International Journal of Mathematics Trends and Technology
ISSN: 2231-5373/ https://doi.org/10.14445/22315373/IJMTT-V69I4P505
Volume 69 Issue 4, 36-47, April 2023

Original Article

Numerical Simulations of Prey-predator System with Holling Type-IV Functional Response

Ashok Munde

Department of Mathematics, NKSPT's Arts, Science and Commerce College, Badnapur, Maharashtra, India

Corresponding Author : mashok.math@gmail.com

Received: 01 March 2023 Revised: 03 April 2023 Accepted: 15 April 2023 Published: 28 April 2023

Abstract

In this paper, we consider a mathematical model for a prey-predator system with a simplified Holling type-IV functional response. Sufficient conditions are derived for the stability of the system around equilibrium points. By numerical simulation, it shows that the system exhibits rich dynamics under different sets of conditions and by taking the different parameter values of α.

AMS Subject Classifications: 92B05, 92D25, 93A30, 93C15, 93D99.
Keywords: Prey-predator model; Holling type-IV functional response; Equilibrium points; Dynamical behaviour.

1 Introduction

In the study of population dynamics, a functional response is the center of attention in many prey-predator mathematical models. Already, many authors proposed prey-predator models with different types of functional responses and studied global stability, limit cycle, Hopf bifurcation and chaotic behavior of their systems see [2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 16, 17, 19, 22, 23, 24, 25]. Holling [11] describes predator-prey model with Holling type-I functional response. Tripathi et al. [21] discussed the dynamics of prey-predator model with Beddington–DeAngelis type function response. Huang and Xiao [13] examined the prey-predator model with Holling type-IV function of the form

$$R(x) = \frac{mx}{a + bx + x^2}. \qquad (R_1)$$

Already, the functional response (R_1) is proposed by Andrews in [1]. Sokol and Howell proposed the experiment by consideration of simplified Holling type-IV function of the form

$$R(x) = \frac{mx}{a + x^2}, \qquad (R_2)$$

Table 1: Definition of parameters in the model.

Parameter	Definition
r	The maximum growth rate of the prey species.
α	The coefficient of intra-specific competition in the prey species
a_1	The maximum predation rate of the predator species.
a_2	The interaction rate of the predator species.
k	The half-saturation constant.
d	The death rate of the predator species.

and found best kinetics parameters by non-linear least squares techniques see [20]. Ruan and Xiao [18] investigated the predator-prey model with simplified Holling type-IV functional response (R_2). They studied the the saddle-node bifurcation, the supercritical and subcritical Hopf bifurcation, and the homoclinic bifurcation of the system.

In this paper, we consider the model equations of a prey-predator system with Holling type-IV functional response as follows:

$$\begin{aligned}\frac{dN_1}{dt} &= \left(r - \alpha N_1 - \frac{a_1 N_2}{k + N_1^2}\right) N_1 \\ \frac{dN_2}{dt} &= \left(\frac{a_2 N_1}{k + N_1^2} - d\right) N_2\end{aligned} \tag{1}$$

where $N_i, i = 1, 2$ represent the population densities of prey and predator respectively and $r, \alpha, a_i(i = 1, 2), k$ and d are assumed to be nonnegative constants defined in Table 1.

The main purpose of this paper is to study the dynamical behaviour of prey-predator system with simplified Holling type-IV functional response. In Section 2, we identified all possible equilibrium points of the model and discussed stability of the system under sufficient conditions. In Section 3, numerical simulations are performed and shows that the system exhibits rich dynamics under different sets of conditions and by taking the different parameter values of α. Section 4 is the concluding section where results are discussed.

2 Equilibria and their stability

In this section, the existence of the equilibrium points of system (1) and their stability of each one are investigated. The system (1) always has trivial equilibrium point $E_0(0, 0)$ and axial equilibrium point $E_1(\frac{r}{\alpha}, 0)$. The co-existence equilibrium point of the system

(1) is the intersection of the nullclines $f_1(N_1, N_2) = 0$ and $f_2(N_1, N_2) = 0$, where

$$f_1(N_1, N_2) = r - \alpha N_1 - \frac{a_1 N_2}{k + N_1^2} \tag{2}$$

$$f_2(N_1, N_2) = \frac{a_2 N_1}{k + N_1^2} - d. \tag{3}$$

From (3), we have

$$N_1 = \frac{a_2 \pm \sqrt{a_2^2 - 4kd^2}}{2d}. \tag{4}$$

From (2) and (4), it is easy to see that if $a_2^2 - 4kd^2 < 0$, the system (1) does not have co-existing interior equilibrium point and if $a_2^2 - 4kd^2 = 0$, the system (1) has an interior equilibrium point $E_2(\widehat{N_1}, \widehat{N_2})$, where

$$\widehat{N_1} = \frac{a_2}{2d}, \quad \widehat{N_2} = \frac{(r - \alpha\widehat{N_1})(k + \widehat{N_1}^2)}{a_1}$$

exist if $\frac{r}{\alpha} > \widehat{N_1}$ and if $a_2^2 - 4kd^2 > 0$, the system (1) has two interior equilibrium points i.e. $E_3(\overline{N_1}, \overline{N_2})$, where

$$\overline{N_1} = \frac{a_2 - \sqrt{a_2^2 - 4kd^2}}{2d}, \quad \overline{N_2} = \frac{(r - \alpha\overline{N_1})(k + \overline{N_1}^2)}{a_1}$$

exist if $\frac{r}{\alpha} > \overline{N_1}$ and $E_4(N_1^*, N_2^*)$, where

$$N_1^* = \frac{a_2 + \sqrt{a_2^2 - 4kd^2}}{2d}, \quad N_2^* = \frac{(r - \alpha N_1^*)(k + N_1^{*2})}{a_1}$$

exists if $\frac{r}{\alpha} > N_1^*$.

For sake of simplicity, we denote $\widetilde{N_3} = \frac{2a_2 - \sqrt{a_2^2 - 4kd^2}}{2d}$ and $\widetilde{N_4} = \frac{2a_2 + \sqrt{a_2^2 - 4kd^2}}{2d}$. The dynamical behavior of the system (1) at the equilibrium points can be studied by computing of the variational matrix J of the form

$$J = \begin{bmatrix} r - 2\alpha N_1 - \frac{a_1(k - N_1^2)N_2}{(k+N_1^2)^2} & -\frac{a_1 N_1}{k+N_1^2} \\ \frac{a_2(k-N_1^2)N_2}{(k+N_1^2)^2} & \frac{a_2 N_1}{k+N_1^2} - d \end{bmatrix} \tag{5}$$

Theorem 1. *The following statements hold for the system (1):*

(i) The trivial equilibrium point E_0 is always a saddle point.

(ii) The axial equilibrium point E_1 is a stable node if $d > \frac{a_2 r\alpha}{k\alpha^2 + r^2}$ and a saddle point if $d < \frac{a_2 r\alpha}{k\alpha^2 + r^2}$.

Proof. (i) From some calculation, eigenvalues of the variational matrix (5) about the equilibrium point E_0 is $\lambda_1 = -d$ and $\lambda_2 = r$. Hence, the equilibrium point E_0 is always a saddle point.

(ii) The variational matrix (5) about the equilibrium point E_1 is given by

$$J(E_1) = \begin{bmatrix} -r & -\frac{a_1 r\alpha}{k\alpha^2+r^2} \\ 0 & \frac{a_2 r\alpha}{k\alpha^2+r^2} - d \end{bmatrix} \tag{6}$$

The eigenvalues of the variational matrix (6) are $\lambda_1 = -r$ and $\lambda_2 = \frac{a_2 r\alpha}{k\alpha^2+r^2} - d$. Hence, the equilibrium point E_1 is a stable node if $d > \frac{a_2 r\alpha}{k\alpha^2+r^2}$ and E_1 becomes a saddle point for $d < \frac{a_2 r\alpha}{k\alpha^2+r^2}$.

□

We have used the results proved in [18] to explore the dynamical behavior of the system (1) about the interior equilibrium points and stated in the following theorems.

Theorem 2. *The following statements hold for the system (1):*

(i) If $4kd^2 < a_2^2 \le \frac{16}{3}kd^2$ *and* $\frac{r}{N_1^*} < \alpha < \frac{r}{\overline{N_1}}$, *then system (1) has three equilibria: two saddle points* $E_0(0,0)$ *and* $E_1(\frac{r}{\alpha},0)$ *and a globally asymptotically stable equilibrium* $E_3(\overline{N_1},\overline{N_2})$ *in the interior of the first quadrant.*

(ii) If $4kd^2 < a_2^2 \le \frac{16}{3}kd^2$ *and* $\frac{r}{\widetilde{N_3}} \le \alpha < \frac{r}{N_1^*}$, *then system (1) has four equilibria: two saddle points* $E_0(0,0)$ *and* $E_4(N_1^*,N_2^*)$, *a stable node* $E_1(\frac{r}{\alpha},0)$ *and a stable equilibrium point* $E_3(\overline{N_1},\overline{N_2})$, *and system (1) has no closed orbit.*

(iii) If $4kd^2 < a_2^2 \le \frac{16}{3}kd^2$ *and* $\frac{rd}{a_2} < \alpha < \frac{r}{\widetilde{N_3}}$ *then system (1) has four equilibria: two saddle points* $E_0(0,0)$ *and* $E_4(N_1^*,N_2^*)$, *a stable equilibrium point* $E_1(\frac{r}{\alpha},0)$ *and a unstable equilibrium point* $E_3(\overline{N_1},\overline{N_2})$, *and system (1) has unique limit cycle exists in the interior of the first quadrant.*

(iv) If $4kd^2 < a_2^2 \le \frac{16}{3}kd^2$ *and* $\frac{r}{\widetilde{N_4}} \le \alpha \le \frac{rd}{a_2}$ *(or* $\alpha < \frac{r}{\widetilde{N_4}}$*) then system (1) has four equilibria: three saddle points* $E_0(0,0), E_3(\overline{N_1},\overline{N_2})$ *and* $E_4(N_1^*,N_2^*)$ *and* $E_1(\frac{r}{\alpha},0)$ *a stable equilibrium and system (1) has no limit cycle exists (or has no closed orbit).*

Theorem 3. *The following statements hold for the system (1):*

(i) If $\frac{16}{3}kd^2 < a_2^2$ *and* $\frac{r}{\widetilde{N_3}} < \alpha < \frac{r}{\overline{N_1}}$ *then system (1) has three equilibria: two saddle points* $E_0(0,0)$ *and* $E_1(\frac{r}{\alpha},0)$ *and* $E_3(\overline{N_1},\overline{N_2})$ *is a stable focus.*

(ii) If $\frac{16}{3}kd^2 < a_2^2$ *and* $\frac{r}{N_1^*} \le \alpha \le \frac{r}{\widetilde{N_3}}$ *then system (1) has three equilibria: two saddle points* $E_0(0,0)$ *and* $E_1(\frac{r}{\alpha},0)$ *and* $E_3(\overline{N_1},\overline{N_2})$ *is a unstable focus and system (1) has unique limit cycle which is stable.*

Stability of equilibria for parameter values $r = 0.6, a_1 = 1, a_2 = 1.1,$ $k = 0.5, d = 0.7$ and different values of α.

Different values of α	E_0	E_1	E_2	E_3	E_4	Remark
0.75	Saddle	Saddle	-	Stable	-	E_3 is globally asymptotically stable.
0.50	Saddle	Stable	-	Stable	Saddle	No closed orbit exists.
0.44	Saddle	Stable	-	Unstable	Saddle	A stable limit cycle around E_3.
0.35	Saddle	Stable	-	Unstable	Saddle	No limit cycle and closed orbit.
0.24	Saddle	Stable	-	Unstable	Saddle	

Table 2: Dynamical Behaviour of the equilibria shown in Figure 1 to 4.

(iii) If $\frac{16}{3}kd^2 < a_2^2$ and $\frac{rd}{a_2} < \alpha < \frac{r}{N_1^*}$ then system (1) has four equilibria: two saddle points $E_0(0,0), E_4(N_1^*, N_2^*)$ and stable equilibrium point $E_1(\frac{r}{\alpha}, 0)$ and $E_3(\overline{N_1}, \overline{N_2})$ is unstable focus.

(iv) If $\frac{16}{3}kd^2 < a_2^2$ and $\frac{r}{\overline{N_4}} \leq \alpha \leq \frac{rd}{a_2}$ (or $\alpha < \frac{r}{\overline{N_4}}$) then system (1) has four equilibria: two saddle points $E_0(0,0), E_4(N_1^*, N_2^*)$ and stable equilibrium point $E_1(\frac{r}{\alpha}, 0)$ and $E_3(\overline{N_1}, \overline{N_2})$ is unstable focus and system (1) has no limit cycle exists (or no closed orbits).

In the next section, we have carried out numerical simulations to prove and verify the statements shown in Theorem 2 and 3.

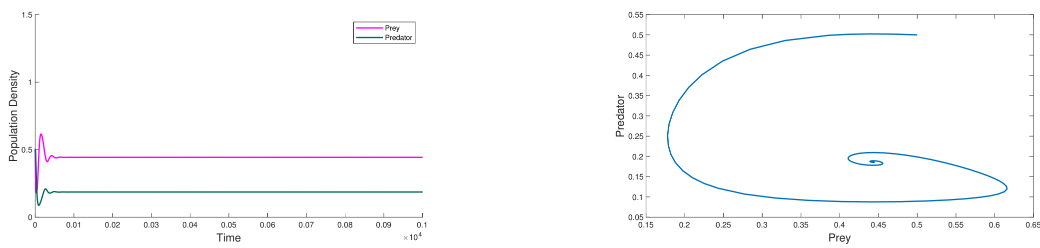
3 Numerical Simulations

We take two sets of parameter values, in the first set we fix $r = 0.6, a_1 = 1, a_2 = 1.1, k = 0.5$ and $d = 0.7$ for numerical simulations with initial condition $N_1(0) = 0.5$ and $N_2(0) = 0.5$ and different values of α. For $\alpha = 0.75$, the interior equilibrium point E_3 is globally asymptotically stable is shown in the Figure-1. Figure-2 represents the interior equilibrium point E_3 is stable focus and system (1) has no closed orbit for $\alpha = 0.50$. If we take $\alpha = 0.44$, then the interior equilibrium point E_3 is unstable and system (1) has unique limit cycle exists as shown in Figure-3. For $\alpha = 0.35$ and 0.24, the interior equilibrium point E_3 is unstable and system (1) has no limit cycle and no closed orbit exists as shown in Figure-4. From the above first set, we have proved the statements in Theorem 2 numerically and the stability of equilibrium points is shown in Table 2.

Stability of equilibria for parameter values $r = 0.6, a_1 = 1, a_2 = 1.1,$ $k = 0.5, d = 0.5$ and different values of α.

Different values of α	E_0	E_1	E_2	E_3	E_4	**Remark**
0.50	Saddle	Saddle	-	Stable	-	No closed orbit exists.
0.44	Saddle	saddle	-	Unstable	-	A stable limit cycle around E_3.
0.30	Saddle	Stable	-	Unstable	Saddle	-
0.24	Saddle	Stable	-	Unstable	Saddle	No limit cycle and closed orbit.
0.17	Saddle	Stable	-	Unstable	Saddle	

Table 3: Dynamical Behaviour of the equilibria shown in Figure 5 to 8.

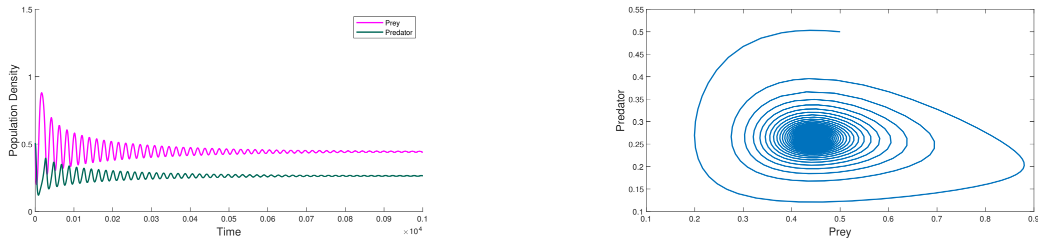


(a) (b)

Figure 1: (1a) time series and (1b) a phase portrait of system (1) for $\alpha = 0.75$.

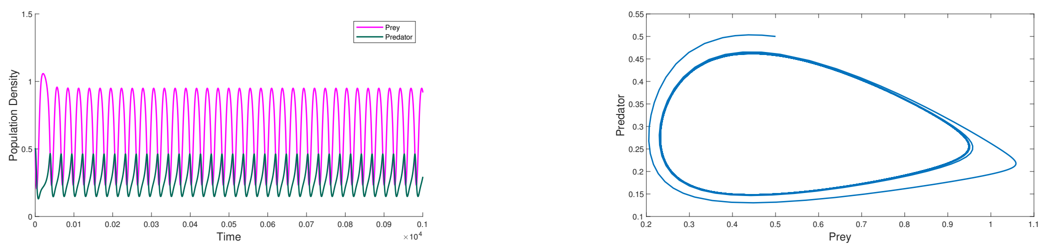
Now, we fix $d = 0.5$ in the second set and remaining parameter values are same as in the first set with initial condition $N_1(0) = 0.5$ and $N_2(0) = 0.5$ and different values of α for the numerical simulations. For $\alpha = 0.50$, the interior equilibrium point E_3 is stable focus and system (1) has no closed orbit as shown in Figure-5. Figure-6 represents the interior equilibrium point E_3 is unstable and system (1) has unique limit cycle exists for $\alpha = 0.44$. If we take $\alpha = 0.30$, then the interior equilibrium point E_3 is unstable as shown in Figure-7. For $\alpha = 0.24$ and 0.17, the interior equilibrium point E_3 is unstable and system (1) has no limit cycle and no closed orbit exists as shown in Figure-8. Based on the parameter values in the second set, we have verified the conditions in Theorem 3 and the stability of equilibrium points is shown in Table 3.

If we take $k = 1.21, \alpha = 0.5$ and the rest of the parameter values are same as in the second set, then we have $a_2^2 - 4kd^2 = 0$. Therefore, the system (1) exhibits three equilibrium points: E_0 is a saddle point, E_1 and E_2 are stable, and dynamical behaviour of the system is shown in Figure-9.



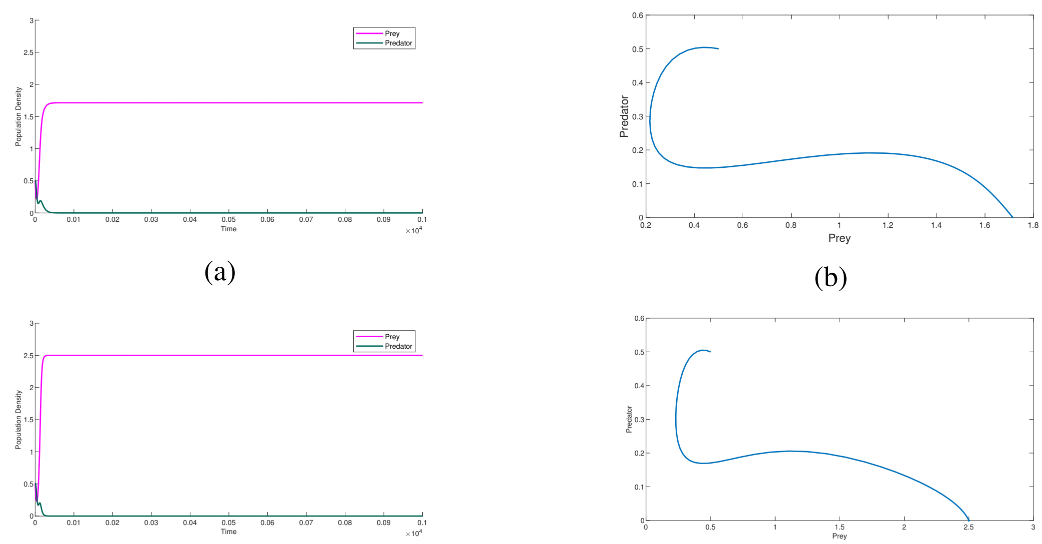
(a) (b)

Figure 2: (2a) time series and (2b) a phase portrait of system (1) for $\alpha = 0.50$.



(a) (b)

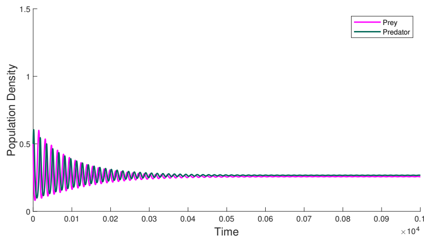
Figure 3: (3a) time series and (3b) a phase portrait of system (1) for $\alpha = 0.44$.



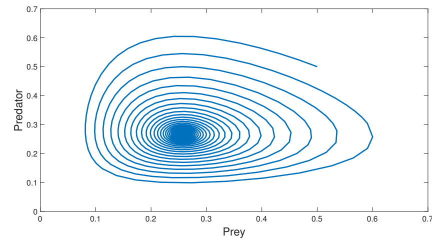
(a) (b)

(c) (d)

Figure 4: (4a) time series and (4b) a phase portrait for $\alpha = 0.35$ and (4c) time series and (4d) a phase portrait of system (1) for $\alpha = 0.24$.

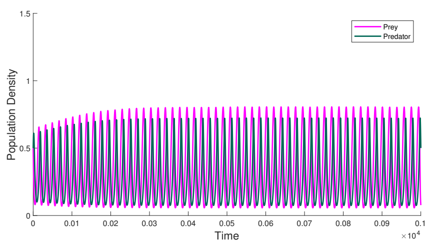


(a)

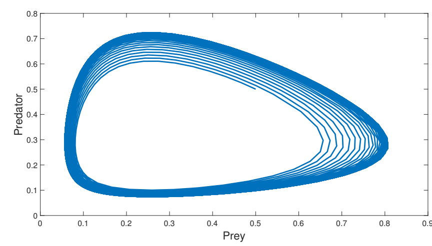


(b)

Figure 5: (5a) time series and (5b) a phase portrait of system (1) for $\alpha = 0.50$.

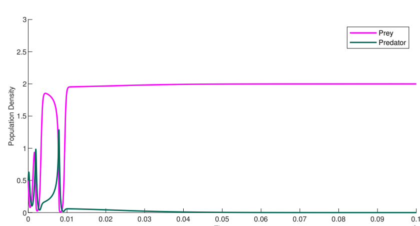


(a)

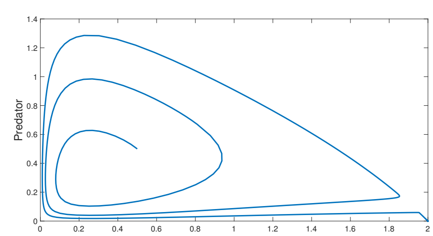


(b)

Figure 6: (6a) time series and (6b) a phase portrait of system (1) for $\alpha = 0.44$.

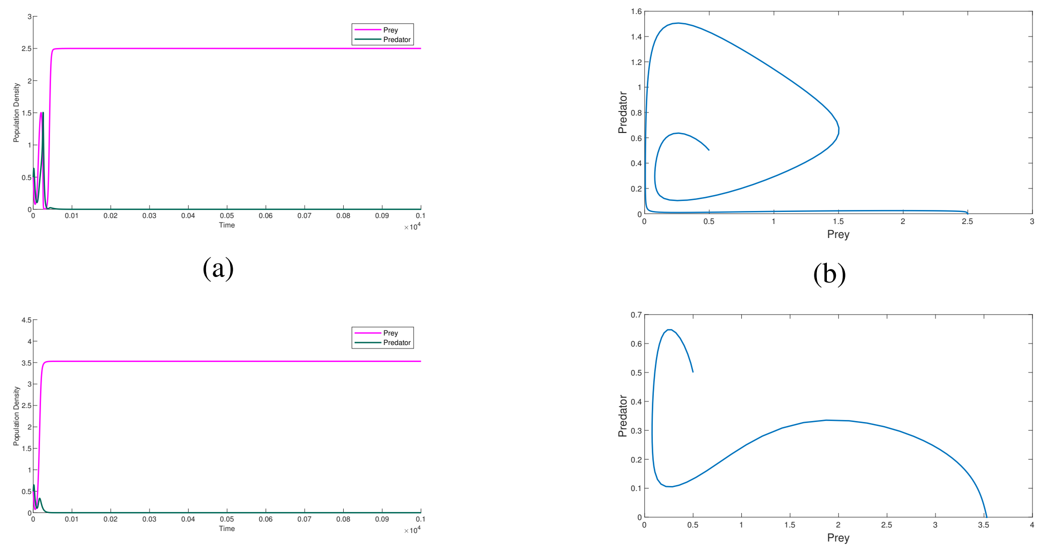


(a)



(b)

Figure 7: (7a) time series and (7b) a phase portrait of system (1) for $\alpha = 0.30$.

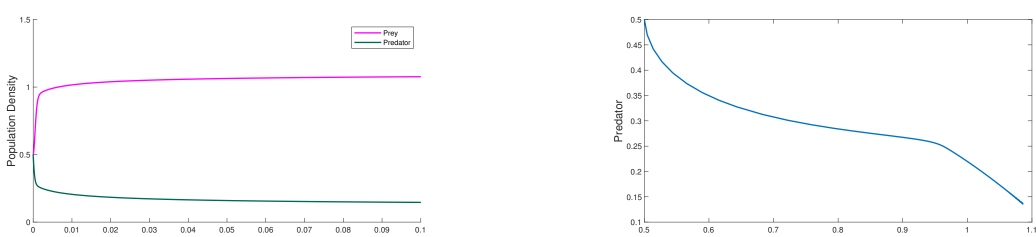


(a) (b) (c) (d)

Figure 8: (8a) time series and (8b) a phase portrait for $\alpha = 0.24$ and (8c) time series and (8d) a phase portrait of system (1) for $\alpha = 0.17$.

4 Conclusion

In this paper, the dynamical behaviour of the prey-predator system have been studied. The interaction between prey and predator is assumed to be governed by simplified Holling type-IV functional response. By using the results of Ruan and Xiao [18], we carried out the numerical simulations and exhibits the dynamical behaviour of the system (1). We consider two sets of parameter values and shows that the dynamical behaviour is rich and very sensitive for different values of α (see Figures 1 to 9).



(a) (b)

Figure 9: (9a) time series and (9b) a phase portrait of system (1) for $r = 0.6, \alpha = 0.5, a_1 = 1, a_2 = 1.1, k = 1.21$ and $d = 0.5$.

Conflicts of Interest

The author declare that there are no conflicts of interest.

Funding Statement

This research work is not supported by any funding agencies.

References

[1] J. F. Andrews, ”A mathematical model for the continuous culture of microoganisms utilizing inhibitory substrates”, *Biotechnol. Bioeng.,* vol. 10, pp. 707-723, 1968. [CrossRef] [Google Scholar] [Publisher Link]

[2] H. Baek and C. Jung, ”Extinction and permanence of a Holling I type impulsive predator-prey model”, *Kyungpook Math. J.,* vol. 49, no. 4 pp. 763-770, 2009. [Google Scholar] [Publisher Link]

[3] H. Baek, ”On the dynamical behavior of a two-prey one-predator system with two-type functional responses”, *Kyungpook Math. J.,* vol. 53, pp. 674-660, 2013. [Google Scholar] [Publisher Link]

[4] P.A. Braza, ”The bifurcation structure of the Holling-Tanner model for predator-prey interactions using two timing”, *SIAM Journal on Applied Mathematics,* vol. 63, no. 3, pp. 889-904, 2003. [CrossRef] [Google Scholar] [Publisher Link]

[5] L. Cheng and L. Zhang, ”Bogdanov–Takens bifurcation of a Holling IV prey–predator model with constant-effort harvesting”, *J. Inequal. Appl.,* vol. 68, 2021. [CrossRef] [Google Scholar] [Publisher Link]

[6] J. M. Cushing, ”Periodic Lotka-Volterra competition equations”, *J. Math. Biol.,* vol. 24, pp. 381-403, 1986. [CrossRef] [Google Scholar] [Publisher Link]

[7] M. Fan and Y. Kuang, ”Dynamics of a nonautonomous predator-prey system with Beddington-DeAnglis functional response”, *Journal of Mathematical Analysis and Applications,* vol. 295, no. 1, pp. 15-39, 2004. [CrossRef] [Google Scholar] [Publisher Link]

[8] H. I. Freedman and R. M. Mathsen, ”Persistence in predator-prey systems with ratio-dependent predator influence”, *Bulletin of Mathematical Biology,* vol. 55, no. 4, pp. 817-827, 1993. [Google Scholar] [Publisher Link]

[9] S. Gakkhar and B. Singh, ”The dynamics of a food web consisting of two preys and a harvesting predator”, *Chaos Solitons and Fractals,* vol. 34, pp. 1346-1356, 2007. [CrossRef] [Google Scholar] [Publisher Link]

[10] S. Gakkhar, B. Singh and R. K. Naji, ”Dynamical Behavior of two predators competing over a single prey”, *Biosystems,* vol. 90, pp. 808-817, 2007. [CrossRef] [Google Scholar] [Publisher Link]

[11] C. S. Holling, ”The Functional Response of Predators to Prey Density and its Role in Mimicry and Population Regulation”, *Mem. Entomolog. Soc. Can.,* vol. 45, pp. 5-60, 1965. [CrossRef] [Google Scholar] [Publisher Link]

[12] Jc. Huang, ”Bifurcations and Chaos in a Discrete Predator-prey System with Holling Type-IV Functional Response”, *Acta Mathematicae Applicatae Sinica, English Series,* vol. 21, pp. 157–176, 2005. [CrossRef] [Google Scholar] [Publisher Link]

[13] Jc. Huang and Dm. Xiao, ”Analyses of Bifurcations and Stability in a Predator-prey System with Holling Type-IV Functional Response”, *Acta Mathematicae Applicatae Sinica, English Series,* vol. 20, pp. 167-178 2004. [CrossRef] [Google Scholar] [Publisher Link]

[14] Li Lin and Z. Wencai, ”Deterministic and stochastic dynamics of a modified Leslie-Gower prey-predator system with simplified Holling-type IV scheme”, *Math. Biosci. Eng.,* vol. 18 no. 3, pp. 2813-2831, 2021. [Google Scholar] [Publisher Link]

[15] D. Mukherjee, ”Fear induced dynamics on Leslie-Gower predator-prey system with Holling-type IV functional response” *Jambura J. Biomath.,* vol. 3 no. 2, pp. 49–57, 2022. [CrossRef] [Google Scholar] [Publisher Link]

[16] R. K. Naji and R. N. Shalan, ”The Dynamics of Holling Type IV Prey Predator Model with Intra- Specific Competition” *Iraqi Journal of Science,* vol. 54 no. 2, pp. 386-396, 2013. [Google Scholar] [Publisher Link]

[17] C. Qiaoling, T. Zhidong and Hu Zengyun, ”Bifurcation and control for a discrete-time prey–predator model with Holling-IV functional response”, *International Journal of Applied Mathematics and Computer Science,* vol. 23 no. 2, pp. 247-261, 2013. [CrossRef] [Google Scholar] [Publisher Link]

[18] S. Ruan and D. Xiao, ”Global Analysis in a Predator-Prey System with Nonmonotonic Functional Response” *SIAM J. Appl. Math.,* vol. 61, pp. 1445-1472, 2001. [CrossRef] [Google Scholar] [Publisher Link]

[19] C. Shen, ”Permanence and global attractivity of the food-chain system with Holling IV type functional response”, *Appl. Math and Comp.,* vol. 194, pp. 179-185, 2007. [CrossRef] [Google Scholar] [Publisher Link]

[20] W. Sokol and J. A. Howell, ”Kinetics of phenol oxidation by washed cells”, *Biot. Bioe.,* vol. 30, pp. 921-927, 1987. [CrossRef] [Google Scholar] [Publisher Link]

[21] J. P. Tripathi S. Abbas and M. Thakur, ”Dynamical analysis of a prey–predator model with Beddington–DeAngelis type function response incorporating a prey refuge”, *Nonlinear Dyn.,* vol. 80, pp. 177–196, 2015. [CrossRef] [Google Scholar] [Publisher Link]

[22] J. P. Tripathi, S. Abbas and M. Thakur, ”Local and global stability analysis of a two prey one predator model with help”, *Commun Nonlinear Sci. Numer. Simulat.,* vol. 19, pp. 3284-3297, 2014. [CrossRef] [Google Scholar] [Publisher Link]

[23] L. Xinxin and H. Qingdao, ”Analysis of optimal harvesting of a predator-prey model with Holling type IV functional response”, *Ecological Complexity,* vol. 42, 100816, 2020. [CrossRef] [Google Scholar] [Publisher Link]

[24] D. Yanfei and Z. Yulin, ”Hopf Cyclicity and Global Dynamics for a Predator–Prey System of Leslie Type with Simplified Holling Type IV Functional Response”, *International Journal of Bifurcation and Chaos,* vol. 28 no. 13, 1850166, 2018. [CrossRef] [Google Scholar] [Publisher Link]

[25] Z. Zhang, R. K. Upadhyay and J. Datta, ”Bifurcation analysis of a modified Leslie–Gower model with Holling type-IV functional response and nonlinear prey harvesting”, *Adv. Differ. Equ.,* vol. 127, 2018. [CrossRef] [Google Scholar] [Publisher Link]