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Original Article

Alternating Group A_8 is Simple using Iwasawa Theorem

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Abstract - *The aim of this article is to prove that the Alternating Group A_8 is simple, using Iwasawa Theorem.*

Keywords - *Alternating Group, Faithful action, Group action, Iwasawa theorem, Primitive action, Simple groups, Transitive action.*

1. Introduction

In group theory, search for finite simple group has been interesting and difficult problem. Iwasawa theorem helps in classifying groups as simple. Alternating groups on n symbols, where $n \geq 5$ are simple. An attempt has been made to prove that alternating group on 8 symbols is simple.

1.1. Preliminaries

Definition 1.1 Group Action. Let G be a group and S be a set. Let Sym(S) denote the set of all bijections from S to S (under composition of functions). We say a group G acts on a set S if there exists a homomorphism $\psi: G \rightarrow Sym(S)$.

Definition 1.2 Faithful Action. A group G is said to act faithfully on S, if for any two distinct g, h ∈ G, there exists x ∈ S such that $gx \neq hx$.

Definition 1.3 Transitive Action. The action of group G on a set S is said to be transitive if $Orb_G(a) = S$, for some (and hence for all) a ∈ S, where $Orb_G(a) = \{g.a \mid g \in G\}$.

Definition 1.4 Doubly Transitive Action. If $|S| \geq 2$ then we say G acts doubly transitive on a set S, if for any (a, b), (c, d) in $S \times S$ with $a \neq b$ and $c \neq d$ there exists x ∈ G such that xa = c and xb = d.

Definition 1.5 Primitive action. Suppose G is transitive on a set S. A block B of imprimitivity for G in S is a subset of S with $|B| \geq 2$, $B \neq S$, such that for all x ∈ G, either xB = B or $xB \cap B = \varphi$. If no such block exists then we say G acts primitively on S.

1.2. Theorems

Theorem 1.1 Suppose that G is transitive on S. The action is primitive if and only if every $Stab_G(a)$,where a ∈ S is a maximal subgroup of G

Theorem 1.2 A_n is generated by 3− cycle

Theorem 1.3 Burnside's theorem in group theory states that if G is a finite group of order $p^a q^b$, where p and q are prime numbers and a, b are non-negative integers, then G is solvable.

Theorem 1.4 The action of group G on a set S is said to be faithful if and only if the homomorphism $\psi: G \rightarrow Sym(S)$ is an injective map.

2. Statement of Iwasawa Theorem

Let G be a group acting on a set S such that the action of G is faithful and primitive on S and G' = G. Fix s in S and set H = $Stab_G(s)$. Suppose there is a solvable normal subgroup K of H such that G = < ∪{K^x : x ∈ G} >. Then G is simple.

3. Proof of A_8 is simple using Iwasawa Theorem

3.1. Claim: A_8 acts on S under the natural action of permutation faithfully, where S is a set of all 4 -sets on 8 symbols.

Proof: Let ψ: $A_8 \to Sym(S)$, be a map defined $\psi(\sigma) = \varphi_\sigma(x) = \sigma(x)$.

It suffices to show that homomorphism ψ has trivial kernal.

Subclaim 1) ψ is group homomorphism.

For $\sigma_1, \sigma_2 \in A_8$, consider $\psi(\sigma_1\, \sigma_2) = \varphi_{\sigma 1\, \sigma 2}\ (x)$, ∀ x ∈ S.

$= \sigma_1\, \sigma_2\, (x)$
$= \sigma_1\, (\sigma_2\, (x))$
$= \varphi_{\sigma 1}\, (\sigma_2\, (x))$
$= \varphi_{\sigma 1}\, (\varphi_{\sigma 2}\, (x))$
$= \varphi_{\sigma 1}\varphi_{\sigma 2}$
$= \psi(\sigma_1)\, \psi(\sigma_2)$

Thus ψ is a group homomorphism.

Subclaim 2) ψ has trivial kernal.

$= \ker(\psi) = \{\sigma \in A_8\, |\psi(\sigma) = \varphi_e\}$
$= \{\sigma \in A_8\, |\, \varphi_\sigma = \varphi_e\}$
$= \{\sigma \in A_8\, |\sigma(x) = e(x) = x, \forall x \in S\}$
$= \{I_{A8}\}$. . . { by property of σ}.

Thus ψ has trivial kernal.

Hence the claim.

3.2. Claim: A_8 acts transitive on S.

Proof: We need to show Orb_{A8} (p) = S, for some p ∈ S.

Without loss of generality, let p = {a, b, c, d}.

As Orb_{A8} (p) ⊆ S, so enough to show that S ⊆ Orb_{A8} (p).

We consider the following cases:

Case i: If all 4 symbols are common in p and 4- set.

Example: {c, a, b, d} ∈ S Choose g = e = identity in A_8.

Then g.{a, b, c, d} = {a, b, c, d} = {c, a, b, d}.

Thus {c, a, b, d} ∈ Orb_{A8} (p).

Case ii: If 3 symbols are common in p and 4- set.

Example: {a, b, c, z} ∈ S.

Choose g = (dz)(ab) ∈ A_8.

Then g.{a, b, c, d} = (dz)(ab).{a, b, c, d} = {a, b, c, z} .

Thus {a, b, c, z} ∈ Orb_{A8}(p).

Case iii: If 2 symbols are common in p and 4- set.

Example: {a, b, z, i} ∈ S.

Choose g = (cz)(di) ∈ A_8.

Then g.{a, b, c, d} = (cz)(di){a, b, c, d} = {a, b, z, i} .

Thus {a, b, z, i} ∈ Orb_{A8}(p)

Case iv: If 1 symbol is common in p and 4- set.

Example: {a, z, i, j} ∈ S. Choose g = (bzci)(dj) ∈ A_8.

Then g.{a, b, c, d} = (bzci)(dj){a, b, c, d} = {a, z, i, j} .

Thus {a, z, i, j} ∈ Orb_{A8} (p).

Case v: If no symbols are common in p and 4- set.
Example: {k, z, i, j} ∈ S. Choose g = (ak)(bz)(ci)(dj) ∈ A_8.
Then g.{a, b, c, d} = (ak)(bz)(ci)(dj){a, b, c, d} = {k, z, i, j}.
Thus {k, z, i, j} ∈ $Orb_{A8}(p)$

Hence from all cases we get S ⊆ $Orb_{A8}(p)$.
Therefore S = $Orb_{A8}(p)$, so we conclude that group action is transitive.

***3.3.* To find stabilizer of an element of S**.
Let p = {a, b, c, d} ∈ S, then to find H = $Stab_{A8}(p)$
H = $Stab_{A8}(p)$ has the following elements.

1) Identity element of A_8.
No. of such elements = 1

2) a) All 3-cycles (ijk), where i, j, k ∈ {i, j, k, l}.
No. of such elements= 8

b) All 3-cycles (abc), where a, b, c ∈ {a, b, c, d}.
No. of such elements= 8

3) Product of 3 cycles i.e., (ijk)(abc).
No. of such elements= 64

4) Product of transpositions.
a) (ij)(kl), where i, j, k, l ∈ {i, j, k, l} .
No. of such elements= 3

b) (ab)(cd), where a, b, c, d ∈ {a, b, c, d} .
No. of such elements= 3

c) (ab)(ij), where a, b ∈ {a, b, c, d} and i, j ∈ {i, j, k, l} .
No. of such elements= 36

5) a) Cycle of the form (ijk)(ab)(cd),where i, j, k ∈ {i, j, k, l}, where a, b, c, d ∈ {a, b, c, d}.
No. of such elements = 24

b) Cycle of the form (abc)(ij)(kl),where i, j, k, l ∈ {i, j, k, l} and a, b, c ∈ {a, b, c, d}.
No. of such elements = 24

6) a) Cycle of the form (ijkl)(ab),where i, j, k, l ∈ {i, j, k, l} and a, b ∈ {a, b, c, d}.
No. of such elements = 36

b) Cycle of the form (abcd)(ij),where i, j, k, l ∈ {i, j, k, l} and a, b, c, d ∈ {a, b, c, d}.
No. of such elements = 36

7) Cycle of the form (ijkl)(abcd),where i, j, k, l ∈ {i, j, k, l} and a, b, c, d ∈ {a, b, c, d}.
No. of such elements = 36

8) Cycle of the form (ij)(kl)(ab)(cd),where i, j, k, l ∈ {i, j, k, l} and a, b, c, d ∈ {a, b, c, d}.
No. of such elements = 9

Thus |H| = |StabA8 (p)| = 288 = $2^5\ 3^2$

3.4. Claim: H is maximal Subgroup of G.

Since $H \subset \langle \cup \{ H^x \mid x \in A_8 \} \rangle \subseteq G$, so enough to show that $\langle \cup \{ H^x \mid x \in A_8 \} \rangle = A_8$.
As $H^x \subseteq A_8$, hence $\langle \cup \{H^x \mid x \in A_8 \} \rangle \subseteq A_8$.
So we show that $A_8 \subseteq \langle \cup \{H^x \mid x \in A_8 \} \rangle$.
As A_8 is generated by 3− cycles so we show that all 3- cycles of A_8 belongs to $\langle \cup \{ H^x \mid x \in A8 \} \rangle$.

a) 3 - cycle with no symbol from {a, b, c, d}.
Example: $(ijk) \in A_8$, choose $x = e$, where e is the identity in A_8 and $h = (ijk) \in H$.
Then $x^{-1} hx = e^{-1} (ijk)e = (ijk) \in \langle \cup \{ H^x \mid x \in A_8 \} \rangle$.

b) 3 - cycle with one symbol from {a, b, c, d}.
Example: $(aij) \in A_8$, choose $x = (ib)(jc)$, and $h = (abc) \in H$.
Then $x^{-1} hx = ((ib)(jc))^{-1} (abc)(ib)(jc) = (aij) \in \langle \cup \{ H^x \mid x \in A_8\} \rangle$.

c) 3 - cycle with two symbols from {a, b, c, d}.
Example: $(abi) \in A_8$, choose $x = (jk)(ic)$, and $h = (abc) \in H$.
Then $x^{-1} hx = ((jk)(ic))^{-1} (abc)(jk)(ic) = (abi) \in \langle \cup \{ H^x \mid x \in A_8\} \rangle$.

d) 3 - cycle with three symbols from {a, b, c, d}.
Example: $(abc) \in A_8$, choose $x = e$, and $h = (abc) \in H$.
Then $x^{-1} hx = (e)^{-1} (abc)(e) = (abc) \in \langle \cup \{ H^x \mid x \in A_8 \} \rangle$.

Thus from all the cases we conclude that $A_8 \subseteq \langle \cup \{ H^x \mid x \in A_8\}\rangle$.
Hence $A_8 = \langle \cup \{ H^x \mid x \in A_8 \} \rangle$.
Therefore we conclude that H is maximal subgroup of G. Hence by Theorem 1.1 group action is primitive on S.

3.5. Claim: $A_8 = A_8$'

We know that $A_8' \subseteq A_8$ . . . {since every derived subgroup is subset of the group}.
It is enough to prove that $A_8 \subseteq A_8'$.
Since A_8 is generated by 3-cycles, therefore it is enough to show that every 3 -cycle in A_8 is contained in A_8'.
Let (ijk) be arbitrary 3-cycle in A_8.
Consider $\sigma = (ilk)$ and $\tau = (kjm)$.
Then $\sigma\tau \sigma^{-1} \tau^{-1} = (ilk)(kjm)(ilk)^{-1} (kjm)^{-1} = (ilk)(kjm)(ikl)(kmj) = (ijk)$.
Hence $(ijk) \in A_8'$.
Since (ijk) was arbitrary element of A_8, Therefore $A_8 \subseteq A_8'$
Thus $A_8 = A_8'$.

3.6. Claim: H is solvable

$|H| = |Stab_{A8} (p)| = 288 = 2^5\, 3^2 = p^a q^b$, hence by Burnside's theorem H is solvable.

3.7. Let K = H.

Since every subgroup is normal to itself, hence K is normal in H.
From Claim 6) K is solvable.
From the steps followed in Claim 4) we conclude that $G = A_8 = \langle \cup \{K^x \mid x \in A_8\} \rangle$.
Hence by Iwasawa theorem we conclude that A_8 is simple.

4. Conclusion

We conclude that alternating group A_8 is simple. Similarly other alternating subgroup A_n, where $n \geq 5$ can also be proved simple using Iwasawa theorem**.**

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