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Original Article

On Posimetrically Equivalent Operators

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Abstract - *In this paper, we generalize metrically equivalent operators to the class of posimetrically equivalent operators. Some basic properties of this class are covered. We also relate this equivalence relation to the class of quasi-p-normal operators. We also relate this class to other equivalence relations such as n-metric equivalence.*

Keywords - *Quasi-p-normal, n-metrically equivalent, Metrically equivalent, Posimetrically equivalent.*

1. Introduction

H symbolizes the complex Hilbert space unless stated otherwise while B (H) is the Banachalgebra of bounded linear operators. The study of operators on the usual Hilbert H space has been intensified over the years. For instance the class of normal operators has been generalized further and diversified into the classes of ; almost normal operators as coveredin [5] , n-normal , (n, m) - normal , mutually normal , quasi -normal , quasi -p-normal , skew-normal , skew quasi-p-normal and perinormal among others . Jibril [6] covered theclass of n-power normal operators. Results linking the class of 3-normal and the 2-normaloperators were struck in [6]. The class of (n, m)-normal operators were covered by Eiman H. and Mustafa [3]. Jibril [4] introduced the class of (Q) operators in 2010. Jibril [4] studied interesting basic properties that this class gets to enjoy. Paramesh [13] later expanded the class (Q) operator to n-power class (Q) operators. Basic properties of thisclass were covered and a result given to show that this class is not generally a normal operator. Manikandan [7] later echoed this class by introducing the class of (n+k) powerclass (Q) for n which is positive definite and $0 \leq k$. New theorems were characterized for this class, in particular, it was shown that if a bounded operator G is (n+k)-normal, then it's (n+k)-power class (Q) . Manikandan [8] furthered results on (n+k) power class (Q). In [8], Manikandan characterized class (Q) operators in terms of complex symmetricoperators. Revathi [14] took the class of class (Q) operators to Quasi-class (Q). Basic properties of this class were investigated and interesting results linking this class with self-adjoint operators were struck. Later on Revathi [15] extended the class of quasi class(Q) Operators into the class of M-quasi class (Q) where M is a bounded operator on theHilbert space *H*. Similarly basic properties of this class were investigated. Wanjala Victor and Nyongesa [18] took the study of class (Q) operators to (α,β)-class (Q) for $0 \leq \alpha \leq 1 \leq \beta$. They investigated some nice algebraic properties of this class, for instance, it was shown that if G is (α,β)-class (Q), then G^* is also (α,β)-class (Q). By relaxing the conditions for (Q) , Wanjala Victor and Beatrice Adhiambo [17] noted that this class coincides with the class of (M,n) operators (*almost class (Q)*) whenever n is equivalent to two (see *[17] under introduction*). K^* Quasi-n-class (Q) operators were covered by Wanjala Victor and Peter Kiptoo Rutto in [19] where they covered some basic properties of this class. Wanjala Victor and Beatrice Adhiambo also introduced the class of (BQ) operators in [23]. G is said to be in (BQ) whenever $G^{*2}G^2$ commutes with (G^*G) .Wanjala Victor and A.M. Nyongesa introduced the class of (Q^*) in [20] , recently Wanjala Victor et al. extended class (Q) to skew quasi-p-class (Q) in [22] . The class of metrically equivalent operators was covered by Nzimbi in [10]. This was expanded to n-metric equivalence and (n, m)-metric equivalence by Wanjala in [26] and [25] respectively. This was later taken to metric equivalence of order n in [24]. This was later analyzed by Wanjala in semi-Hilbert space in [16]. Mahmoud introduced Quasi equivalent operators in [9], this was enhanced by Nzimbi [1] to Unitary quasi equivalence which was later generalized by Wanjala in [21].Nearly equivalent operators was covered by Sadoon in [12].In this paper we extend metrically equivalent operators to the class of posimetrically equivalent operators.

Definition 1. [3] An operator G ∈ B (H) is (n, m)-normal if G*m Gn = Gn G*m .

Definition 2. [2] An operator G ∈ B (H) is:

(i) (n,p)-quasinormal operator if Gn(G∗G)p = (G∗G)p Gn

(ii) Self-adjoint when G∗ = G.

Definition 3. [26] An operator G ∈B (H) is:

(i) An orthogonal projection when G*=G (idempotent) and G2 = I.
(ii) Unitary if G*G = GG* = I.

(iii) Polar decomposition if G = US, where U is partial isometry and S is a positive operator.

Definition 4. [2] An operator $G \in B(H)$ is:
(i) Quasi-p-normal if $G*G (G + G*) = (G + G*) GG*$.

(ii) Quasi-Isometry $G*2G2 = G*G = I$.

(iii) 2-self-adjoint if $G*2 = G2$.

Definition 5. ([10]) Two bounded linear operators S and G are said to be metrically equivalent if $S*S = G*G$.

Definition 6. ([25]) Two bounded linear operators S and G are said to be (n, m)-metrically equivalent if $S*mSn = G*mGn$ for positive integers n and m.

Definition 7. ([26]) Two bounded linear operators S and G are said to be n-metrically equivalent if $S*Sn = G*Gn$ for a positive integer n.

Definition 8. ([11]) Two bounded linear operators S and G are said to be almost similarly equivalent if there exists an invertible operator say N such that; $S*S = N-1 (G*G) N$ and $S* + S = N-1 (G* + G) N$.

2. Main Results

Definition 9. Two operators $S \in B(H)$ and $T \in B(H)$ are said to be Posimetrically equivalent denoted by $S \sim p\ T$ if $(S*S)(S + S*) = (T*T)(T + T*)$.

Theorem 10. If G is quasi-p-normal and $S \in B(H)$ is unitarily equivalent to G, then S is quasi-p-normal.

Proof. Suppose $S = U^*GU$ where U is unitary and S is quasi-p-normal, then

$$\begin{aligned}(S^*S)(S + S^*) &= ((U^*G^*U)(U^*GU))(U^*GU + U^*G^*U)\\ &= ((U^*G^*UU^*GU)(U^*GU + U^*G^*U)\\ &= (U^*G^*GU)(U^*GU + U^*G^*U)\\ &= (U^*GU + U^*G^*U)(U^*G^*GU)\\ &= (SU^*U + S^*U^*U)(S^*U^*US)\\ &= (S + S^*)(S^*S).\end{aligned}$$

hence the proof.

Corollary 11. An operator $S \in B(H)$ is quasi-p-normal if and only if S and S * are posimetricaly equivalent.

Proof. The proof follows from Theorem 10

Lemma 12. Let $S, G \in B(H)$ be bounded linear operators with $S \sim p\ G$, then;

1. S is isometric whenever G is isometric
2. S is a contraction whenever G is a contraction
3. λS and λG are p-metrically equivalent for any $\lambda \in R$
4. The restriction S/M of S and G/M of G to any closed subspace M of H that reduces S

Proof. The proof for 1 and 2 is trivial, for 3 and 4, we have;3). Since S and G are Posimetrically equivalent we have;

$$\begin{aligned}(S^*S)(S + S^*) &= (G^*G)(G + G^*)\\ &= ((\lambda S)^*(\lambda S))(\lambda S + \lambda S^*) = ((\lambda G)^*(\lambda G))(\lambda G + \lambda G^*)\\ &= |\lambda|^2 (S^*S)\lambda(S + S^*) = |\lambda|^2 (G^*G)\lambda(G + G^*)\\ &= |\lambda|^3 (S^*S)(S + S^*) = |\lambda|^3 (G^*G)(G + G^*)\end{aligned}$$

Hence λS and λG are posimetrically equivalent.

For (4) we have; $((S/M)^*(S/M))((S/M) + (S/M)^*) = ((G/M)^*(G/M))((G/M) + (G/M)^*)$

$$(S^*S/M)(S/M + S^*/M) = (G^*G/M)(G/M + G^*/M)$$

$$((S^*S)(S + S^*))/M = ((G^*G)(G + G^*))/M$$

$$(S^*S)/M(S + S^*)/M = (G^*G)/M(G + G^*)/M$$

$$((S^*)/M(S)/M)((S/M) + (S^*/M)) = ((G^*/M)(G)/M))((G/M) + (G^*/M))$$

Theorem 13. Let S, G ∈ B (H) be posimetrically equivalent. If S and G complex symmetric operators, then (S∗S) (S + S∗) = (G∗G) (G + G∗) holds.

Proof. If S and G are complex symmetric operators, then; $S^* = CSC$, $S = CS^*C$ and $G^* = CGC$, $G = CG^*C$ with $C^2 = I$. It then implies that

$$(S^*S)(S + S^*) = (CSCCS^*C)(CS^*C + CSC) \tag{0.16}$$
$$= (CSS^*C)(CS^* + SC) \tag{0.17}$$
$$= (CS^*SC)(CS + S^*C) \tag{0.18}$$
$$= (C(S^*S))((S + S^*)C) \tag{0.19}$$
$$= C^2(S^*S)(S + S^*) \tag{0.20}$$

Similarly;

$$(G^*G)(G + G^*) = (CGCCG^*C)(CG^*C + CGC) \tag{0.21}$$
$$= (CGG^*C)(CG^* + GC) \tag{0.22}$$
$$= (CG^*GC)(CG + G^*C) \tag{0.23}$$
$$= (C(G^*G))((G + G^*)C) \tag{0.24}$$

$$C^2(S^*S)(S + S^*) = C^2(G^*G)(G + G^*) \tag{0.25}$$
$$(S^*S)(S + S^*) = (G^*G)(G + G^*) \tag{0.26}$$

As required.

Theorem 14. If S and G are Posimetrically equivalent with polar decompositions S = U|S| and G = U| G|, then |S|3 = |G|3 if and only if U|S| = |S|U and U|G| = |G|U.

Proof. Since S and G are Posimetrically equivalent;

$$(S^*S)(S + S^*) = (G^*G)(G + G^*) \tag{0.28}$$
$$(U^*|S|U|S|)(U|S| + U^*|S|) = (U^*|G|U|G|)(U|G| + U^*|G|) \tag{0.29}$$
$$(|S|U^*U|S|)(U|S| + U^*|S|) = (|G|U^*U|G|)(U|G| + U^*|G|) \tag{0.30}$$
$$(|S|^2)(U|S| + U^*|S|) = (|G|^2)(U|G| + U^*|G|) \tag{0.31}$$
$$U|S|^3 + U^*|S|^3 = U|G|^3 + U^*|G|^3 \tag{0.32}$$
$$U|S|^3 + |S|^3U^* = U|G|^3 + |G|^3U^* \tag{0.33}$$

Pre-multiplying both the left and right hand side of 0.33 by U^* and post-multiplying thesame by U;

$$U^*U|S|^3 + |S|^3U^*U = U^*U|G|^3 + |G|^3U^*U \tag{0.34}$$
$$2|S|^3 = 2|G|^3 \tag{0.35}$$
$$|S|^3 = |G|^3 \tag{0.36}$$

hence the proof.

2.1. Relationship between Posimetric Equivalence and Other Equivalence Relations

Remark 15. The following result establishes the relationship between Posimetric equivalence and 2-metrically equivalent operators, a subclass of n-metrically equivalent operators. That two Posimetrically equivalent operators are 2-metrically provided they are self-adjoint.

Theorem 16. If S, G ∈ B (H) are Posimetrically, then they are 2-metrically equivalent provided S and G are self-adjoint.

Proof. By assumption;

$$(S^*S)(S+S^*) = (G^*G)\,(G+G^*) \tag{0.37}$$

Since S and G are self-adjoint;

$$(S^*S)(S+S) = (G^*G)(G+G) \tag{0.38}$$

$$2(S^*S)S = 2(G^*G)G \tag{0.39}$$

$$S^*S^2 = G^*G^2 \tag{0.40}$$

Remark 17. The following result establishes the relationship between the class of Posimetrically equivalent operators and the class of metrically equivalent operators. That two Posimetrically equivalent operators are metrically equivalent provided they are idempotent.

Theorem 18. Let S, G ∈ B (H) be Posimetrically equivalent, then they metrically equivalent provided they are idempotent.

Proof. Since S and G are Posimetrically equivalent, we have;

$$(S^*S)(S+S^*) = (G^*G)\,(G+G^*) \tag{0.41}$$

$$(S^*SS) + (S^*SS^*) = (G^*GG) + (G^*GG^*) \tag{0.42}$$

$$(S^*S^2) + (S^{*2}S) = (G^*G^2) + (G^{*2}G) \tag{0.43}$$

Since S and G are idempotent; $S = SS$, $S^* = S^*S^*$ and $G = GG$, $G^* = G^*G^*$, hence;

$$(S^*S) + (S^*S) = (G^*G) + (G^*G) \tag{0.44}$$

$$2(S^*S) = 2(G^*G) \tag{0.45}$$

$$S^*S = G^*G. \tag{0.46}$$

hence the proof.

Remark 19. The result below establishes the relationship between posimetrically equivalent operators and almost similarly equivalent operators.

Theorem 20. Let S, G ∈ B(H) be two similar Posimetrically equivalent operators , then they are almost similarly equivalent provided they are isometries and $S + S* = S* + S$ and $G + G* = G* + G$.

Proof. Since S and G are similar Posimetrically equivalent, then there exists an invertible operator N such that;

$$(S^*S)(S+S^*) = N^{-1}\,((G^*G)\,(G+G^*))\,N \tag{0.47}$$

$$S^*SS + S^*SS^* = N^{-1}\,(G^*GG + G^*GG^*)\,N \tag{0.48}$$

Pre-multiplying and post-multiplying both the left hand side of 0.48 by S^* and S andright hand side by G^* and G respectively we get;

$$S^*S^*SS + S^*SS^*S = N^{-1}\,(G^*G^*GG + G^*GG^*G)\,N \tag{0.49}$$

$$S^{*2}S^2 + S^{*2}S^2 = N^{-1}\,(G^{*2}G^2 + T^{*2}G^2)\,N \tag{0.50}$$

$$(S^*S)(S^*S + S^*S) = N^{-1}((G^*G)(G^*G + G^*G))N \tag{0.51}$$

Since S and G are isometries;
From 0.53 and 0.57, S and G are almost similarly equivalent

3. Conclusion

It has been shown that the class of posimetrically equivalent operators is related to the classes of almost similar operators and metrically equivalent operators from the result. This study will be useful in telecommunication industry through solving the pick and novenlinna problem which is useful in code processing through analyzation of signals by using properties of posimetrically equivalent operators.

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