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Original Article

Solutions of Fuzzy Fractional Abel Differential Equations using Residual Power Series Method

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Abstract *- This paper describes the computed approach of such fuzzy fractional Abel differential equation (FFADE) according to a specific make using the extended power series (PS) formula in case of nonlinear. The methodology is based on applying representational processing to create a fractional power series solution in the form of a residual power series (RPS) with the smallest number of computations possible. The suggested approach is consistent with the initial problem difficulty, and the results are promising. The effective computational examples offered to ensure the method and explain the numerical expressions of the analytic solution to its potency, flexibility, and efficiency towards answering similar fractional equations. To demonstrate answer, visual and numerical data were provided and statistically evaluated.*

Keywords *- Abel initial value problems, Fuzzy, Fractional differential equations, Strongly generalized differentiability, Residual power series.*

1. Introduction

The purpose of the article is to improve the use of the RPS approach to create and identify several fractional PS solutions of the fractional in the Caputo concept. The fundamental benefit of this approach is its efficiency in finding the equations of elements of the mathematical formulation by using simply differential operators, as opposed to certain other well-known methodologies that need integral operators, which is challenging in the fractional situation. Furthermore, the suggested approach may be employed in frame domain adaptation and can even be utilized without regard to the nature of the equation or the kind of categorization. The reader is invite to read for further information on the RPS approach [1, 2, 3, 5, 4]. Systems of fuzzy fractional differential equations are founded in mathematical modeling, astrophysics, technology, biotechnology, computational analysis of life events uncertain or ambiguous, etc. Because it is frequently hard to obtain restricted solutions to solutions of fuzzy fractional differential equations experienced in reality, various writers have investigated these issues with value utilizing mathematical approaches. As a result, a category of differential equation systems has a prominent position in the mathematical modeling literature[6, 7, 8, 9].

Some compared with the standard are represented mathematically by fractional ODEs. Indeed, this will contribute to a better understanding of these real-world systems, decrease the skill required and facilitate a control scheme sans sacrificing some characteristics. Fractional ODEs have received a lot of interest since they are frequently adopted to represent different included flow fields, data processing, operations research, statistical inference, probability concept, a chance for success, and economics. Although analytical and mathematical approaches are significant in fractional differential equation areas, fractional should be solved. In most circumstances, the fractional is obtained analytically and the result is given in a linear system, in which the solution of such an equation is always required owing to interests. As a result, effective and dependable computer simulation is essential. In more realistic settings, the fractional is typically approximated using numerical approaches. In any case, several authors have addressed the approximate solution to the fractional using well-known methods [10, 11, 12, 13, 14].

The aim of this research is always to improve the need for the RPS method to evaluate the solution of fuzzy FADEs in the power series but include appropriate controls under strongly generalized differentiation. We consider the following nonlinear fuzzy FADE:

$$D^{\alpha}\tilde{g}(t) = A\tilde{g}^{3}(t) + B\tilde{g}^{2}(t) + E\tilde{g}(t) + F, 0 < \beta \leq 1, 0 \leq t \leq R. \tag{1.1}$$

with the fuzzy initial condition

$$\tilde{g}(0) = \tilde{g}_0, \tag{1.2}$$

where $\tilde{g}^3(t) \neq 0$, A, B, E and F $\in \Re$, $\tilde{g}_0$ is arbitrary fuzzy number, D^α is the caputo fractional derivative for order α and $\tilde{g}(t)$ is unknown fuzzy function of the crisp variable t. However, assume IVPs (1.1) and (1.2) each $t > 0$ has a unique fuzzy solution. Doing R_F is used to refer to a set of all fuzzy numbers defined in R. The RPS calculation is a novel numeric plan created to examine and decipher the arrangement of first and second request dubious IVPs. This strategy used to provide power series and fractional power series solutions to a few problems that arise in the field of design and science. The proposed approach targets constructing an answer of a power series development just as limiting remaining blunder capacities for processing the obscure coefficients of power series by applying a specific differential administrator without linearly or constraint on the structure [1, 19, 25-28]. Again, we refer to [15, 16, 24, 21] to see numerous qualities to show and reconsider some radical strategies for managing the various problems that occur in ordinary miracles.

The following is how this article is arranged. In next section provides definitions and theorems for Caputo's fractional derivative operator and residual power series. Section 3 presents the major theoretical conclusion, which is a formulation of the fuzzy fractional Abel differential equation. The basic methodology discussed in Section 4, where the residual power series approach for the fuzzy fractional Abel differential equations is used to demonstrate the proposed procedure's high performance and dependability. The paper concludes with some numerical examples and a conclusion.

2. Preliminaries

The importance definitions and related properties of the hypothesis of fuzzy calculus are reviews in this part. As a rule, a fuzzy number u is a fuzzy subset of R with normal, closed, convex, curved, and upper semi-continuous membership function of bounded support.

Definition 2.1[20] Let the membership function $u: S \to [0,1]$. Where S is characterized nonempty set, $u(s)$ is the degree of membership of set. A fuzzy set u is called convex if $u, s, t \in \Re$ and $\lambda \in [0,1]$,$u(\lambda s + (1-\lambda)t) \geq \min\{u(s), u(t)\}$ is called upper semi-continuous. If for each $r \in [0,1]$,$\{s \in \Re | u(s) \geq r\}$ is closed set, if $\{s \in \Re | u(s) = 1\}$ is normal set, if $\{s \in \Re | u(s) > 0\}$ is support of a fuzzy set.

Definition 2.2 [20] Let u is a fuzzy number iff $[u]^r$ is compact convex subset of $\Re$ for $r \in [0,1]$ and $[u]^1 \neq \phi$. If u is a fuzzy number, then $[u]^r = [u_1(r), u_2(r)]$, for each $s \in [u]^r$, $r \in [0,1]$, where $u_1(r)$ =min{s} , $u_2(r)$ =max{s} and $[u]^r$ is called $r-$cut representation form.

Theorem 2.3[20] Let $u_1, u_2: [0,1] \to \Re$ satisfy the below conditions:

1. u_1 is a bounded non decreasing function,
2. u_2 is a bounded non increasing function,
3. $u_1(1) \leq u_2(1)$,
4. $\lim_{r\to k^-} u_1(r) = u_1(k)$ and $\lim_{r\to k^-} u_2(r) = u_2(k), k \in (0,1]$,
5. $\lim_{r\to 0^+} u_1(r) = u_1(0)$ and $\lim_{r\to 0^+} u_2(r) = u_2(0)$.

Then $u: \Re \to [0,1]$, defined by $u(s) = \sup\{r | u_1(r) \leq s \leq u_2(r)\}$ is a fuzzy number with parameter $[u_1(r), u_2(r)]$.

Definition 2.4[20] If u and v are two fuzzy numbers, for each $r \in [0,1]$, we've

1. $[u + v]^r = [u]^r + [v]^r = [u_{1r} + v_{1r}, u_{2r} + v_{2r}]$,
2. $[\lambda u]^r = \lambda[u]^r = [\min\{\lambda u_{1r}, \lambda u_{2r}\}, \max\{\lambda u_{1r}, \lambda u_{2r}\}]$,
3. $uv]^r = [u]^r[v]^r = [\min\{u_{1r}v_{1r}, u_{1r}v_{2r}, u_{2r}v_{1r}, u_{2r}v_{2r}\},$
$\max\{u_{1r}v_{1r}, u_{1r}v_{2r}, u_{2r}v_{1r}, u_{2r}v_{2r}\}]$,
4. $u = v$ iff $[u]^r = [v]^r$ iff $u_{1r} = v_{1r}$ and $u_{2r} = v_{2r}$,

collection of all fuzzy numbers with addition and scalar multiplication is a convex cone.

Definition 2.5[23] Let u, v and w $\in \Re_F^*$, such that $u = v + w$; then w is called the Hukuhara differentiable of u and v, denoted by $u \ominus v$. Let $u \ominus v \neq u + (-1)v = u - v$ is Hukuhara differentiable, then $[u \ominus v]^r = [u_{1r} - v_{1r}, u_{2r} - v_{2r}]$.

Definition 2.6[17] Let g is strongly differentiable at $t_0 \in [a, b]$ and $g: [a, b] \to \Re_F$ such that

1. For each $h > 0$, the Hukuhara differences $g(t_0 + h) \ominus g(t_0), g(t_0) \ominus g(t_0 - h)$ and

$$\lim_{h\to 0^+} \frac{g(t_0+h)\ominus g(t_0)}{h} = \lim_{h\to 0^+} \frac{g(t_0)\ominus g(t_0-h)}{h}$$
$$= g'(t_0)$$

2. For each $h > 0$, the Hukuhara differences $g(t_0) \ominus g(t_0+h), g(t_0-h) \ominus g(t_0)$ and
$\lim_{h\to 0^+} \frac{g(t_0)\ominus g(t_0+h)}{-h} = \lim_{h\to 0^+} \frac{g(t_0-h)\ominus g(t_0)}{-h}$
$= g'(t_0)$

Theorem 2.7 [18] For each $r \in [0,1], g:[a,b] \to \Re_F$ and $[g(t)]^r = [g_{1r}(t), g_{2r}(t)]$. Such that g_{1r} and g_{2r} are differentiable functions on $[a,b]$

1. If g is (1) $-$differentiable on $[a,b]$ then $[g'(t)]^r = [g'_{1r}(t), g'_{2r}(t)]$,
2. If g is (2) $-$differentiable on $[a,b]$ then $[g'(t)]^r = [g'_{2r}(t), g'_{1r}(t)]$.

Theorem 2.8[18] Let $g:[a,b] \to \Re_F$ be a fuzzy-valued function. For fixed $t_0 \in [a,b]$ and $\epsilon > 0$ if there exist $\delta > 0$ such that $|t-t_0| < \delta$ which implies $d(g(t), g(t_0)) < \epsilon$, then we say that g is continuous at t_0.

Definition 2.9[28] Let $g \in L^F(I)$. The Riemann-Liouville fractional integral of order α of the fuzzy number valued function g is defined as $J^\alpha g(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \frac{g(\xi)}{(x-\xi)^{1-\alpha}} d\xi$, $x > a$ where $\Gamma(\alpha)$ is the well-known Gamma function.

Definition 2.10[29, 30, 26] Let $g \in AC(I)$, then Riemann-Liouville fractional derivative of order α of the crisp function g exists almost every where on I and can be represented by ${}_a^{RL}D^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx}\int_a^x g(\xi)(x-\xi)^{-\alpha} d\xi$

Note that Riemann-Liouville fractional derivative of order α of g is the first order derivative of the fractional integral $1-\alpha$ of g.

Definition 2.11[29, 30, 26] Let $g \in AC(I)$. Then Caputo fractional derivative of order α of the crisp function g exists almost everywhere on I and can be represented by ${}_a^{C}D^\alpha g(x) = \frac{1}{\Gamma(1-\alpha)}\int_a^x g'(\xi)(x-\xi)^{-\alpha} d\xi$

Note that Caputo fractional derivative of order α of g is the fractional integral $1-\alpha$ of the first order derivative of g.

Definition 2.12[22] Let $g \in (AC)^F(I)$ and $G(x) = \frac{1}{\Gamma(1-\alpha)}\int_a^x g(\xi)(x-\xi)^{-\alpha} d\xi$, for $x > a$. G is (1)-differentiable, then Riemann-Liouville fractional derivative of order α of the fuzzy number valued function g exists, ${}_a^{RL}D_1^\alpha g(x) = \frac{d}{dx}G(x)$. G is (2)-differentiable, then Riemann-Liouville fractional derivative of order α of the fuzzy number valued function g exists, ${}_a^{RL}D_2^\alpha g(x) = \frac{d}{dx}G(x)$.

3. Formulation of Fuzzy Fractional Abel Differential Equation

Consider the nonlinear fuzzy fractional Abel differential equations,

$$D^\alpha \tilde{g}(t) = A\tilde{g}^3(t) + B\tilde{g}^2(t) + E\tilde{g}(t) + F, 0 < \alpha \le 1, 0 \le t \le R. \quad (3.1)$$

with the fuzzy initial condition

$$\tilde{g}(0) = \tilde{g}_0, \quad (3.2)$$

where $\tilde{g}^3(t) \neq 0, A, B, E$ and $F \in \Re$, $\tilde{g}(t) = [0,T] \to \Re_F$ and $\tilde{g}_0 \in \Re_F$.

To construct the section of fuzzy fractional Abel differential equation(FFADE) (3.1) based on the type of differentiability and fuzzy initial condition (3.2), we consider the $r-$ cut level representation of $D^\alpha\tilde{g}(t), \tilde{g}^3(t), \tilde{g}^2(t), \tilde{g}(t)$ and $\tilde{g}(0)$ as $[D^\alpha g_{1r}t), D^\alpha g_{2r}(t)]$, $[g_{1r}^3 t), g_{2r}^3(t)]$, $[g_{1r}^2 t), g_{2r}^2(t)]$, $[g_{1r}t), g_{2r}(t)]$, $[g_{0,1r}t), g_{0,2r}(t)]$, respectively. Consequently, the FFADEs (3.1) and (3.2) should be written with the parametric form as follows:

$$[D^\alpha\tilde{g}(t)]^r = A[\tilde{g}^3(t)]^r + B[\tilde{g}^2(t)]^r + E[\tilde{g}(t)]^r + F, t > 0. \quad (3.3)$$

with the fuzzy initial condition

$$[\tilde{g}(0)]^r = [\tilde{g}_0]^r. \quad (3.4)$$

Now, the algorithm presents us the residual power series strongly differentiability for solving initial value problems (3.3) and (3.4) in r-cut level representation that converted to crisp systems of ODEs. To obtain the fuzzy solution $\tilde{g}(t)$ for the initial value problems (3.3) and (3.4), two cases are considered according to kinds of differentiability, where $\tilde{g}(t)$ is either $(1)-$ differentiable or $(2)-$ differentiable.

Case 1: If $\tilde{g}(t)$ is $(1)-$ differentiable, then initial value problems (3.3) and (3.4) can be converted into the following crisp system:

$$\begin{aligned} D^{\alpha}g_{1r}(t) &= Ag_{1r}^{3}(t) + Bg_{1r}^{2}(t) + Eg_{1r}(t) + F, \\ D^{\alpha}g_{2r}(t) &= Ag_{2r}^{3}(t) + Bg_{2r}^{2}(t) + Eg_{2r}(t) + F, \end{aligned} \tag{3.5}$$

with the fuzzy initial condition

$$\begin{aligned} g_{1r}(0) &= g_{0,1r}, \\ g_{2r}(0) &= g_{0,2r}, \end{aligned} \tag{3.6}$$

Consequently, the following steps should be taken:

- Solve the system (3.5) and (3.6) using the procedure of residual power series algorithm.
- Ensure that the solution $[g_{1r}t), g_{2r}(t)]$ and $[D^{\alpha}g_{1r}(t), D^{\alpha}g_{2r}(t)]$ are valid r-cut level sets, $\forall r \in [0,1]$.
- Obtain the (1)-solution $\tilde{g}$ whose r- cut level representation is $[g_{1r}(t), g_{2r}(t)]$.

Case 2: If $\tilde{g}(t)$ is $(2)-$ differentiable, then initial value problems (3.3) and (3.4) can be converted into the following crisp system:

$$\begin{aligned} D^{\alpha}g_{1r}(t) &= Ag_{2r}^{3}(t) + Bg_{2r}^{2}(t) + Eg_{2r}(t) + F, \\ D^{\alpha}g_{2r}(t) &= Ag_{1r}^{3}(t) + Bg_{1r}^{2}(t) + Eg_{1r}(t) + F, \end{aligned} \tag{3.7}$$

with the fuzzy initial condition

$$\begin{aligned} g_{1r}(0) &= g_{0,1r}, \\ g_{2r}(0) &= g_{0,2r}, \end{aligned} \tag{3.8}$$

Consequently, the following steps should be taken:

- Solve the system (3.7) and (3.8) using the procedure of residual power series algorithm.
- Ensure that the solution $[g_{1r}t), g_{2r}(t)]$ and $[D^{\alpha}g_{1r}(t), D^{\alpha}g_{2r}(t)]$ are valid r-cut level sets, $\forall r \in [0,1]$.
- Obtain the (2)-solution $\tilde{g}$ whose r- cut level representation is $[g_{2r}(t), g_{1r}(t)]$.

4. The Residual Power Series Method for the Fuzzy Fractional Abel Differential Equation

In this section, we seek to obtain the (1)- solution for the fuzzy fractional Abel differential equation (3.3) and (3.4) by employing the procedures of residual power series method. Further, same procedure can be followed (2)-differentiable, we assume that $\tilde{g}(t)$ is (1)- differentiable, therefore the solutions of equations (3.5) and (3.6) at $t = 0$ have the following forms:

$$\begin{aligned} g_{1r}(t) &= \sum_{k=0}^{\infty} a_k \frac{t^{k\alpha}}{\Gamma(1+k\alpha)}, \\ g_{2r}(t) &= \sum_{k=0}^{\infty} b_k \frac{t^{k\alpha}}{\Gamma(1+k\alpha)}. \end{aligned} \tag{4.1}$$

By using the initial conditions $g_{1r}(0) = g_{0,1r} = p_0$ and $g_{2r}(0) = g_{0,2r} = q_0$ as initial approximations. Then, the expression of (4.1) can be written as:

$$\begin{aligned} g_{1r}(t) &= g_{0,1r} + \sum_{k=1}^{\infty} a_k \frac{t^{k\alpha}}{\Gamma(1+k\alpha)}, \\ g_{2r}(t) &= g_{0,2r} + \sum_{k=1}^{\infty} b_k \frac{t^{k\alpha}}{\Gamma(1+k\alpha)}. \end{aligned} \tag{4.2}$$

Consequently, the $i^{th}-$ truncated series solutions of $g_{1r}(t)$ and $g_{2r}(t)$ can be written as:

$$g_{i,1r}(t) = g_{0,1r} + \sum_{k=1}^{i} a_k \frac{t^{k\alpha}}{\Gamma(1+k\alpha)},$$
$$g_{i,2r}(t) = g_{0,2r} + \sum_{k=1}^{i} b_k \frac{t^{k\alpha}}{\Gamma(1+k\alpha)}, \tag{4.3}$$

According the residual power series approach, the $i^{th}-$ residual functions of system (3.5) and (3.6) are defined by

$$Res_{i,1r}(t) = D^{\alpha} g_{1r}(t) - A g_{1r}^3(t) - B g_{1r}^2(t) - E g_{1r}(t) - F,$$
$$Res_{i,2r}(t) = D^{\alpha} \tilde{g}_{2r}(t) - A g_{2r}^3(t) - B g_{2r}^2(t) - E g_{2r}(t) - F. \tag{4.4}$$

where the $\infty^{th}-$ residual functions are given by

$$Res_{\infty,1r}(t) = lim_{i\to\infty} Res_{i,1r}(t) = D^{\alpha} g_{1r}(t) - A g_{1r}^3(t) - B g_{1r}^2(t) - E g_{1r}(t) - F,$$
$$Res_{\infty,2r}(t) = lim_{i\to\infty} Res_{i,2r}(t) = D^{\alpha} g_{2r}(t) - A g_{2r}^3(t) - B g_{2r}^2(t) - E g_{2r}(t) - F, \tag{4.5}$$

As in residual power series, put $Res_{\infty,ir}(t) = 0$ for each $t \in [0, R]$, R is radius of convergence and $i = \{1,2\}$, which are infinitely differentiable functions at $t = 0$. Then we get $\frac{d^{k-1}}{dt^{k-1}} Res_{\infty,ir}(0) = \frac{d^{k-1}}{dt^{k-1}} Res_{k,ir}(0) = 0$, for $k = 1,2,3, \dots j$. The residual power series algorithm basic fact the parameters a_k and b_k , $k \geq 1$.

To find the coefficients a_1 and b_1 , substitute $g_{1,1r}(t) = g_{0,1r} + a_1 \frac{t^{\alpha}}{\Gamma(1+\alpha)}$ and $g_{1,2r}(t) = g_{0,2r} + b_1 \frac{t^{\alpha}}{\Gamma(1+\alpha)}$ to apply the residual functions, $Res_{1,1r}(t)$ and $Res_{1,2r}(t)$, at $i = 1$ of (4.4) we get:

$$Res_{1,1r}(t) = D^{\alpha} g_{1,1r}(t) - A g_{1,1r}^3(t) - B g_{1,1r}^2(t) - E g_{1,1r}(t) - F,$$
$$= D^{\alpha}\left(g_{0,1r} + a_1 \frac{t^{\alpha}}{\Gamma(1+\alpha)}\right) - A\left(g_{0,1r} + a_1 \frac{t^{\alpha}}{\Gamma(1+\alpha)}\right)^3 - B\left(g_{0,1r} + a_1 \frac{t^{\alpha}}{\Gamma(1+\alpha)}\right)^2$$
$$-E\left(g_{0,1r} + a_1 \frac{t^{\alpha}}{\Gamma(1+\alpha)}\right) - F,$$

$$Res_{1,2r}(t) = D^{\alpha} g_{1,2r}(t) - A g_{1,2r}^3(t) - B g_{1,2r}^2(t) - E g_{1,2r}(t) - F,$$
$$= D^{\alpha}\left(g_{0,2r} + b_1 \frac{t^{\alpha}}{\Gamma(1+\alpha)}\right) - A\left(g_{0,2r} + b_1 \frac{t^{\alpha}}{\Gamma(1+\alpha)}\right)^3 - B\left(g_{0,2r} + b_1 \frac{t^{\alpha}}{\Gamma(1+\alpha)}\right)^2$$
$$-E\left(g_{0,2r} + b_1 \frac{t^{\alpha}}{\Gamma(1+\alpha)}\right) - F,$$

Using the residual power series fact that $\alpha = 1$, $Res_{1,1r}(0) = 0$ and $Res_{1,2r}(0) = 0$ in (4.6) it yields that $a_1 = A g_{0,1r}^3 + B g_{0,1r}^2 + E g_{0,1r} + F$ and $b_1 = A g_{0,2r}^3 + B g_{0,2r}^2 + E g_{0,2r} + F$. Then the first approximations are:

$$g_{1,1r}(t) = a_0 + (A a_0^3 + B a_0^2 + E a_0 + F) \frac{t^{\alpha}}{\Gamma(1+\alpha)},$$
$$g_{1,2r}(t) = b_0 + (A b_0^3 + B b_0^2 + E b_0 + F) \frac{t^{\alpha}}{\Gamma(1+\alpha)}. \tag{4.6}$$

For $i = 2$, the second approximations are:

$$g_{2,1r}(t) = a_0 + (A a_0^3 + B a_0^2 + E a_0 + F) \frac{t^{\alpha}}{\Gamma(1+\alpha)} + a_2 \frac{t^{2\alpha}}{\Gamma(1+2\alpha)},$$
$$g_{2,2r}(t) = b_0 + (A b_0^3 + B b_0^2 + E b_0 + F) \frac{t^{\alpha}}{\Gamma(1+\alpha)} + b_2 \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}. \tag{4.7}$$

The residual functions, $Res_{2,1r}(t)$ and $Res_{2,2r}(t)$ of (4.4) such that

$$Res_{2,1r}(t) = D^{\alpha} g_{2,1r}(t) - A g_{2,1r}^3(t) - B g_{2,1r}^2(t) - E g_{2,1r}(t) - F,$$
$$= D^{\alpha}\left(a_0 + (A a_0^3 + B a_0^2 + E a_0 + F) \frac{t^{\alpha}}{\Gamma(1+\alpha)} + a_2 \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right)$$
$$-A\left(a_0 + (A a_0^3 + B a_0^2 + E a_0 + F) \frac{t^{\alpha}}{\Gamma(1+\alpha)} + a_2 \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right)^3$$
$$-B\left(a_0 + (A a_0^3 + B a_0^2 + E a_0 + F) \frac{t^{\alpha}}{\Gamma(1+\alpha)} + a_2 \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right)^2$$

$$-E\left(a_0+(Aa_0^3+Ba_0^2+Ea_0+F)\frac{t^\alpha}{\Gamma(1+\alpha)}+a_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right)-F,$$
$$Res_{2,2r}(t)=D^\alpha g_{2,2r}(t)-Ag_{2,2r}^3(t)-Bg_{2,2r}^2(t)-Eg_{2,2r}(t)-F,$$
$$=D^\alpha\left(b_0+(Ab_0^3+Bb_0^2+Eb_0+F)\frac{t^\alpha}{\Gamma(1+\alpha)}+b_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right)$$
$$-A\left(b_0+(Ab_0^3+Bb_0^2+Eb_0+F)\frac{t^\alpha}{\Gamma(1+\alpha)}+b_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right)^3$$
$$-B\left(b_0+(Ab_0^3+Bb_0^2+Eb_0+F)\frac{t^\alpha}{\Gamma(1+\alpha)}+b_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right)^2$$
$$-E\left(b_0+(Ab_0^3+Bb_0^2+Eb_0+F)\frac{t^\alpha}{\Gamma(1+\alpha)}+b_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right)-F.$$

Now, differentiable both sides of $Res_{2,1r}(t)$ and $Res_{2,2r}(t)$ such that

$$\frac{d}{dt}Res_{2,1r}(t)=\frac{d}{dt}\left[D^\alpha g_{2,1r}(t)-Ag_{2,1r}^3(t)-Bg_{2,1r}^2(t)-Eg_{2,1r}(t)-F\right],$$
$$=\frac{d}{dt}\left[D^\alpha\left(a_0+(Aa_0^3+Ba_0^2+Ea_0+F)\frac{t^\alpha}{\Gamma(1+\alpha)}+a_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right)\right]$$
$$-\frac{d}{dt}\left[A\left(a_0+(Aa_0^3+Ba_0^2+Ea_0+F)\frac{t^\alpha}{\Gamma(1+\alpha)}+a_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right)^3\right]$$
$$-\frac{d}{dt}\left[B\left(a_0+(Aa_0^3+Ba_0^2+Ea_0+F)\frac{t^\alpha}{\Gamma(1+\alpha)}+a_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right)^2\right]$$
$$-\frac{d}{dt}\left[E\left(a_0+(Aa_0^3+Ba_0^2+Ea_0+F)\frac{t^\alpha}{\Gamma(1+\alpha)}+a_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right)\right]-\frac{d}{dt}[F],$$

$$\frac{d}{dt}Res_{2,2r}(t)=\frac{d}{dt}\left[D^\alpha g_{2,2r}(t)-Ag_{2,2r}^3(t)-Bg_{2,2r}^2(t)-Eg_{2,2r}(t)-F\right],$$
$$=\frac{d}{dt}\left[D^\alpha\left(b_0+(Ab_0^3+Bb_0^2+Eb_0+F)\frac{t^\alpha}{\Gamma(1+\alpha)}+b_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right)\right]$$
$$-\frac{d}{dt}\left[A\left(b_0+(Ab_0^3+Bb_0^2+Eb_0+F)\frac{t^\alpha}{\Gamma(1+\alpha)}+b_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right)^3\right]$$
$$-\frac{d}{dt}\left[B\left(b_0+(Ab_0^3+Bb_0^2+Eb_0+F)\frac{t^\alpha}{\Gamma(1+\alpha)}+b_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right)^2\right]$$
$$-\frac{d}{dt}\left[E\left(b_0+(Ab_0^3+Bb_0^2+Eb_0+F)\frac{t^\alpha}{\Gamma(1+\alpha)}+b_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right)\right]-\frac{d}{dt}[F],$$

by using residual power series facts $\alpha=1$, $\frac{d}{dt}Res_{2,1r}(0)=0$ and $\frac{d}{dt}Res_{2,2r}(0)=0$, it can be deduced that the residual functions

$$a_2=\frac{3}{2}Aa_0^2a_1+Ba_0a_1+\frac{1}{2}Ea_1,$$
$$b_2=\frac{3}{2}Ab_0^2b_1+Bb_0b_1+\frac{1}{2}Eb_1. \tag{4.8}$$

Then the second approximations are:

$$g_{2,1r}(t)=a_0+(Aa_0^3+Ba_0^2+Ea_0+F)\frac{t^\alpha}{\Gamma(1+\alpha)}+\left(\frac{3}{2}Aa_0^2a_1+Ba_0a_1+\frac{1}{2}Ea_1\right)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)},$$
$$g_{2,2r}(t)=b_0+(Ab_0^3+Bb_0^2+Eb_0+F)\frac{t^\alpha}{\Gamma(1+\alpha)}+\left(\frac{3}{2}Ab_0^2b_1+Bb_0b_1+\frac{1}{2}Eb_1\right)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}. \tag{4.9}$$

For $i=3$, the third approximations are $g_{3,1r}(t)$ and $g_{3,2r}(t)$ into the residual functions, $Res_{3,1r}(t)$ and $Res_{3,2r}(t)$ of (4.4) utilized the residual power series facts $\alpha=1$, $\frac{d^2}{dt^2}Res_{3,1r}(0)=0$ and $\frac{d^2}{dt^2}Res_{3,2r}(0)=0$. Then we get the third coefficients are given by

$$a_3=A(a_2a_0^2+a_1^2a_0)+\frac{1}{3}B(2a_0a_2+a_1^2)+\frac{1}{3}Ea_2,$$
$$b_3=A(b_2b_0^2+b_1^2b_0)+\frac{1}{3}B(2b_0b_2+b_1^2)+\frac{1}{3}Eb_2. \tag{4.10}$$

For $i=4$, the fourth approximations are $g_{4,1r}(t)$ and $g_{4,2r}(t)$ into the residual functions, $Res_{4,1r}(t)$ and $Res_{4,2r}(t)$ of (4.4) utilized the residual power series facts $\alpha=1$, $\frac{d^3}{dt^3}Res_{4,1r}(0)=0$ and $\frac{d^3}{dt^3}Res_{4,2r}(0)=0$. Then we get the third coefficients are given by

$$a_4 = \frac{1}{4}A(a_1^3 + 6a_0a_1a_2 + 3a_0^2a_3) + \frac{1}{2}B(a_1a_2 + a_0a_3) + \frac{1}{4}Ea_3,$$
$$b_4 = \frac{1}{4}A(b_1^3 + 6b_0b_1b_2 + 3b_0^2b_3) + \frac{1}{2}B(b_1b_2 + b_0b_3) + \frac{1}{4}Eb_3. \quad (4.11)$$

By continuing the same procedures upto arbitrary coefficients order $\boldsymbol{i = n}$ using residual power series facts $\frac{d^{(n-1)}}{dt^{(n-1)}}\boldsymbol{Res_{n,1r}(0) = 0}$ and $\frac{d^{(n-1)}}{dt^{(n-1)}}\boldsymbol{Res_{n,2r}(0) = 0}$, it can be deduced that the residual functions $\boldsymbol{a_n}$ and $\boldsymbol{b_n}$ can be obtained. Similarly, $\boldsymbol{\tilde{g}(t)}$ is $(\mathbf{2})-$ solution for the $(\mathbf{2})-$ differentiable fuzzy fractional Abel differential equation $(\mathbf{3.3})$ and $(\mathbf{3.4})$ can be obtained.

5. Experimental Study

Example 5.1 *Consider the following fractional Abel initial value problem,*

$$D^\alpha \tilde{g}(t) - 3\tilde{g}(t)^3 + \tilde{g}(t) = 0, t > 0, \quad (5.1)$$

with the fuzzy initial condition

$$[\tilde{g}(0)]^r = [\frac{7}{24} + \frac{1}{24}u, \frac{101}{300} - \frac{1}{300}u], u \in [0,1]. \quad (5.2)$$

In particular for $u = 1$ and $\alpha = 1$ the solution of (5.1) with crisp initial condition $\tilde{g}(0) = \frac{1}{3}$ as follows:

$$\tilde{g}(t) = \frac{1}{\sqrt{6e^{2t}+3}}. \quad (5.3)$$

we represent the parametric forms of (5.1) as follows:

$$D^\alpha g_{1r}(t) = 3g_{1r}(t)^3 - g_{1r}(t),$$
$$D^\alpha g_{2r}(t) = 3g_{2r}(t)^3 - g_{2r}(t), \quad (5.4)$$

with the fuzzy initial condition

$$g_{1r}(0) = \frac{7}{24} + \frac{1}{24}u,$$
$$g_{2r}(0) = \frac{101}{300} - \frac{1}{300}u. \quad (5.5)$$

By using the initial conditions $g_{1r}(0) = g_{0,1r} = a_0$ and $g_{2r}(0) = g_{0,2r} = b_0$ as initial approximations, the expression of (5.5) can be written as $g_{1r}(0) = \frac{7}{24} + \frac{1}{24}u$ and $g_{2r}(0) = \frac{101}{300} - \frac{1}{300}u$, the residual power series solutions $D^\alpha g_{1r}(t)$ and $D^\alpha g_{2r}(t)$ of system (5.4) can be written as:

$$g_{1r}(t) = \frac{7}{24} + \frac{1}{24}u + a_1\frac{t^\alpha}{\Gamma(1+\alpha)} + a_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \cdots + a_i\frac{t^{i\alpha}}{\Gamma(1+i\alpha)} + \cdots,$$
$$g_{2r}(t) = \frac{101}{300} - \frac{1}{300}r + b_1\frac{t^\alpha}{\Gamma(1+\alpha)} + b_2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \cdots + b_i\frac{t^{i\alpha}}{\Gamma(1+i\alpha)} + \cdots. \quad (5.6)$$

By utilizing the residual power series $\frac{d^{(i-1)}}{dt^{(i-1)}}Res_{i,1r}(0) = 0$ and $\frac{d^{(i-1)}}{dt^{(i-1)}}Res_{i,2r}(0) = 0$, for $i = 1,2$ ..., the terms of a_i and b_i are:

$a_0 = \frac{7}{24} + \frac{1}{24}u,$
$a_1 = \frac{1}{4608}(u+7)(u^2 + 14u - 143),$
$a_2 = \frac{1}{589824}(u-1)(u+15)(u+7)(u^2 + 14u - 143),$
$a_3 = \frac{1}{339738624}(5u^4 + 140u^3 + 702u^2 - 3892u - 13339)(u+7)(u^2 + 14u - 143),$
and
$b_0 = \frac{101}{300} - \frac{1}{300}u,$
$b_1 = \frac{1}{9000000}(101-u)(u^2 - 202u - 19799),$
$b_2 = \frac{1}{180000000000}(101-u)(u^2 - 202u - 19799)(1-u)(201-u),$
$b_3 = \frac{1}{3240000000000000}(101-u)(u^2 - 202u - 19799)(u^4 - 404u^3 + 37206u^2 + 726796u - 80763599),$ and so on.

If $u = 1$, then the residual power series solution becomes

$$g(t) = \frac{1}{3} - \frac{2}{9}\frac{t^{\alpha}}{\Gamma(1+\alpha)} + \frac{4}{81}\frac{t^{3\alpha}}{\Gamma(1+3\alpha)} - \frac{2}{243}\frac{t^{4\alpha}}{\Gamma(1+4\alpha)} + \cdots \ldots \quad (5.7)$$

The numerical results of Example 1 for various t in [0,1] is shown in Table 1, Table 2 and Fig 1.

Table 1. Value of g(t)

	$\alpha = 0.2$		$\alpha = 0.4$	
t	**Exact solution**	**RPSM solution**	**Exact solution**	**RPSM solution**
0	0.333333333333333	0.333333333333333	0.333333333333333	0.333333333333333
0.1	0.193471007156861	0.201381245662826	0.237563672647012	0.241882871969703
0.2	0.176519388144558	0.184715229698909	0.210321281312679	0.215958907891209
0.3	0.166105051148641	0.174111175460565	0.191911496037013	0.197998426415079
0.4	0.158527534841319	0.166165375323268	0.177828824029250	0.183874656100862
0.5	0.152558461080159	0.159740970196098	0.166395195450085	0.172075077162908
0.6	0.147631044092847	0.154310348890382	0.156774182849010	0.161853823923007
0.7	0.143435215689562	0.149583289081796	0.148480659421844	0.152781573923248
0.8	0.139782263095155	0.145382236233200	0.141205233022639	0.144586015862536
0.9	0.136548548404761	0.141590285096808	0.134737216623282	0.137082431575621
1.0	0.133648555869576	0.138126192142471	0.128926056443408	0.130139175738835

Table 2. Value of g(t)

	$\alpha = 0.8$		$\alpha = 1$	
t	**Exact solution**	**RPSM solution**	**Exact solution**	**RPSM solution**
0	0.333333333333333	0.333333333333333	0.333333333333333	0.333333333333333
0.1	0.295753483812600	0.296385790774987	0.311159544858791	0.311357990397805
0.2	0.268689364116527	0.270118516859413	0.289266951196178	0.289875994513032
0.3	0.245311385555235	0.247276980269422	0.267905825643850	0.268886111111111
0.4	0.224535908022159	0.226618045743047	0.247285140856786	0.248386282578875
0.5	0.205865356048560	0.207576575382591	0.227568600277633	0.228373628257888
0.6	0.188992508866941	0.189825838487919	0.208874799971873	0.208844444444444
0.7	0.173694594369442	0.173150976821573	0.191280479364827	0.189794204389575
0.8	0.159794571311227	0.157398565824515	0.174825723730926	0.171217558299040
0.9	0.147144243984156	0.142452511017569	0.159520099726950	0.153108333333333
1.0	0.135615881953843	0.128221075728880	0.145348934559835	0.135459533607682

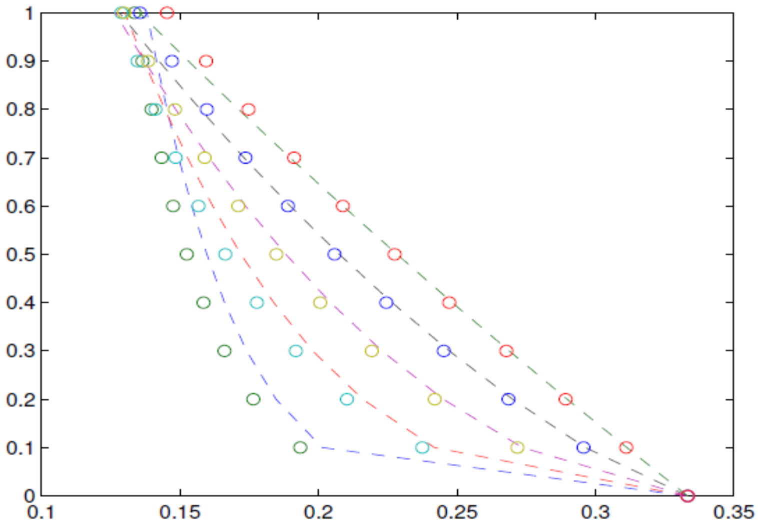

Fig. 1 Value of f(x)

Example 5.2

Consider the following fractional Abel initial value problem,

$$D^\alpha \tilde{g}(t) + \tilde{g}(t)^3 - \tilde{g}(t) = 0, t > 0, \tag{5.8}$$

with the fuzzy initial condition

$$[\tilde{g}(0)]^r = [\frac{7}{24} + \frac{1}{24}u, \frac{101}{300} - \frac{1}{300}u], u \in [0,1]. \tag{5.9}$$

In particular for $u = 1$and $\alpha = 1$ the solution of (5.8) with crisp initial condition $\tilde{g}(0) = \frac{1}{3}$ can be found as:

$$\tilde{g}(t) = \frac{e^t}{\sqrt{e^{2t}+8}}. \tag{5.10}$$

we represent the parametric forms of (5.8) as follows:

$$\begin{aligned} D^\alpha g_{1r}(t) &= g_{1r}(t) - g_{1r}(t)^3, \\ D^\alpha g_{2r}(t) &= g_{1r}(t) - g_{2r}(t)^3. \end{aligned} \tag{5.11}$$

with the fuzzy initial condition

$$\begin{aligned} g_{1r}(0) &= \frac{7}{24} + \frac{1}{24}u, \\ g_{2r}(0) &= \frac{101}{300} - \frac{1}{300}u. \end{aligned} \tag{5.12}$$

By using the initial conditions $g_{1r}(0) = g_{0,1r} = a_0$ and $g_{2r}(0) = g_{0,2r} = b_0$ as initial approximations. Then, the expression of (5.9) can be written as $g_{1r}(0) = \frac{7}{24} + \frac{1}{24}u$ and $g_{2r}(0) = \frac{101}{300} - \frac{1}{300}u$, the residual power series solutions $D^\alpha g_{1r}(t)$ and $D^\alpha g_{2r}(t)$ of system (5.11) can be written as:

$$\begin{aligned} g_{1r}(t) &= \frac{7}{24} + \frac{1}{24}u + a_1 \frac{t^\alpha}{\Gamma(1+\alpha)} + a_2 \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \cdots + a_i \frac{t^{i\alpha}}{\Gamma(1+i\alpha)} + \cdots, \\ g_{2r}(t) &= \frac{101}{300} - \frac{1}{300}u + b_1 \frac{t^\alpha}{\Gamma(1+\alpha)} + b_2 \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \cdots + b_i \frac{t^{i\alpha}}{\Gamma(1+i\alpha)} + \cdots. \end{aligned} \tag{5.13}$$

By utilizing the residual power series $\frac{d^{(i-1)}}{dt^{(i-1)}} Res_{i,1r}(0) = 0$ and $\frac{d^{(i-1)}}{dt^{(i-1)}} Res_{i,2r}(0) = 0$, for $i = 1,2$ ..., the terms of a_i and b_i are:

$$\begin{aligned} a_0 &= \frac{7}{24} + \frac{1}{24}u, \\ a_1 &= \frac{1}{13824}(u+7)(17-u)(u+31), \\ a_2 &= \frac{1}{5308416}(u+7)(u-17)(u+31)(u^2+14u-143), \\ a_3 &= \frac{1}{9172942848}(5u^4+140u^3-834u^2-25396u+9701)(u+7)(17-u)(u+31), \end{aligned}$$

and

$$\begin{aligned} b_0 &= \frac{101}{300} - \frac{1}{300}u, \\ b_1 &= \frac{1}{27000000}(u-101)(u+199)(u-401), \\ b_2 &= \frac{1}{1620000000000}(401-u)(u-101)(u+199)(u^2-202u-19799), \\ b_3 &= \frac{1}{87480000000000000}(u-101)(u+199)(u-401)(u^4-404u^3-10794u^2+10422796u-90411599), \end{aligned}$$

and so on.

If $u = 1$, then the residual power series solution

$$g(t) = \frac{1}{3} + \frac{8}{27}\frac{t^\alpha}{\Gamma(1+\alpha)} + \frac{8}{81}\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} - \frac{16}{2187}\frac{t^{3\alpha}}{\Gamma(1+3\alpha)} - \frac{440}{19683}\frac{t^{4\alpha}}{\Gamma(1+4\alpha)} + \cdots ... \tag{5.14}$$

The numerical results of Example 2 for various t in [0,1] is shown in Table 3 and Fig 3.

Table 3. Value of g(t)

	$\alpha = 0.2$		$\alpha = 0.4$		$\alpha = 0.8$		$\alpha = 1$	
t	**Exact**	**RPSM**	**Exact**	**RPSM**	**Exact**	**RPSM**	**Exact**	**RPSM**
0	0.3333	0.3333	0.3333	0.3333	0.3333	0.3333	0.3333	0.3333
0.1	0.5750	0.5792	0.4844	0.4826	0.3865	0.3854	0.3639	0.3634
0.2	0.6143	0.6225	0.5385	0.5370	0.4294	0.4263	0.3964	0.3945
0.3	0.6396	0.6517	0.5785	0.5785	0.4701	0.4646	0.4307	0.4266
0.4	0.6587	0.6744	0.6111	0.6135	0.5095	0.5018	0.4665	0.4596
0.5	0.6740	0.6932	0.6389	0.6444	0.5479	0.5383	0.5035	0.4936
0.6	0.6869	0.7094	0.6631	0.6724	0.5851	0.5745	0.5415	0.5286
0.7	0.6980	0.7237	0.6846	0.6982	0.6210	0.6104	0.5799	0.5645
0.8	0.7077	0.7365	0.7039	0.7223	0.6554	0.6463	0.6183	0.6013
0.9	0.7164	0.7483	0.7213	0.7451	0.6881	0.6822	0.6561	0.6391
1.0	0.7243	0.7591	0.7372	0.7666	0.7190	0.7180	0.6929	0.6777

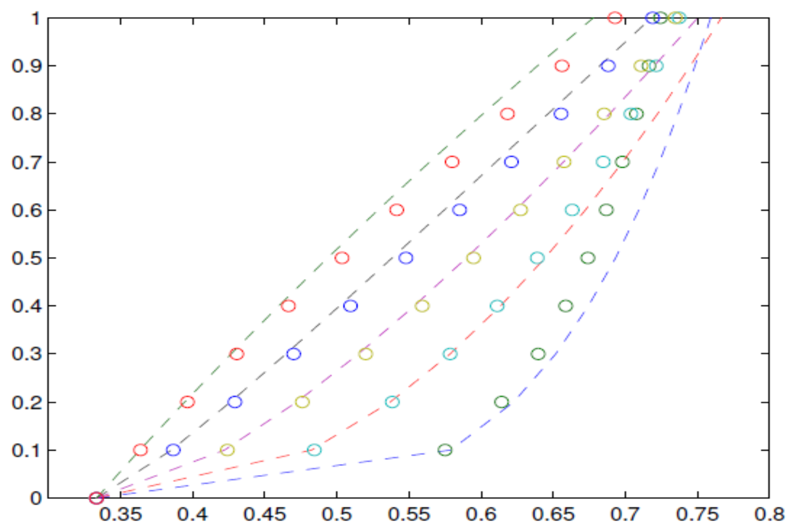

Fig. 2 Value of f(x)

6. Conclusion

In this paper, under strongly generalized differentiability, the RPS application to study the exact solutions for fuzzy FADE. When selecting a suitable fuzzy rule base from becoming extracted and submitted under an obstruction, the approach used explicitly. Mathematical research has demonstrated the stability and efficiency of the novel process. The results show these for nonlinear fuzzy fractional differential equations with fewer simulations reduce time RPS process is effective and successful. About through a saw that, it must be established also that RPS results are effective calculated by other approaches.

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"Author 1 and Author 2 contributed equally to this work".

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