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Original Article

Comparative Analysis of Heterogeneous Feedback Queue Model in Fuzzy Environment using L-R method and α-cut method

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Abstract - *The present paper is an attempt to discuss the behavior of a heterogeneous queue model in fuzzy environment. The objective of the paper is to discuss the various queue characteristics in fuzzy environment by using L-R method and α-cut method. The model consists of one server which is commonly connected to two heterogeneous servers. A numerical example is illustrated to check the validity and difference between all queue characteristics by applying both the techniques.*

Keywords - *Queuing, Feedback, Heterogeneous server, Fuzzy Environment, α-cut, L-R Method.*

1. Introduction

As we know that queues in fuzzy environment are more real than crisp queues, so many researchers developed queue models in fuzzy environment by using different approaches and different types of fuzzy numbers. It was Lie & Lie who introduced fuzzy queue models in 1989.Ritha W., Vinnarsi S.(2017) and Mukeba J.B.(2016) proposed L-R method for fuzzy numbers. Singh T.P., Kususm (2012), Singh T.P., Arti(2014), Mittal M.(2015) discussed fuzzy queue networks by using α-cut approach. Gupta D., Saini V.(2022) analyze bi-tandem queue network in fuzzy environment by using α-cut approach. Gupta D., Saini V., Tripathi A.K.(2022) analyze heterogeneous feedback queue model in fuzzy environment by using L-R method.

Gupta D., Saini V. (2021) discussed a heterogeneous feedback queue system with feedback in stochastic environment. Present paper is an extension. In this paper we try to analyze the behavior of queue characteristics in a heterogeneous queue system in fuzzy environment by using two different approaches which are α-cut approach and L-R approach.

2. Definitions

Below are some definitions as per the knowledge from Mukeba J.P. et al (2015**)**

2.1. Fuzzy Number

Let $\tilde{M}$ be a fuzzy set in the universe U, then $\tilde{M}$ is said to be a fuzzy number if and only if U is equal to R, $\tilde{M}$ is normal and convex, membership function $n_{\tilde{M}}$ of $\tilde{M}$ is piecewise continuous and also there exists one and only one x $\in$ R such that $n_{\tilde{M}}(x) = 1$

2.2. Triangular Fuzzy Number

Let $\tilde{M}$ be a fuzzy number then $\tilde{M}$ is triangular fuzzy if and only if there exists numbers $a_1 < b_1 < c_1$ such that

$$n_{\tilde{M}}(x_1) = \begin{cases} \dfrac{x_1 - a_1}{b_1 - a_1} \; for(a_1 \le x_1 \le b_1) \\ \dfrac{c_1 - x_1}{c_1 - b_1} \; for(b_1 \le x_1 \le c_1) \\ 0(otherwise) \end{cases}$$

2.3. Fuzzy Arithmetic Operations

Let $\tilde{M} = (a_1, b_1, c_1)$ and $\tilde{N} = (a_2, b_2, c_2)$ be two triangular fuzzy numbers, then the basic arithmetic operations on these triangular fuzzy numbers are

(i) Sum = $\tilde{M} + \tilde{N} = (a_1 + a_2, b_1 + b_2, c_1 + c_2)$

(ii) Difference $= \tilde{M} - \tilde{N} = (a_1 - a_2, b_1 - b_2, c_1 - c_2)$ if DP($\tilde{M}$) ≥ DP($\tilde{N}$), where $\text{DP}(\tilde{\text{M}}) = \frac{c_1 - a_1}{2}$ and $DP(\tilde{N}) = \frac{c_2 - a_2}{2}$, DP is the difference point of a triangular fuzzy numbers, otherwise $\tilde{A} - \tilde{B} = (a_1 - c_2, b_1 - b_2, c_1 - a_2)$

(iii) Multiplication = $\tilde{M} \times \tilde{N} = (a_1 b_2 + b_1 a_2 - b_1 b_2, b_1 b_2, c_1 b_2 + b_1 c_2 - b_1 b_2)$

(iv) Division = $\tilde{M} / \tilde{N} = (\frac{2a_1}{a_2 + c_2}, \frac{b_1}{b_2}, \frac{2c_1}{a_2 + c_2})$

2.4. Defuzzification of Triangular Fuzzy Numbers

Let $\tilde{M} = (a_1, b_1, c_1)$ be the triangular fuzzy number. Then by Yager's [1981] formula, crisp $M = \left(\frac{a_1 + 2b_1 + c_1}{4} \right)$

2.5. L-R Fuzzy Number

A fuzzy number $\tilde{M}$ is said to be L-R fuzzy if and only if there exists real numbers m_1 , a_1>0, a_2>0 and two functions L and R, from R →[0,1], both of which are positive, continuous and decreasing, such as L(0) = R(0) =1

L(1)=0, L(x_1)>0, $\lim_{x_1 \to \infty} L(x_1) = 0$

R(1)=0, R(x_1)>0, $\lim_{x_1 \to \infty} R(x_1) = 0$

$$n_{\tilde{M}}(x) = \begin{cases} L\left(\frac{m_1 - x_1}{a_1} \right) for(x_1 \in [m_1 - a_1, m_1]) \\ R\left(\frac{x_1 - m_1}{a_2} \right) for(x_1 \in [m_1, m_1 + a_2]) \\ 0(otherwise) \end{cases}$$

If $\tilde{M}$ is a L-R fuzzy number, then it can be denoted as $\tilde{M}$ = (m_1, a_1, b_1) LR , which is L-R representation of $\tilde{M}$ where m_1 is the modal value, a_1 is left spread and b_1 is right spread of $\tilde{M}$, and support of $\tilde{M}$ = Supp($\tilde{M}$)= ($m_1 - a_1, m_1 - b_1$)

2.6. L-R Fuzzy Arithmetic

Let $\tilde{M}$ = **(** m_1, a_1, b_1 **) LR** , $\tilde{N}$ = **(** n_1, c_1, d_1 **) LR** be two L-R fuzzy numbers then we can define arithmetic operations such as, addition, subtraction, multiplication and division as,

$\tilde{M} + \tilde{N}$ = **(** $m_1 + n_1, a_1 + c_1, b_1 + d_1$ **) LR**

$\tilde{M} - \tilde{N}$ = **(** $m_1 - n_1, a_1 + d_1, b_1 + c_1$ **) LR**

$$\tilde{M}..\tilde{N} = (m_1 n_1, m_1 c_1 + n_1 a_1 - a_1 c_1,, m_1 d_1 + n_1 b_1 + b_1 d_1)_{LR}$$

$$\tilde{M} / \tilde{N} =$$

$$(m_1/n_1, m_1 d_1/n_1(n_1 + d_1) + a_1/n_1 - a_1 d_1/n_1(n_1 + d_1), m_1 c_1/n_1(n_1 - c_1) + b_1/n_1 - b_1 c_1/n_1(n_1 - c_1))_{LR}$$

2.7. Alpha Cuts of a Fuzzy Set

If M is a fuzzy set in X, and α is any real number such that α ∈ [0,1], then the α-cut of fuzzy set M is, $\alpha_M = \{x \in X : \mu_M(x) \geq \alpha\}$, it is also known as weak α-cut.

The strong α-cut of fuzzy set M is, $\alpha +_M = \{x \in X : \mu_M(x) > \alpha\}$

3. Model Description

The model consists three service channels S_1, S_2, S_3 .Service channels S_2 and S_3 are heterogeneous which are commonly connected to service channel S_1 .Initially the customer arrives at service channel S_1 and after being served the customer may move to either S_2 or S_3 and if customer is unsatisfied then he can revisit the server maximum one time.

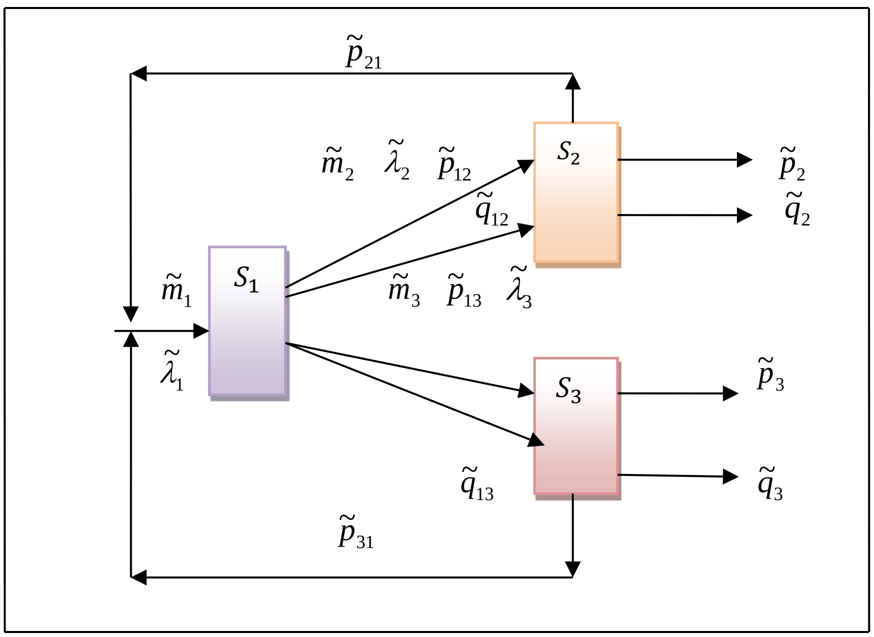

Fig. 1 Fuzzy heterogeneous feedback queue model

4. Notations

Following are some notations used throughout whole work

$\tilde{m}_i$ = fuzzy number of arriving customers, where i=1,2,3

$\tilde{\lambda}$ = fuzzy arrival rate

$\tilde{\mu}$ = fuzzy service rate

$\tilde{a}, \tilde{b}, \tilde{c}$ = fuzzy probabilities of leaving the servers at first time

$\tilde{a}', \tilde{b}', \tilde{c}'$ = fuzzy probabilities of leaving the servers at second time

$\tilde{p}_{ij}$ = fuzzy probability of moving first time from one server to another in the state (i.j)

$\tilde{q}_{ij}$ = fuzzy probability of moving first time from one server to another in the state (i.j)

$\tilde{L}_i$ = fuzzy partial queue length of the server, where i=1,2,3

$\tilde{L}$ =fuzzy average queue length of the system

5. Solution of the Mathematical model

From the work done by Saini V., Gupta D (2021) utilization factors for all the three servers of present model in stochastic environment are as follows,

$$\rho_1 = \frac{(\lambda_1 + \lambda_2 b p_{21} + \lambda_3 c p_{31})}{\mu_1 \{1 - (a p_{12} + a' q_{12}) b p_{21} - (a p_{13} + a' q_{13}) c p_{31}\}}$$

$$\rho_2 = \frac{\lambda_1 (a p_{12} + a' q_{12}) + \lambda_2 \{1 - (a p_{13} + a' q_{13}) c p_{31}\} + \lambda_3 c p_{31} (a p_{12} + a' q_{12})}{\mu_2 \{1 - (a p_{12} + a' q_{12}) b p_{21} - (a p_{13} + a' q_{13}) c p_{31}\}}$$

$$\rho_3 = \frac{\lambda_1 (a p_{13} + a' q_{13}) + \lambda_2 \{(a p_{13} + a' q_{13}) b p_{21}\} + \lambda_3 \{1 - (a p_{12} + a' q_{12})\} b p_{21}}{\mu_3 \{1 - (a p_{12} + a' q_{12}) b p_{21} - (a p_{13} + a' q_{13}) c p_{31}\}}$$

With the condition that the solution for the model exists if, $\rho_1, \rho_2, \rho_3 < 1$

In fuzzy environment, since all the parameters will be fuzzy so, the utilization factors will be as follows,

$$\tilde{\rho}_1 = \frac{(\tilde{\lambda}_1 + \tilde{\lambda}_2 \tilde{b} \tilde{p}_{21} + \tilde{\lambda}_3 \tilde{c} \tilde{p}_{31})}{\tilde{\mu}_1 \{1 - (\tilde{a} \tilde{p}_{12} + \tilde{a}' \tilde{q}_{12}) \tilde{b} \tilde{p}_{21} - (\tilde{a} \tilde{p}_{13} + \tilde{a}' \tilde{q}_{13}) \tilde{c} \tilde{p}_{31}\}}$$

$$\tilde{\rho}_2 = \frac{\tilde{\lambda}_1 (\tilde{a} \tilde{p}_{12} + \tilde{a}' \tilde{q}_{12}) + \tilde{\lambda}_2 \{1 - (\tilde{a} \tilde{p}_{13} + \tilde{a}' \tilde{q}_{13}) \tilde{c} \tilde{p}_{31}\} + \tilde{\lambda}_3 \tilde{c} \tilde{p}_{31} (\tilde{a} \tilde{p}_{12} + \tilde{a}' \tilde{q}_{12})}{\tilde{\mu}_2 \{1 - (\tilde{a} \tilde{p}_{12} + \tilde{a}' \tilde{q}_{12}) \tilde{b} \tilde{p}_{21} - (\tilde{a} \tilde{p}_{13} + \tilde{a}' \tilde{q}_{13}) \tilde{c} \tilde{p}_{31}\}}$$

$$\tilde{\rho}_3 = \frac{\tilde{\lambda}_1 (\tilde{a} \tilde{p}_{13} + \tilde{a}' \tilde{q}_{13}) + \tilde{\lambda}_2 \{(\tilde{a} \tilde{p}_{13} + \tilde{a}' \tilde{q}_{13}) \tilde{b} \tilde{p}_{21}\} + \tilde{\lambda}_3 \{1 - (\tilde{a} \tilde{p}_{12} + \tilde{a}' \tilde{q}_{12})\} \tilde{b} \tilde{p}_{21}}{\tilde{\mu}_3 \{1 - (\tilde{a} \tilde{p}_{12} + \tilde{a}' \tilde{q}_{12}) \tilde{b} \tilde{p}_{21} - (\tilde{a} \tilde{p}_{13} + \tilde{a}' \tilde{q}_{13}) \tilde{c} \tilde{p}_{31}\}}$$

6. Evaluation of queue characteristics by L-R method and α-cut method

In this section we will apply L-R method and α-cut method for fuzzy numbers to evaluate utilization factors and queue lengths for all the three servers.

6.1. Evaluation using L-R method

We take value of each fuzzy parameter as triangular fuzzy numbers in table 1, satisfying the following conditions

$$\tilde{p}_{12} + \tilde{p}_{13} = 1, \tilde{q}_{12} + \tilde{q}_{13} = 1$$

$$\tilde{p}_2 + \tilde{p}_{21} = 1, \tilde{p}_3 + \tilde{p}_{31} = 1$$

$$\tilde{a} \tilde{p}_{12} + \tilde{a} \tilde{p}_{13} + \tilde{a}' \tilde{q}_{12} + \tilde{a}' \tilde{q}_{13} = 1$$

$$\tilde{b} \tilde{p}_2 + \tilde{b} \tilde{p}_{21} + \tilde{b}' \tilde{q}_2 = 1$$

$$\tilde{c} \tilde{p}_3 + \tilde{c} \tilde{p}_{31} + \tilde{c}' \tilde{q}_3 = 1$$

Table 1. Numerical values

No. of customers	Arrival Rate	Service Rate	Probabilities	
$\tilde{m}_1$ =(11,12,13)	$\tilde{\lambda}_1$ =(1,2,3)	$\tilde{\mu}_1$ =(10,11,12)	$\tilde{p}_{12}$ =(.3,.4,.5)	$\tilde{a}$ = (.2,.4,.6)
$\tilde{m}_2$ =(7,8,9)	$\tilde{\lambda}_2$ =(2,3,4)	$\tilde{\mu}_2$ =(11,12,13)	$\tilde{p}_{13}$ = (.7,.6,.5)	$\tilde{a}'$ = (.8,.6,.4)
$\tilde{m}_3$ =(6,7,8)	$\tilde{\lambda}_3$ =(1,2,3)	$\tilde{\mu}_3$ =(12,13,14)	$\tilde{p}_{21}$ =(.3,.2,.1)	$\tilde{b}$ = (.8,.7,.6)
			$\tilde{p}_{31}$ =(.3,.2,.1)	$\tilde{b}'$ = (.2,.3,.4)
			$\tilde{q}_{12}$ =(.2,.3,.4)	$\tilde{c}$ = (.9,.8,.7)
			$\tilde{q}_{13}$ =(.8,.7,.6)	$\tilde{c}'$ = (.1,.2,.3)

Table 2. L-R representation of triangular fuzzy parameters

Arrival Rate	Service Rate	Probabilities	
$\tilde{\lambda}_1$ =(2,1,1)$_{LR}$	$\tilde{\mu}_1$ =(11,1,1)$_{LR}$	$\tilde{p}_{12}$ = (.4,.1,.1)$_{LR}$	$\tilde{a}$ = (.4,.2,.2)$_{LR}$
$\tilde{\lambda}_2$ =(3,1,1)$_{LR}$	$\tilde{\mu}_2$ =(12,1,1)$_{LR}$	$\tilde{p}_{13}$ = (.6,.1,.1)$_{LR}$	$\tilde{a}'$ = (.6,.2,.2)$_{LR}$
$\tilde{\lambda}_3$ =(2,1,1)$_{LR}$	$\tilde{\mu}_3$ =(13,1,1)$_{LR}$	$\tilde{p}_{21}$ =(.2,.1,.1)$_{LR}$	$\tilde{b}$ = (.7,.1,.1)$_{LR}$
		$\tilde{p}_{31}$ =(.2,.1,.1)$_{LR}$	$\tilde{b}'$ = (.3,.1,.1)$_{LR}$
		$\tilde{q}_{12}$ =(.3,.1,.1)$_{LR}$	$\tilde{c}$ = (.8,.1,.1)$_{LR}$
		$\tilde{q}_{13}$ =(.7,.1,.1)$_{LR}$	$\tilde{c}'$ = (.2,.1,.1)$_{LR}$

Using values from above table 2, we find the values of utilization factors for all the three servers $\tilde{\rho}_1, \tilde{\rho}_2, \tilde{\rho}_3$ as given below:

$\tilde{\rho}_1$ = (0.2942, 0.2062, 0.3630)$_{LR}$

$\tilde{\rho}_2$ = (0.3417, 0.2207, 0.4927)$_{LR}$

$\tilde{\rho}_3$ = (0.3181, 0.2363, 0.5199)$_{LR}$

Modal values of utilization factors $\tilde{\rho}_1, \tilde{\rho}_2, \tilde{\rho}_3$ are as follows

$\tilde{\rho}_1$ = 0.2942

$\tilde{\rho}_2$ = 0.3417

$\tilde{\rho}_3$ = 0.3181

Also,

Supp ($\tilde{\rho}_1$) = (0.2942-0.2062, 0.2942+ 0.3630) = (0.088, 0.6572)

Supp ($\tilde{\rho}_2$) = (0.3417-0.2207, 0.3417+0.4927) = (0.121, 0.8344)

Supp ($\tilde{\rho}_3$) = (0.3181-0.2363, 0.3181+0.5199) = (0.0818, 0.838)

Now the values of queue lengths $\tilde{L}_1, \tilde{L}_2, \tilde{L}_3$ for all the three servers are,

$$\tilde{L}_1 = \frac{\tilde{\rho}_1}{1-\tilde{\rho}_1} = (0.4168, 0.3346, 0.8986)_{LR}$$

$$\tilde{L}_2 = \frac{\tilde{\rho}_2}{1-\tilde{\rho}_2} = (0.5191, 0.414, 1.3877)_{LR}$$

$$\tilde{L}_3 = \frac{\tilde{\rho}_3}{1-\tilde{\rho}_3} = (0.4666, 0.3984, 1.4141)_{LR}$$

Modal values of queue lengths for servers S_1, S_2, S_3 are,

$\tilde{L}_1$ = 0.4168

$\tilde{L}_2$ = 0.5191

$\tilde{L}_3$ = 0.4666

Supp ($\tilde{L}_1$) = (0.4168-0.3346, 0.4168+ 0.8986) = (0.0822, 1.3154)

Supp ($\tilde{L}_2$) = (0.5191-0.4140, 0.5191+1.3877) = (0.1051, 1.9068)

Supp ($\tilde{L}_3$) = (0.4666-0.3984, 0.4666+1.4141) = (0.0682, 1.8807)

6.2. Evaluation using α-cut method

From knowledge of α-cut approach given by Sharma S. et al(2015), $\tilde{\lambda}_1$, $\tilde{\lambda}_2$, $\tilde{\lambda}_3$, $\tilde{\mu}_1$, $\tilde{\mu}_2$, $\tilde{\mu}_3$, $\tilde{a}$, $\tilde{a}'$, $\tilde{b}$, $\tilde{b}'$, $\tilde{c}$, $\tilde{c}'$, $\tilde{p}_{12}$, $\tilde{p}_{13}$, $\tilde{p}_{21}$, $\tilde{p}_{31}$, $\tilde{q}_{12}$, $\tilde{q}_{13}$ can be defined as follows

$\tilde{\lambda}_i = (\lambda_i^1, \lambda_i^2, \lambda_i^3)$, $\tilde{\mu}_i = (\mu_i^1, \mu_i^2, \mu_i^3)$, $\tilde{p}_{ij} = (p_{ij}^1, p_{ij}^2, p_{ij}^3)$, $\tilde{a} = (a^1, a^2, a^3)$, $\tilde{b} = (b^1, b^2, b^3)$,

$\tilde{c} = (c^1, c^2, c^3)$, $\tilde{a}' = (a'^1, a'^2, a'^3)$, $\tilde{b}' = (b'^1, b'^2, b'^3)$, $\tilde{c}' = (c'^1, c'^2, c'^3)$ for all values of i and j.

Now fuzzy utilization factors for various servers can be defined by,

$\tilde{\rho}_1 = (\rho_1^1, \rho_1^2, \rho_1^3)$

Where,

$$\rho_1^1 = \frac{\lambda_1^1 + \lambda_2^1 b^1 p_{21}^1 + \lambda_3^1 c^1 p_{31}^1}{\mu_1^3\left[\left\{1 - (a^3 p_{12}^3 + a'^3 q_{12}^3)(b^3 p_{21}^3)\right\} - \left\{(a^3 p_{13}^3 + a'^3 q_{13}^3)(c^3 p_{31}^3)\right\}\right]}$$

$$\rho_1^2 = \frac{\lambda_1^2 + \lambda_2^2 b^2 p_{21}^2 + \lambda_3^2 c^2 p_{31}^2}{\mu_1^2\left[\left\{1 - (a^2 p_{12}^2 + a'^2 q_{12}^2)(b^2 p_{21}^2)\right\} - \left\{(a^2 p_{13}^2 + a'^2 q_{13}^2)(c^2 p_{31}^2)\right\}\right]}$$

$$\rho_1^3 = \frac{\lambda_1^3 + \lambda_2^3 b^3 p_{21}^3 + \lambda_3^3 c^3 p_{31}^3}{\mu_1^1\left[\left\{1 - (a^1 p_{12}^1 + a'^1 q_{12}^1)(b^1 p_{21}^1)\right\} - \left\{(a^1 p_{13}^1 + a'^1 q_{13}^1)(c^1 p_{31}^1)\right\}\right]}$$

$\tilde{\rho}_2 = (\rho_2^1, \rho_2^2, \rho_2^3)$

Where,

$$\rho_2^1 = \frac{\lambda_1^1 (a^1 p_{12}^1 + a'^1 q_{12}^1) + \lambda_2^1 \left\{1 - \left((a^1 p_{13}^1 + a'^1 q_{13}^1)\right) c^1 p_{31}^1\right\} + \lambda_3^1 \left(a^1 p_{12}^1 + a'^1 q_{12}^1\right) c^1 p_{31}^1}{\mu_2^3\left[\left\{1 - (a^3 p_{12}^3 + a'^3 q_{12}^3)(b^3 p_{21}^3)\right\} - \left\{(a^3 p_{13}^3 + a'^3 q_{13}^3)(c^3 p_{31}^3)\right\}\right]}$$

$$\rho_2^2 = \frac{\lambda_1^2 (a^2 p_{12}^2 + a'^2 q_{12}^2) + \lambda_2^2 \left\{1 - \left((a^2 p_{13}^2 + a'^2 q_{13}^2)\right) c^2 p_{31}^2\right\} + \lambda_3^2 \left(a^2 p_{12}^1 + a'^2 q_{12}^2\right) c^2 p_{31}^2}{\mu_2^2\left[\left\{1 - (a^2 p_{12}^2 + a'^2 q_{12}^2)(b^2 p_{21}^2)\right\} - \left\{(a^2 p_{13}^2 + a'^2 q_{13}^2)(c^2 p_{31}^2)\right\}\right]}$$

$$\rho_2^{31} = \frac{\lambda_1^3 (a^3 p_{12}^3 + a'^3 q_{12}^3) + \lambda_2^3 \left\{1 - \left((a^3 p_{13}^3 + a'^3 q_{13}^3)\right) c^3 p_{31}^3\right\} + \lambda_3^3 \left(a^3 p_{12}^3 + a'^3 q_{12}^3\right) c^3 p_{31}^3}{\mu_2^1\left[\left\{1 - (a^1 p_{12}^1 + a'^1 q_{12}^1)(b^1 p_{21}^1)\right\} - \left\{(a^1 p_{13}^1 + a'^1 q_{13}^1)(c^1 p_{31}^1)\right\}\right]}$$

$\tilde{\rho}_3 = (\rho_3^1, \rho_3^2, \rho_3^3)$

$$\rho_3^1 = \frac{\lambda_1^1 (a^1 p_{13}^1 + a'^1 q_{13}^1) + \lambda_2^1 \left\{\left((a^1 p_{13}^1 + a'^1 q_{13}^1)\right) b^1 p_{21}^1\right\} + \lambda_3^1 \left\{1 - \left(a^1 p_{12}^1 + a'^1 q_{12}^1\right) b^1 p_{21}^1\right\}}{\mu_3^3\left[\left\{1 - (a^3 p_{12}^3 + a'^3 q_{12}^3)(b^3 p_{21}^3)\right\} - \left\{(a^3 p_{13}^3 + a'^3 q_{13}^3)(c^3 p_{31}^3)\right\}\right]}$$

$$\rho_3^2 = \frac{\lambda_1^2(a^2 p_{13}^2 + a'^2 q_{13}^2) + \lambda_2^2\left\{\left((a^2 p_{13}^2 + a'^2 q_{13}^2)\right)b^2 p_{21}^2\right\} + \lambda_3^2\left\{1 - \left(a^2 p_{12}^2 + a'^2 q_{12}^2\right)b^2 p_{21}^2\right\}}{\mu_3^2\left[\left\{1 - (a^2 p_{12}^2 + a'^2 q_{12}^2)(b^2 p_{21}^2)\right\} - \left\{(a^2 p_{13}^2 + a'^2 q_{13}^2)(c^2 p_{31}^2)\right\}\right]}$$

$$\rho_3^3 = \frac{\lambda_1^3(a^3 p_{13}^3 + a'^3 q_{13}^3) + \lambda_2^3\left\{\left((a^3 p_{13}^3 + a'^3 q_{13}^3)\right)b^3 p_{21}^3\right\} + \lambda_3^3\left\{1 - \left(a^3 p_{12}^3 + a'^3 q_{12}^3\right)b^3 p_{21}^3\right\}}{\mu_3^1\left[\left\{1 - (a^1 p_{12}^1 + a'^1 q_{12}^1)(b^1 p_{21}^1)\right\} - \left\{(a^1 p_{13}^1 + a'^1 q_{13}^1)(c^1 p_{31}^1)\right\}\right]}$$

Now using values from table 1,

$\rho_1^1 = 0.1560$, $\rho_1^2 = 0.2696$, $\rho_1^3 = 0.4684$, $\rho_2^1 = 0.1539$, $\rho_2^2 = 0.3625$, $\rho_2^3 = 0.6572$, $\rho_3^1 = 0.1606$, $\rho_3^2 = 0.2963$, $\rho_3^3 = 0.5280$

$\tilde{\rho}_1 = (\rho_1^1, \rho_1^2, \rho_1^3) = (0.1560, 0.2696, 0.4684)$

$\tilde{\rho}_2 = (\rho_2^1, \rho_2^2, \rho_2^3) = (0.1529, 0.3417, 0.6592)$

$\tilde{\rho}_3 = (\rho_3^1, \rho_3^2, \rho_3^3) = (0.1606, 0.2963, 0.5280)$

$$\tilde{L}_1 = \frac{\tilde{\rho}_1}{1 - \tilde{\rho}_1} = (0.2268, 0.3691, 0.6810)$$

$$\tilde{L}_2 = \frac{\tilde{\rho}_2}{1 - \tilde{\rho}_2} = (0.2570, 0.5507, 1.1046)$$

$$\tilde{L}_3 = \frac{\tilde{\rho}_3}{1 - \tilde{\rho}_3} = (0.2449, 0.4211, 0.8252)$$

Most possible values of utilization factors and partial queue lengths for three servers are,

$\tilde{\rho}_1 = 0.2696$, $\tilde{\rho}_2 = 0.3417$, $\tilde{\rho}_3 = 0.2963$

$\tilde{L}_1 = 0.3691$, $\tilde{L}_2$ 0.5507, $\tilde{L}_3 = 0.4211$

7. Results

- According to subsection 4.1 modal values for utilization factors are $\tilde{\rho}_1 = 0.2942$, $\tilde{\rho}_2 = 0.3417$, $\tilde{\rho}_3 = 0.3181$ and queue lengths for the servers are, $\tilde{L}_1 = 0.4168$, $\tilde{L}_2 = 0.5191$, $\tilde{L}_3 = 0.4666$.
- According to subsection 4.2 most possible values for utilization factors and queue lengths for the servers are, $\tilde{\rho}_1 = 0.2696$, $\widetilde{\rho_2}$ $\tilde{\rho}_2 = 0.3417$, $\tilde{\rho}_3 = 0.2963$, $\tilde{L}_1 = 0.3691$, $\tilde{L}_2 = 0.5507$, $\tilde{L}_3 = 0.4211$
- As a comparison of both the methods, we see that the values of utilization factors and queue lengths are higher in case of L-R method except the values for 2nd server.

8. Conclusion

In the present work, we used α-cut method and L-R method to find queue characteristics for a heterogeneous queue model in fuzzy environment. As such we cannot conclude which method is better than the other. We can only conclude that values of various queue characteristics obtained by L-R method is higher than the values obtained by α-cut method. Also while calculating the various queue characteristics we find that L-R method is simpler than α-cut method. A more accurate comparison can be done by applying these two methods for finding queue characteristics of complex queue networks with parallel and bi-serial servers in fuzzy environment.

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