$(\lambda+8)^4(\lambda+4)^4(\lambda-24\pm 2\sqrt{241}).$$

Thus, spectrum of $DS(SD_2(C_4))$ is

$$\begin{pmatrix} -8 & -4 & 24+\sqrt{241} & 24-\sqrt{241} \\ 4 & 4 & 1 & 1 \end{pmatrix}$$

Calculating absolute sum of eigen values, we get

$$E_{DS}(C(K_4)) = 2(24+\sqrt{241}).$$

3 Conclusion

The energy of a graph is one of the emerging concepts in graph theory which serves as a frontier between chemistry and mathematics. In this paper, we compute the Degree sum energy of $m-$ splitting graph of $r-$regular graph and central graph of $r-$ regular graph. We define $m-$semi shadow graph and compute the Degree sum energy of $m-$semi shadow graph of $r-$regular graph. It is interesting to compute different graph energies for the different families of graphs.

References

[1] M.M. Ali Akbar, K. Kaliraj and Vernold J. Vivin, "On equitable coloring of central graphs and total graphs", *Electronic Notes in Discrete Mathematics*, vol. 33, pp. 1-6, 2009.

[2] David W. Lewis, "Matrix Theory", *Allied Publishers Ltd*, 1995.

[3] G. V. Ghodasara and M. J. Patel, "Some bistar related square sum graphs", *International Journal of Mathematics Trends and Technology*, vol. 8, no. 1, pp. 47-57, 2013.

[4] H. S. Ramane, D. S. Revankar and J. B. Patil, "Bounds for the degree sum eigen values and degree sum energy of a graph", *International Journal of Pure and Applied Mathematical Sciences*, vol. 6, no. 2, pp. 161-167, 2013.

[5] I. Gutman, The energy of a graph, *Berlin Mathmatics-Statistics Forschungszentrum*, vol. 103, pp. 1-22, 1978.

[6] I. Gutman, H. S. Ramane, "Research on Graph Energies in 2019", *MATCH Communications in Mathematical and in Computer Chemistry*, vol. 84, pp. 277-292, 2020.

[7] I. Shparlinski, "On the energy of some circulant graphs", *Linear Algebra and its Applications*, vol. 414, pp. 378-382, 2006.

[8] J. A. Bondy and U.S. Murty, "Graph Theory with Applications", *Elsevier Science Publication*, 1982.

[9] M. J. Patel and G. V. Ghodasara, "Some Results on Degree Sum Energy of a Graph", *Recent Advancements in Graph Theory,CRC Press*, pp. 1-7, 2017.

[10] S. Jog, R. Kotambari, "Degree Sum Energy of Some Graphs", *Annals of Pure and Applied Mathematics*, vol. 11, pp. 17-27, 2016.

[11] S. K. Vaidya, K. M. Popat, "Energy of m-splitting and m-shadow graphs", *Far East Journal of Mathematical Sciences*, vol. 102, pp. 571-1578, 2017.

[12] S. M. Hosamani, H. S. Ramane, "On Degree Sum Energy of a Graph", *European Journal of Pure and Applied Mathematics*, vol. 9, no. 3, pp. 340-345, 2016.

[13] S. P. Hande, S. R. Jog and D. S. Revankar, "Bounds for the degree sum eigenvalue and degree sum energy of a common neighborhood graph", *International Journal of Graph Theory*, vol. 1, no. 4, pp. 131-136, 2013.

[14] S. R. Jog, S. P. Hande and D. S. Revankar, "Degree sum polynomial of graph valued functions on regular graphs", *International Journal of Graph Theory*, vol. 1, no. 3, pp. 108-115, 2013.