International Journal of Mathematics Trends and Technology
ISSN: 2231-5373/ https://doi.org/10.14445/22315373/IJMTT-V69I4P507

Volume 69 Issue 4, 54-68, April 2023

Original Article

Some New Families of Edge, Face Edge and Total Face Edge Sum Divisor Cordial Graphs

G. Vijayalakshmi[1], M. Mohamed Sheriff[2], P. Lawrence Rozario Raj[3]

[1]School of Mathematics, Madurai Kamaraj University, Madurai, Tamil Nadu, India.
[2]P.G. and Research Department of Mathematics, Hajee Karutha Rowther Howdia College, Uthamapalayam, Affiliated to Madurai Kamaraj University, Tamil Nadu, India.
[3]P.G. and Research Department of Mathematics, St. Joseph's College, Tiruchirappalli, Affiliated to Bharathidasan University, Tiruchirappalli, India.

Received: 05 March 2023 Revised: 05 April 2023 Accepted: 17 April 2023 Published: 30 April 2023

Abstract *- In this paper, we investigate some new families of edge sum divisor cordial graphs, face edge sum divisor cordial graphs and total face edge sum divisor cordial graphs.*

Keywords *- Edge sum divisor cordial labelling, Edge sum divisor cordial graph, face edge sum divisor cordial graph, total face edge sum divisor cordial graph, switching of any vertex.*

1. Introduction

We begin with a simple, finite, planar, undirected graph. A (p,q) planar graph G means a graph G = (V, E), where V is the set of vertices with |V| = p, E is the set of edges with |E| = q, and F is the set of interior faces of G with |F| = the number of interior faces of G. For standard terminology and notations related to graph theory we refer to Harary [8], Chartrand [4] and Bondy and Murthy [1]. For number theory, we refer to Burton [2], and for graph labelling, we refer to Gallian [6]. In 1967, Rosa[18] introduced the concept of β-valuation of graph G. Initially, Rosa named the above-defined labelling as β-valuation, but Golomb[7] renamed β - valuation as graceful labelling. In [3], Cahit introduced the concept of cordial labelling of graphs. Lawrence et al. discussed various types of splitting cordial graphs in [9]. In [25], Yilmaz et al. introduced the concept of the E-cordial labelling of graphs. The concept of total edge product cordial labelling was introduced by Vaidya et al. [20]. Varatharajan et al.[21] introduced the concept of divisor cordial labelling of graphs. The sum divisor cordial labelling concept was introduced by Lourdusamy et al. in [12]. Lawrence et al. introduced the concept of face and total face product cordial labelling of graphs in [11]. The concept of face edge and total face edge product cordial labelling of graphs was introduced by Lawrence et al. in [10]. Mohamed Sheriff et al. introduced the concept of face integer cordial labelling of graphs in [13]. In [14], Mohamed Sheriff et al. introduced the concept of face integer edge cordial labelling of graphs in [15]. In [5], Farhana Abbas et al. introduced the concept of total face integer edge cordial labelling of graphs. Mohamed Sheriff et al. introduced the concept of face sum divisor cordial labelling of the graph in [15]. The face sum divisor cordial labelling of cycle-related graphs is discussed in [17]. In [16], the face sum divisor cordial labelling of wheel-related graphs is studied. In [23], Vijayalakshmi et al. introduced the concept of edge sum divisor cordial labelling of graphs. Vijayalakshmi et al. introduced the concept of total face edge sum divisor cordial labelling of graphs in [22]. In [24], Vijayalakshmi et al. introduced the concept of face edge sum divisor cordial labelling of graphs. In this paper, we investigate some new families of edge sum divisor cordial graphs, face edge sum divisor cordial graphs and total face edge sum divisor cordial graphs.

Definition 1.1 Let a and b be two integers. If a divides b, there is a positive integer k such that b = ka. It is denoted by a|b. If a does not divide b, then we denote $a \nmid b$.

Definition 1.2 Let G = (V(G), E(G)) be a simple graph and $f : V(G) \rightarrow \{1,2,\ldots,|V(G)|\}$ be a bijection. For each edge uv, assign the label 1 if $2|(f(u)+f(v))$ and the label 0 otherwise. The function f is called a sum divisor cordial labelling if $|e_f(0)-e_f(1)| \leq 1$. A graph which admits a sum divisor cordial labelling is called a sum divisor cordial graph.

Definition 1.3 Let G = (V(G),E(G)) be a simple graph and $f : E(G) \rightarrow \{1,2,...,|E(G)|\}$ be a bijection. For each vertex v, assign the label 1 if $2 \mid f(a_1)+f(a_2)+\ldots+f(a_s)$ and the label 0 otherwise where $a_1,a_2,\ldots, a_s$ are edges incident with the vertex v. The function f is called an edge sum divisor cordial labelling if the number of vertices labelled with 0 and the number of vertices labelled with 1 differ by at most 1. A graph which admits an edge sum divisor cordial labelling is called an edge sum divisor cordial graph.

Definition 1.4 A face edge sum divisor cordial labelling of a graph G with edge set E is a bijection f from E(G) to {1,2,…, |E(G)|} such that a vertex v is assigned the label 1 if 2 divides $f(a_1)+f(a_2)+\ldots+f(a_s)$ and 0 otherwise where $a_1,a_2,\ldots, a_s$ are edges incident with the vertex v and for face f is assigned the label 1 if 2 divides $f(b_1)+f(b_2)+\ldots+f(b_t)$ and 0 otherwise, where $b_1,b_2,\ldots, b_t$ are edges corresponding to the face f. Also, the number of vertices labelled with 0 and the number of vertices labelled with 1 differ by at most 1, and the number of faces labelled with 0 and the number of faces labelled with 1 differ by at most 1. A graph which admits a face-edge sum divisor cordial labelling is called a face-edge sum divisor cordial graph.

Definition 1.5 A total face edge sum divisor cordial labelling of a graph G with edge set E is a bijection g from E(G) to {1,2,…, |E(G)|} such that a vertex v is assigned the label 1 if 2 divides $f(a_1)+f(a_2)+\ldots+f(a_s)$ and 0 otherwise where $a_1,a_2,\ldots,a_s$ as are edges incident with the vertex v and for face f is assigned the label 1 if 2 divides $f(b_1)+f(b_2)+\ldots+f(b_t)$ and 0 otherwise, where $b_1,b_2,\ldots,b_t$ are edges corresponding to the face f. Also, the number of vertices and faces labelled with 0 and the number of vertices and faces labelled with 1 differ by at most 1. A graph which admits a total face edge sum divisor cordial labelling is called a total face edge sum divisor cordial graph.

Definition 1.6 [19] For graph G, the splitting graph S′(G) of a graph G is obtained by adding a new vertex v′ corresponding to each vertex v of G such that N(v) = N(v′).

Theorem 1.1 [23] Necessary condition for a graph G with n vertices to admit an edge sum divisor cordial labelling is that $n \not\equiv 2 \pmod 4$.

2. Main Theorems

Theorem 2.1 The graph obtained by switching of an apex vertex in Helm H_n admits an edge sum divisor cordial labelling for $n \geq 3$.

Proof. Let $v,v_1,v_2,\ldots,v_{2n}$ be vertices and $e_1,e_2,\ldots,e_{3n}$ be edges of Helm H_n, where v is the apex vertex in H_n. G_v denotes the graph, which is obtained by switching of a vertex v of H_n. Let G be a graph G_v. Now $v_1,v_2,\ldots,v_{2n}$ be vertices and $e_1,e_2,\ldots,e_{3n}$ be edges of G_v, where $e_i = vv_{n+i}$ for i = 1,2,…,n, $e_{n+i} = v_iv_{i+1}$ for i = 1,2,…,n–1, $e_{2n} = v_nv_1$ and $e_{2n+i} = v_iv_{n+i}$ for i = 1,2,…,n.

Then |V(G)| = 2n+1 and |E(G)| = 3n. Define $f : E(G) \to \{1,2,\ldots, |E(G)|\}$ as follows.

$f(e_i) = 2i-1$, for $1 \le i \le n$, $f(e_{n+i}) = 2i$, for $1 \le i \le n$, $f(e_{2n+i}) = 2n+i$, for $1 \le i \le n$

Then induced vertex labels are $f^*(v) = 0$, if n is odd and $f^*(v) = 1$, if n is even

For n is odd

$$f^*(v_i) = 0, \text{ for } 1 \le i \le \frac{n+1}{2},\ f^*(v_{\frac{n+1}{2}+i}) = 1, \text{ for } 1 \le i \le \frac{n-1}{2},$$

$$f^*(v_{n+i}) = 1, \text{ for } 1 \le i \le \frac{n+1}{2} \text{ and } f^*(v_{\frac{3n+1}{2}+i}) = 0, \text{ for } 1 \le i \le \frac{n-1}{2}$$

For n is odd $f^*(v_{2i-1}) = 0$, for $1 \le i \le n$ and $f^*(v_{2i}) = 1$, for $1 \le i \le n$

For n is even

$$f^*(v_{2i-1}) = 0, \text{ for } 1 \le i \le \frac{n}{2},\ f^*(v_{2i}) = 1, \text{ for } 1 \le i \le \frac{n}{2},\ f^*(v_{n+2i-1}) = 1, \text{ for } 1 \le i \le \frac{n}{2},\ f^*(v_{n+2i}) = 0, \text{ for } 1 \le i \le \frac{n}{2}$$

In view of the above-defined labelling pattern, we have $v_f(1) = n+1$ and $v_f(0) = n$, when n is even. Then $| v_f(0) - v_f(1) | \le 1$.

$v_f(1) = n$ and $v_f(0) = n+1$, when n is odd. Then $| v_f(0) - v_f(1) | \le 1$.

Thus, the graph obtained by switching of an apex vertex in Helm H_n admits an edge sum divisor cordial labelling for $n \ge 3$.

Illustration 2.1 The switching of an apex vertex in Helm H_4 and its edge sum divisor cordial labelling is shown in Figure 2.1.

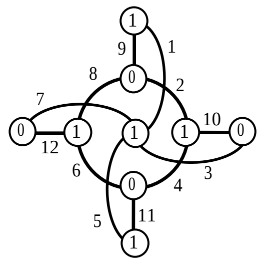

Fig. 2.1

Theorem 2.2 The graph obtained by switching of an apex vertex in Closed Helm CH_n admits an edge sum divisor cordial labelling for $n \geq 3$.

Proof. Let $v, v_1, v_2, ..., v_{2n}$ be vertices and $e_1, e_2, ..., e_{4n}$ be edges of Closed Helm CH_n, where v is the apex vertex in CH_n. G_v denotes the graph, which is obtained by switching of a vertex v of CH_n. Let G be a graph G_v. Now $v_1, v_2, ..., v_{2n}$ be vertices and $e_1, e_2, ..., e_{4n}$ be edges of G_v, where $e_i = vv_{n+i}$ for $i = 1, 2, \ldots, n$, $e_{n+i} = v_i v_{i+1}$ for $i = 1, 2, \ldots, n-1$, $e_{2n} = v_n v_1$, $e_{2n+i} = v_i v_{n+i}$ for $i = 1, 2, \ldots, n$, $e_{3n+i} = v_{n+i} v_{n+i+1}$ for $i = 1, 2, \ldots, n-1$ and $e_{4n} = v_{2n} v_{n+1}$. Then $|V(G)| = 2n+1$ and $|E(G)| = 4n$.

Define $f : E(G) \rightarrow \{1, 2, \ldots, |E(G)|\}$ as follows.

$f(e_i) = 2i-1$, for $1 \leq i \leq n$, $f(e_{n+i}) = 2i$, for $1 \leq i \leq n$

$f(e_{2n+i}) = 2n+2i-1$, for $1 \leq i \leq n$ and $f(e_{3n+i}) = 2n+2i$, for $1 \leq i \leq n$

Then induced vertex labels are $f^*(v) = 0$, if n is odd and $f^*(v) = 1$, if n is even

$f^*(v_i) = 0$, for $1 \leq i \leq n$ and $f^*(v_{n+i}) = 1$, for $1 \leq i \leq n$

In view of the above-defined labelling pattern, we have $v_f(1) = n+1$ and $v_f(0) = n$, when n is even. Then $|v_f(0) - v_f(1)| \leq 1$.

$v_f(1) = n$ and $v_f(0) = n+1$, when n is odd. Then $|v_f(0) - v_f(1)| \leq 1$.

Thus, the graph obtained by switching of an apex vertex in Closed Helm CH_n admits an edge sum divisor cordial labelling for $n \geq 3$.

Illustration 2.2 The switching of an apex vertex in Closed Helm CH_4 and its edge sum divisor cordial labelling is shown in Figure 2.2.

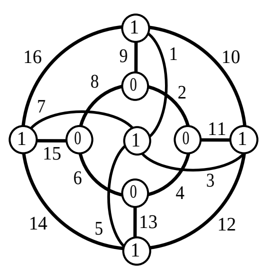

Fig. 2.2

Theorem 2.3 The graph obtained by duplication of an arbitrary vertex of C_n admits edge sum divisor cordial graph for all $n \geq 3$ and $n \not\equiv 1 \pmod 4$.

Proof. Let $v_1, v_2, ..., v_n$ be vertices and $e_1, e_2, ..., e_n$ be an edge of cycle C_n. Let G be the graph obtained by duplicating an arbitrary vertex of C_n. Without loss of generality, let this vertex be v_2, and the newly added vertex be v_2'.

Let $v_1, v_2, ..., v_n, v_2'$ be vertices and $e_1, e_2, ..., e_n, e', e''$ be an edges of G, where $e_i = v_i v_{i+1}$ for $i = 1, 2, \ldots, n-1$, $e_n = v_n v_1$, $e' = v_n v_2'$ and $e'' = v_2 v_2'$. Then $|V(G)| = n+1$ and $|E(G)| = n+2$.

To define $f: E(G) \rightarrow \{0, 1\}$, three cases are to be considered.

Case 1: n = 3

$f(e_1) = 1$, $f(e_2) = 3$, $f(e_3) = 2$, $f(e') = 4$ and $f(e'') = 5$.

Then induced vertex labels are $f^*(v_1) = 1$, $f^*(v_2) = 0$, $f^*(v_3) = 1$ and $f^*(v_2') = 0$.

In view of the above-defined labelling pattern, we have $v_f(0) = v_f(1) = 2$. Then $|v_f(0) - v_f(1)| \leq 1$.

Thus the graph obtained by duplication of an arbitrary vertex of cycle C_3 is an edge sum divisor cordial graph.

Case 2: n = 4

$f(e_1) = 1, f(e_2) = 3, f(e_3) = 2, f(e_4) = 4, f(e') = 5$ and $f(e'') = 6$.

Then induced vertex labels are $f^*(v'_2) = 0$, $f^*(v_1) = 1$, $f^*(v_3) = 0$ and $f^*(v_{2i}) = 1$, for $1 \le i \le 2$

In view of the above-defined labelling pattern, we have $v_f(1) = v_f(0)+1 = 3$. Then $|v_f(0) - v_f(1)| \le 1$.

Thus the graph obtained by duplication of an arbitrary vertex of cycle C_4 is an edge sum divisor cordial graph.

Case 3: n > 4

Sub Case 3.1: $n \equiv 0 \pmod 4$

$f(e_1) = 1, f(e_2) = 3, f(e_3) = 2, f(e_4) = 4, f(e_{i+4}) = f(e_i)+4$, for $1 \le i \le n-4$, $f(e') = n+1$ and $f(e'') = n+2$.

Then induced vertex labels are $f^*(v'_2) = 0$, $f^*(v_1) = 1$, $f^*(v_{2i-1}) = 0$, for $2 \le i \le \frac{n}{2}$ and $f^*(v_{2i}) = 1$, for $1 \le i \le \frac{n}{2}$

In view of the above-defined labelling pattern, we have $v_f(1) = v_f(0)+1 = \frac{n+2}{2}$. Then $|v_f(0) - v_f(1)| \le 1$.

Thus the graph obtained by duplication of an arbitrary vertex of cycle C_n is an edge sum divisor cordial graph, where $n \equiv 0 \pmod 4$.

Sub Case 3.2: $n \equiv 1 \pmod 4$

It is not possible to have an edge sum divisor labelling of the graph obtained by duplication of an arbitrary vertex of cycle C_n for $n \equiv 1 \pmod 4$ by Theorem 1.1.

Thus the graph obtained by duplication of an arbitrary vertex of cycle C_n is not an edge sum divisor cordial graph, where $n \equiv 1 \pmod 4$.

Sub Case 3.3: $n \equiv 2 \pmod 4$

$f(e_1) = 1, f(e_2) = 3, f(e_3) = 2, f(e_4) = 4, f(e_{i+4}) = f(e_i)+4$, for $1 \le i \le n-5$, $f(e_n) = n$, $f(e') = n+1$ and $f(e'') = n+2$.

Then induced vertex labels are $f^*(v'_2) = 0$, $f^*(v_1) = 1$, $f^*(v_{2i-1}) = 0$ for $2 \le i \le \frac{n}{2}$, $f^*(v_{2i}) = 1$ for $1 \le i \le \frac{n-2}{2}$ and $f^*(v_n) = 0$. In view of the above-defined labelling pattern, we have $v_f(0) = v_f(1)+1 = \frac{n+2}{2}$. Then $|v_f(0) - v_f(1)| \le 1$.

Thus the graph obtained by duplication of an arbitrary vertex of cycle C_n is an edge sum divisor cordial graph, where $n \equiv 2 \pmod 4$.

Sub Case 3.4: $n \equiv 3 \pmod 4$

$f(e_1) = 1, f(e_2) = 3, f(e_3) = 2, f(e_4) = 4, f(e_{i+4}) = f(e_i)+4$, for $1 \le i \le n-4$, $f(e') = n+1$ and $f(e'') = n+2$.

Then induced vertex labels are $f^*(v'_2) = 0$, $f^*(v_1) = 0$, $f^*(v_3) = 1$, $f^*(v_{2i-1}) = 0$ for $3 \le i \le \frac{n+1}{2}$ and $f^*(v_{2i}) = 1$ for $1 \le i \le \frac{n-1}{2}$. In view of the above-defined labelling pattern, we have $v_f(0) = v_f(1) = \frac{n+1}{2}$. Then $|v_f(0) - v_f(1)| \le 1$.

Thus the graph obtained by duplication of an arbitrary vertex of cycle C_n is an edge sum divisor cordial graph, where $n \equiv 3 \pmod 4$.

Therefore, the graph obtained by duplication of an arbitrary vertex of cycle C_n is an edge sum divisor cordial graph for all $n \ge 3$ such that $n \not\equiv 1 \pmod 4$.

Illustration 2.3 The graph obtained by duplication of an arbitrary vertex of cycle C_7 and its edge sum divisor cordial labelling is shown in Figure 2.3.

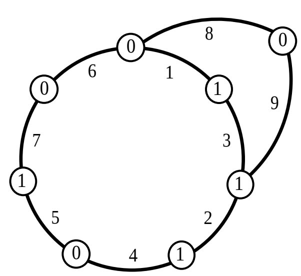

Fig. 2.3

Theorem 2.4 The graph obtained by duplication of an arbitrary edge of C_n admits an edge sum divisor cordial graph for all $n \geq 3$ and $n \not\equiv 0 (\text{mod } 4)$.

Proof. Let $v_1, v_2, ..., v_n$ be vertices and $e_1, e_2, ..., e_n$ be an edges of cycle C_n. Let G be the graph obtained by duplicating an arbitrary edge of C_n. Without loss of generality, let this vertex be e_1, and the newly added edge be e'_1, where $e'_1 = v'_1 \, v'_2$.

Let $v_1, v_2, ..., v_n$, v'_1, v'_2 be vertices and $e_1, e_2, ..., e_n$, e'_1, e', e'' be an edges of G, where $e_i = v_i v_{i+1}$ for $i = 1,2,\ldots,n-1$, $e_n = v_n v_1$, $e' = v_n v'_1$ and $e'' = v_3 v'_2$. Then $|V(G)| = n+1$ and $|E(G)| = n+2$. To define $f: E(G) \to \{0, 1\}$, two cases are to be considered.

Case 1: n = 3

$f(e_1) = 1$, $f(e_2) = 3$, $f(e_3) = 2$, $f(e') = 4$, $f(e'_1) = 6$ and $f(e'') = 5$.

Then induced vertex labels are $f^*(v_1) = 1$, $f^*(v_2) = 1$, $f^*(v_3) = 0$, $f^*(v'_1) = 1$ and $f^*(v'_2) = 0$.

In view of the above-defined labelling pattern, we have $v_f(1) = v_f(0)+1 = 3$. Then $|v_f(0) - v_f(1)| \leq 1$.

Thus the graph obtained by duplication of an arbitrary edge of cycle C_3 is an edge sum divisor cordial graph.

Case 2: $n \geq 4$

Sub Case 2.1: $n \equiv 0$ (mod 4)

It is not possible to have an edge sum divisor labelling of the graph obtained by duplication of an arbitrary edge of cycle C_n for $n \equiv 0$(mod 4) by Theorem 1.1. Thus the graph obtained by duplication of an arbitrary edge of cycle C_n is an edge sum divisor cordial graph, where $n \equiv 0$ (mod 4).

Sub Case 2.2: $n \equiv 1$ (mod 4)

$f(e_1) = 1$, $f(e_2) = 3$, $f(e_3) = 2$, $f(e_4) = 4$, $f(e_{i+4}) = f(e_i)+4$, for $1 \leq i \leq n-4$, $f(e') = n+1$, $f(e'_1) = n+2$ and $f(e'') = n+3$.

Then induced vertex labels are $f^*(v'_1) = 0$, $f^*(v'_2) = 0$, $f^*(v_1) = 1$, $f^*(v_{2i-1}) = 0$, for $2 \leq i \leq \frac{n+1}{2}$ and $f^*(v_{2i}) = 1$, for $1 \leq i \leq \frac{n-1}{2}$. In view of the above-defined labelling pattern, we have $v_f(0) = v_f(1)+1 = \frac{n+3}{2}$. Then $|v_f(0) - v_f(1)| \leq 1$.

Thus the graph obtained by duplication of an arbitrary edge of cycle C_n is an edge sum divisor cordial graph, where $n \equiv 1$ (mod 4).

Sub Case 2.3: $n \equiv 2$ (mod 4)

$f(e_1) = 1$, $f(e_2) = 3$, $f(e_3) = 2$, $f(e_4) = 4$, $f(e_{i+4}) = f(e_i)+4$, for $1 \leq i \leq n-4$, $f(e') = n$, $f(e'_1) = n+3$ and $f(e'') = n+2$.

Then induced vertex labels are $f^*(v'_1) = 0$, $f^*(v'_2) = 0$, $f^*(v_1) = 1$, $f^*(v_{2i-1}) = 0$, for $2 \leq i \leq \frac{n}{2}$ and $f^*(v_{2i}) = 1$, for $1 \leq i \leq \frac{n}{2}$. In view of the above-defined labelling pattern, we have $v_f(0) = v_f(1) = \frac{n}{2}$. Then $|v_f(0) - v_f(1)| \leq 1$.

Thus the graph obtained by duplication of an arbitrary edge of cycle C_n is an edge sum divisor cordial graph, where $n \equiv 2$ (mod 4).

Sub Case 2.4: $n \equiv 3$ (mod 4)

$f(e_1) = 1$, $f(e_2) = 3$, $f(e_3) = 2$, $f(e_4) = 4$, $f(e_{i+4}) = f(e_i)+4$, for $1 \leq i \leq n-4$, $f(e') = n$, $f(e'_1) = n+3$ and $f(e'') = n+2$.

Then induced vertex labels are $f^*(v'_1) = 0$, $f^*(v'_2) = 1$, $f^*(v_1) = 1$, $f^*(v_{2i-1}) = 0$, for $2 \leq i \leq \frac{n-1}{2}$ and $f^*(v_{2i}) = 1$, for $1 \leq i \leq \frac{n-1}{2}$. In view of the above-defined labelling pattern, we have $v_f(1) = v_f(0)+1 = \frac{n+3}{2}$. Then $|v_f(0) - v_f(1)| \leq 1$.

Thus the graph obtained by duplication of an arbitrary edge of cycle C_n is an edge sum divisor cordial graph, where $n \equiv 3$ (mod 4).

Therefore, the graph obtained by duplication of an arbitrary edge of cycle C_n is an edge sum divisor cordial graph for all $n \geq 3$ such that $n \not\equiv 0$ (mod 4).

Illustration 2.4 The graph obtained by duplication of an arbitrary edge of cycle C_7 and its edge sum divisor cordial labeling is shown in Figure 2.4.

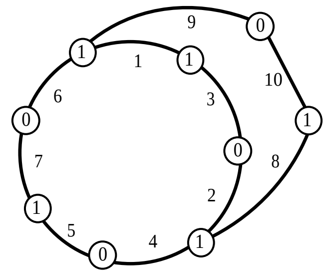

Fig. 2.4

Theorem: 2.5 Switching of any vertex in cycle C_n admits face edge sum divisor cordial labelling for $n \geq 5$ such that $n \not\equiv 2 \pmod 4$.

Proof.

Let $v_1,v_2,\ldots,v_n$ be the successive vertices of C_n. G_v denotes the graph, which is obtained by switching of a vertex v of C_n. Without loss of generality, let the switched vertex be v_1. Let G be a graph G_{v_1}. Then $v_1,v_2,\ldots,v_n$ are vertices, $e_1,e_2,\ldots,e_{2n-5}$ are edges and $f_1,f_2,\ldots, f_{n-4}$ are the interior faces of G. $e_i = v_1v_{i+2}$, for $1 \leq i \leq n-3$, $e_{n-3+i} = v_{i+1}v_{i+2}$, for $1 \leq i \leq n-2$ and $f_i = e_ie_{n-2+i}\,e_{i+1}$ for $1 \leq i \leq n-4$. Then $|V(G)| = n$, $|E(G)| = 2n-5$ and $|F(G)| = n-4$.

Define $g : E(G) \to \{1,2,\ldots, |E(G)|\}$ as follows.

Case 1: n = 5

$g(e_1) = 2$, $g(e_2) = 1$, $g(e_3) = 3$, $g(e_4) = 5$ and $g(e_5) = 4$.

Then induced vertex labels are $g^*(v_1) = 0$, $g^*(v_2) = 0$, $g^*(v_3) = 1$, $g^*(v_4) = 1$ and $g^*(v_5) =1$.

Also, the induced face labels are $g^{**}(f_i) = 1$, if i is odd and $g^{**}(f_i) = 0$, if i is even.

In view of the above defined labeling pattern we have $v_g(1) = v_g(0)+1 = 3$ and $f_g(1) = 1$, $f_g(0) = 0$

Then $|v_g(0) - v_g(1)| \leq 1$ and $|f_g(0) - f_g(1)| \leq 1$. Switching of any vertex in cycle C_5 is face edge sum divisor cordial graph.

Case 2 : n > 5.

Subcase 2.1 : $n \equiv 0 \pmod 4$

$g(e_1) = 2$, $g(e_2) = 1$ and $g(e_{i+2}) = g(e_i)+2$, for $1 \leq i \leq n-5$, $g(e_{n-2}) = n-3$, $g(e_{n-2+i}) = g(e_{n-3})+i$, for $1 \leq i \leq n-3$.

Then induced vertex labels are $g^*(v_1) = 1$, $g^*(v_2) = 0$, $g^*(v_3) = 1$, $g^*(v_{2i-1}) = 0$, for $3 \leq i \leq \frac{n}{2}$, $g^*(v_{2i}) = 1$, for $2 \leq i \leq \frac{n-2}{2}$ and $g^*(v_n) = 0$. Also the induced face labels are $g^{**}(f_i) = 1$, if i is odd and $g^{**}(f_i) = 0$, if i is even.

In view of the above defined labeling pattern we have $v_g(1) = v_g(0) = \frac{n}{2}$ and $f_g(1) = f_g(0) = \frac{n-4}{2}$

Then $|v_g(0) - v_g(1)| \leq 1$ and $|f_g(0) - f_g(1)| \leq 1$.

Switching of any vertex in cycle C_n is face edge sum divisor cordial graph for $n \equiv 0 \pmod 4$.

Subcase 2.2: $n \equiv 1 \pmod 4$

$g(e_1) = 2$, $g(e_2) = 1$ and $g(e_{i+2}) = g(e_i)+2$, for $1 \leq i \leq n-5$, $g(e_{n-2}) = n-2$, $g(e_{n-1}) = n$, $g(e_n) = n-1$ and $g(e_{n+i}) = g(e_{n-2+i})+2$, for $1 \leq i \leq n-5$.

Then induced vertex labels are $g^*(v_1) = 0$, $g^*(v_2) = 0$, $g^*(v_3) = 1$, $g^*(v_{2i-1}) = 0$, for $3 \leq i \leq \frac{n-1}{2}$, $g^*(v_{2i}) = 1$, for $2 \leq i \leq \frac{n-1}{2}$ and $g^*(v_n) = 1$. Also, the induced face labels are $g^{**}(f_i) = 1$, if i is odd and $g^{**}(f_i) = 0$, if i is even.

In view of the above defined labeling pattern we have $v_g(1) = v_g(0)+1 = \frac{n+1}{2}$ and $f_g(1) = f_g(0)+1 = \frac{n-3}{2}$

Then $|v_g(0) - v_g(1)| \leq 1$ and $|f_g(0) - f_g(1)| \leq 1$.

Switching of any vertex in cycle C_n is face edge sum divisor cordial graph for $n \equiv 1 \pmod 4$.

Sub Case 2.3 : $n \equiv 2 \pmod 4$

It is not possible to have an edge sum divisor labelling of switching of any vertex in cycle C_n for $n \equiv 2 \pmod 4$ by Theorem 1.1. Thus switching of any vertex in cycle C_n does not admit face edge sum divisor cordial labelling for $n \equiv 2 \pmod 4$.

Sub Case 2.4 : $n \equiv 3 \pmod 4$

$g(e_1) = 2$, $g(e_2) = 1$ and $g(e_{i+2}) = f(e_i)+2$, for $1 \le i \le n-5$, $g(e_{n-2}) = n-2$, $g(e_{n-2+i}) = n-2+i$, for $1 \le i \le n-3$.

Then induced vertex labels are $g^*(v_1) = 1$, $g^*(v_2) = 0$, $g^*(v_{2i-1}) = 0$, for $2 \le i \le \frac{n+1}{2}$ and $g^*(v_{2i}) = 1$ for $2 \le i \le \frac{n-1}{2}$.

Also the induced face labels are $g^{**}(f_i) = 0$, if i is odd and $g^{**}(f_i) = 1$, if i is even.

In view of the above defined labeling pattern we have $v_g(0) = v_g(1)+1 = \frac{n+1}{2}$ and $f_g(0) = f_g(1)+1 = \frac{n-3}{2}$

Then $|v_g(0) - v_g(1)| \le 1$ and $|f_g(0) - f_g(1)| \le 1$.

Switching of any vertex in cycle C_n is face edge sum divisor cordial graph for $n \equiv 3 \pmod 4$.

Therefore, the switching of any vertex in cycle C_n is face edge sum divisor cordial graph for all $n \ge 3$ such that $n \not\equiv 2 \pmod 4$.

Illustration 2.5 Switching of any vertex in cycle C_8 and its face edge sum divisor cordial labelling is shown in Figure 2.5.

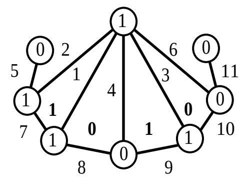

Fig. 2.5

Theorem: 2.6 Switching of a pendent vertex in path P_n admits face edge sum divisor cordial graph for $n \ge 4$.

Proof.

Let $v_1, v_2, \ldots, v_n$, be the vertices of path P_n. v_1 and v_n are pendent vertex of path P_n. Without loss of generality, let the switched vertex be v_1. The graph G is obtained by switching of a pendent vertex v_1 in path P_n.

The $v_1, v_2, \ldots, v_n$ are vertices, $e_1, e_2, \ldots, e_{2n-4}$ are edges and $f_1, f_2, \ldots, f_{n-3}$ are the interior faces of G. $e_i = v_1v_{i+2}$, for $1 \le i \le n-2$, $e_{n-2+i} = v_{i+1}v_{i+2}$, for $1 \le i \le n-2$ and $f_i = e_ie_{n-2+i}\, e_{i+1}$ for $1 \le i \le n-3$. Then $|V(G)| = n$, $|E(G)| = 2n-4$ and $|F(G)| = n-3$.

Define $g : E(G) \to \{1,2,\ldots, |E(G)|\}$ as follows.

Case 1: n = 4

$g(e_1) = 1$, $g(e_2) = 2$, $g(e_3) = 4$ and $g(e_4) = 3$.

Then induced vertex labels are $g^*(v_1) = 0$, $g^*(v_2) = 1$, $g^*(v_3) = 1$ and $g^*(v_4) = 0$.

Also, the induced face labels are $g^{**}(f_1) = 1$.

In view of the above defined labeling pattern we have $v_g(1) = v_g(0) = 2$ and $f_g(1) = 1$, $f_g(0) = 0$.

Then $|v_g(0) - v_g(1)| \le 1$ and $|f_g(0) - f_g(1)| \le 1$.

Thus switching of a pendent vertex in path P_4 is face edge sum divisor cordial graph.

Case 1: n = 5

$g(e_1) = 1$, $g(e_2) = 2$, $g(e_3) = 3$, $g(e_4) = 4$, $g(e_5) = 5$ and $g(e_6) = 6$.

Then induced vertex labels are $g^*(v_1) = 1$, $g^*(v_2) = 1$, $g^*(v_3) = 1$, $g^*(v_4) = 0$ and $g^*(v_5) = 0$.

Also, the induced face labels are $g^{**}(f_1) = 1$ and $g^{**}(f_2) = 0$.

In view of the above defined labeling pattern we have $v_g(1) = v_g(0)+1 = 3$ and $f_g(1) = f_g(0) = 1$.

Then $|v_g(0) - v_g(1)| \le 1$ and $|f_g(0) - f_g(1)| \le 1$.

Thus switching of a pendent vertex in path P_5 is face edge sum divisor cordial graph.

Case 2 : n > 5.

Subcase 2.1 : $n \equiv 0 \pmod 4$

$g(e_1) = 1$, $g(e_2) = 2$, and $g(e_{i+2}) = g(e_i)+2$, for $1 \le i \le n-4$.

$g(e_{n-1}) = n$, $g(e_n) = n-1$ and $g(e_{n+i}) = g(e_{n-2+i})+2$, for $1 \le i \le n-4$.

Then induced vertex labels are $g^*(v_1) = 0$, $g^*(v_2) = 1$, $g^*(v_{2i-1}) = 1$, for $2 \le i \le \frac{n}{2}$ and $g^*(v_{2i}) = 0$, for $2 \le i \le \frac{n}{2}$.

Also, the induced face labels are $g^{**}(f_i) = 1$, if i is odd and $g^{**}(f_i) = 0$, if i is even.

In view of the above defined labeling pattern we have $v_g(1) = v_g(0) = \frac{n}{2}$ and $f_g(1) = f_g(0)+1 = \frac{n-2}{2}$

Then $|v_g(0) - v_g(1)| \le 1$ and $|f_g(0) - f_g(1)| \le 1$.
Thus switching of a pendent vertex in path P_n is face edge sum divisor cordial graph for $n \equiv 0 \pmod 4$.
Subcase 2.2: $n \equiv 1 \pmod 4$

$g(e_i) = i$, for $1 \le i \le 2n - 4$.

Then induced vertex labels are $g^*(v_1) = 1$, $g^*(v_2) = 1$, $g^*(v_{2i-1}) = 1$, for $2 \le i \le \frac{n-1}{2}$, $g^*(v_{2i}) = 0$, for $2 \le i \le \frac{n-1}{2}$ and $g^*(v_n) = 0$.
Also, the induced face labels are $g^{**}(f_i) = 1$, if i is odd and $g^{**}(f_i) = 0$, if i is even.
In view of the above defined labeling pattern we have $v_g(1) = v_g(0)+1 = \frac{n+1}{2}$ and $f_g(1) = f_g(0) = \frac{n-2}{2}$
Then $|v_g(0) - v_g(1)| \le 1$ and $|f_g(0) - f_g(1)| \le 1$.
Thus switching of a pendent vertex in path P_n is face edge sum divisor cordial graph for $n \equiv 1 \pmod 4$.
Sub Case 2.3 : $n \equiv 2 \pmod 4$
It is not possible to have an edge sum divisor labelling of switching of a pendent vertex in path P_n for $n \equiv 2 \pmod 4$ by Theorem 1.1. Thus switching of a pendent vertex in path P_n does not admit face edge sum divisor cordial labelling for $n \equiv 2 \pmod 4$.
Sub Case 2.4: $n \equiv 3 \pmod 4$

$g(e_i) = i$, for $1 \le i \le 2n - 4$.

Then induced vertex labels are $g^*(v_1) = 0$, $g^*(v_2) = 1$, $g^*(v_{2i-1}) = 1$, for $2 \le i \le \frac{n-1}{2}$ and $g^*(v_{2i}) = 0$, for $2 \le i \le \frac{n-1}{2}$ and $g^*(v_n) = 0$.
Also, the induced face labels are $g^{**}(f_i) = 1$, if i is odd and $g^{**}(f_i) = 0$, if i is even.
In view of the above defined labeling pattern we have $v_g(0) = v_g(1)+1 = \frac{n+1}{2}$ and $f_g(1) = f_g(0) = \frac{n-3}{2}$
Then $|v_g(0) - v_g(1)| \le 1$ and $|f_g(0) - f_g(1)| \le 1$.
Switching of a pendent vertex in path P_n is face edge sum divisor cordial graph for $n \equiv 3 \pmod 4$.

Therefore, the switching of a pendent vertex in path P_n is face edge sum divisor cordial graph for all $n \ge 4$ such that $n \not\equiv 2 \pmod 4$.

Illustration 2.6 switching of a pendent vertex in path P_7 and its face edge sum divisor cordial labelling is shown in Figure 2.6.

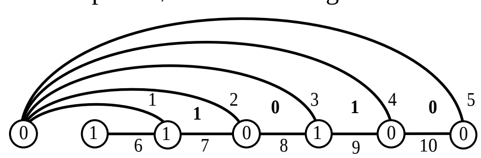

Fig. 2.6

Theorem: 2.7 The graph DT_n is face edge sum divisor cordial labelling for $n \ge 2$ and $n \not\equiv 0 \pmod 4$.
Proof.

Let $v_1,v_2,\ldots,v_n$ be the successive vertices of P_n. Let G be a graph DT_n. Then $v_1,v_2,\ldots,v_n$, $u_1,u_2,\ldots,u_{n-1}$ and $w_1,w_2,\ldots,w_{n-1}$ are vertices, $e_1,e_2,\ldots,e_{5n-5}$ are edges and $f_1,f_2,\ldots,f_{2n-2}$ are the interior faces of G. $e_i = v_iv_{i+1}$, for $1 \le i \le n-1$, $e_{n-1+2i-1} = v_iu_i$, for $1 \le i \le n-1$, $e_{n-1+2i} = u_iv_{i+1}$, for $1 \le i \le n-1$, $e_{3n-3+2i-1} = v_iw_i$, for $1 \le i \le n-1$, $e_{3n-3+2i} = w_iv_{i+1}$, for $1 \le i \le n-1$ and $f_i = e_ie_{n+i}e_{n+i-1}$ for $1 \le i \le n-1$ and $f_{n-1+i} = e_ie_{2n+i-1}e_{2n+i-2}$ for $1 \le i \le n-1$. Then $|V(G)| = 3n-2$, $|E(G)| = 5n-5$ and $|F(G)| = 2n-2$.
Define $g : E(G) \to \{1, 2, \ldots, 5n-5\}$ as follows.

Case 1: $n = 2$

$g(e_1) = 1$, $g(e_2) = 3$, $g(e_3) = 5$, $g(e_4) = 1$ and $g(e_5) = 2$.

Then induced vertex labels are $g^*(v_1) = 1$, $g^*(v_2) = 0$, $g^*(u_1) = 1$ and $g^*(w_1) = 0$.
Also, the induced face labels are $g^{**}(f_1) = 1$ and $g^{**}(f_2) = 0$.
In view of the above defined labeling pattern we have $v_g(1) = v_g(0) = 2$ and $f_g(1) = f_g(0) = 1$.
Then $|v_g(0) - v_g(1)| \le 1$ and $|f_g(0) - f_g(1)| \le 1$. Therefore the graph DT_2 is face edge sum divisor cordial graph.

Case 2: n = 3

$g(e_1) = 9$, $g(e_2) = 10$, $g(e_3) = 5$, $g(e_4) = 7$, $g(e_5) = 6$, $g(e_6) = 8$, $g(e_{i+6}) = i$, for $1 \le i \le 4$.
Then induced vertex labels are $g^*(v_i) = 0$, for i = 1,2, $g^*(v_3) = 1$, $g^*(u_i) = 1$, for i = 1,2, $g^*(w_i) = 0$, for i = 1,2,.
Also the induced face labels are $g^{**}(f_i) = 0$, for i = 1,4 and $g^{**}(f_i) = 1$, for i = 2,3.
In view of the above defined labeling pattern we have $v_g(1) = 3$, $v_g(0) = 4$, $f_g(1) = 2$ and $f_g(0) = 2$.
Then $|v_g(0) - v_g(1)| \le 1$ and $|f_g(0) - f_g(1)| \le 1$. Thus the graph DT_3 is face edge sum divisor cordial graph.

Case 3: $n \equiv 0 \pmod 4$ and n > 3.

It is not possible to have an edge sum divisor labelling of DT_n for $n \equiv 0 \pmod 4$ by Theorem 1.1. Thus DT_n does not admit face edge sum divisor cordial labelling for $n \equiv 0 \pmod 4$. Thus the graph DT_n is face edge sum divisor cordial graph for $n \equiv 0 \pmod 4$.

Case 4 : n = 5

$g(e_1) = 17$, $g(e_2) = 19$, $g(e_3) = 18$, $g(e_4) = 20$, $g(e_9) = 9$, $g(e_{10}) = 11$, $g(e_{11}) = 10$, $g(e_{12}) = 12$, $g(e_{i+4}) = g(e_i)+4$, for $9 \le i \le 12$, $g(e_{12+i}) = i$, for $1 \le i \le 8$.

Then induced vertex labels are $g^*(v_{2i-1}) = 0$, for $1 \le i \le 2$, $g^*(v_{2i}) = 1$, for $1 \le i \le 2$, $g^*(u_i) = 1$, for $1 \le i \le n-1$, $g^*(w_i) = 0$, for $1 \le i \le n-1$.

Also the induced face labels are $g^{**}(f_1) = 0$, $g^{**}(f_2) = 0$, $g^{**}(f_3) = 1$, $g^{**}(f_4) = 1$, $g^{**}(f_5) = 1$, $g^{**}(f_6) = 1$, $g^{**}(f_7) = 0$, $g^{**}(f_8) = 0$. In view of the above defined labeling pattern we have $v_g(1) = 7$, $v_g(0) = 6$, $f_g(0) = 4$ and $f_g(1) = 4$.
Then $|v_g(0) - v_g(1)| \le 1$ and $|f_g(0) - f_g(1)| \le 1$. Thus the graph DT_5 is face edge sum divisor cordial graph.

Case 5: n > 5.
Subcase 5.1: $n \equiv 1 \pmod 4$

$g(e_1) = 4n-3$, $g(e_2) = 4n-1$, $g(e_3) = 4n-2$, $g(e_4) = 4n$, $g(e_{i+4}) = g(e_i)+4$, for $1 \le i \le n-5$, $g(e_{2n-1}) = 2n-1$, $g(e_{2n}) = 2n+1$, $g(e_{2n+1}) = 2n$, $g(e_{2n+2}) = 2n+2$, $g(e_{i+4}) = g(e_i)+4$, for $2n-1 \le i \le 4n-8$, $g(e_{3n-3+i}) = i$, for $1 \le i \le 2n-2$.

Then induced vertex labels are $g^*(v_{2i-1}) = 0$, for $1 \le i \le \frac{n-1}{2}$, $g^*(v_{2i}) = 1$, for $1 \le i \le \frac{n-1}{2}$ and $g^*(v_n) = 1$, $g^*(u_i) = 1$, for $1 \le i \le n-1$, $g^*(w_i) = 0$, for $1 \le i \le n-1$.

Also the induced face labels are $g^{**}(f_1) = 0$, $g^{**}(f_2) = 0$, $g^{**}(f_3) = 1$, $g^{**}(f_4) = 1$, $g^{**}(f_{i+4}) = g^{**}(f_i)$ for $1 \le i \le n-5$, $g^{**}(f_n) = 1$, $g^{**}(f_{n+1}) = 1$, $g^{**}(f_{n+2}) = 0$, $g^{**}(f_{n+3}) = 0$, $g^{**}(f_{i+4}) = g^{**}(f_i)$ for $n \le i \le 2n-6$.
In view of the above defined labeling pattern we have $v_g(1) = \frac{3n-1}{2}$, $v_g(0) = \frac{3n-3}{2}$, $f_g(0) = n-1$ and $f_g(1) = n-1$.
Then $|v_g(0)-v_g(1)| \le 1$ and $|f_g(0)-f_g(1)| \le 1$. Thus DT_n is the face edge sum divisor cordial graph for $n \equiv 1 \pmod 4$ and n > 5.
Sub Case 5.2 : $n \equiv 2 \pmod 4$

$g(e_1) = 4n-4$, $g(e_2) = 4n-2$, $g(e_3) = 4n-3$, $g(e_4) = 4n-1$, $g(e_{i+4}) = g(e_i)+4$, for $1 \le i \le n-5$, $g(e_{2n-1}) = 2n-1$, $g(e_{2n}) = 2n+1$, $g(e_{2n+1}) = 2n$, $g(e_{2n+2}) = 2n+2$, $g(e_{i+4}) = g(e_i)+4$, for $2n-1 \le i \le 4n-8$, $g(e_{3n-3+i}) = i$, for $1 \le i \le 2n-2$.

Then induced vertex labels are $g^*(v_1) = 1$, $g^*(v_{2i}) = 1$, for $1 \le i \le \frac{n-2}{2}$, $g^*(v_{2i+1}) = 0$, for $1 \le i \le \frac{n-2}{2}$ and $g^*(v_n) = 0$, $g^*(u_i) = 1$, for $1 \le i \le n-1$, $g^*(w_i) = 0$, for $1 \le i \le n-1$.

Also the induced face labels are $g^{**}(f_1) = 1$, $g^{**}(f_2) = 1$, $g^{**}(f_3) = 0$, $g^{**}(f_4) = 0$, $g^{**}(f_{i+4}) = g^{**}(f_i)$ for $1 \le i \le n-5$, $g^{**}(f_n) = 0$, $g^{**}(f_{n+1}) = 0$, $g^{**}(f_{n+2}) = 1$, $g^{**}(f_{n+3}) = 1$, $g^{**}(f_{i+4}) = g^{**}(f_i)$ for $n \le i \le 2n-6$.
In view of the above defined labeling pattern we have $v_g(1) = \frac{3n-2}{2}$, $v_g(0) = \frac{3n-2}{2}$, $f_g(0) = n-1$ and $f_g(1) = n-1$.
Then $|v_g(0)-v_g(1)| \le 1$ and $|f_g(0)-f_g(1)| \le 1$. Thus DT_n is face edge sum divisor cordial graph for $n \equiv 2 \pmod 4$ and n > 5.
Subcase 5.3: $n \equiv 3 \pmod 4$

$g(e_1) = 4n-3$, $g(e_2) = 4n-1$, $g(e_3) = 4n-2$, $g(e_4) = 4n$, $g(e_{i+4}) = g(e_i)+4$, for $1 \le i \le n-6$, $g(e_{n-1}) = 5n-5$, $g(e_{2n-1}) = 2n-1$, $g(e_{2n}) = 2n+1$, $g(e_{2n+1}) = 2n$, $g(e_{2n+2}) = 2n+2$, $g(e_{i+4}) = g(e_i)+4$, for $2n-1 \le i \le 4n-8$, $g(e_{3n-3+i}) = i$, for $1 \le i \le 2n-2$.

Then induced vertex labels are $g^*(v_{2i-1}) = 0$, for $1 \le i \le \frac{n-1}{2}$, $g^*(v_{2i}) = 1$, for $1 \le i \le \frac{n-3}{2}$, $g^*(v_{n-1}) = 0$ and $g^*(v_n) = 1$, $g^*(u_i) = 1$, for $1 \le i \le n-1$, $g^*(w_i) = 0$, for $1 \le i \le n-1$.

Also the induced face labels are $g^{**}(f_1) = 0$, $g^{**}(f_2) = 0$, $g^{**}(f_3) = 1$, $g^{**}(f_4) = 1$, $g^{**}(f_{i+4}) = g^{**}(f_i)$ for $1 \le i \le n-6$,

$g^{**}(f_{n-1}) = 1$, $g^{**}(f_n) = 1$, $g^{**}(f_{n+1}) = 1$, $g^{**}(f_{n+2}) = 0$, $g^{**}(f_{n+3}) = 0$, $g^{**}(f_{i+4}) = g^{**}(f_i)$ for $n \le i \le 2n-7$, $g^{**}(f_{2n-2}) = 0$.

In view of the above defined labeling pattern we have $v_g(1) = \frac{3n-3}{2}$, $v_g(0) = \frac{3n-1}{2}$, $f_g(0) = n-1$ and $f_g(1) = n-1$.

Then $|v_g(0) - v_g(1)| \le 1$ and $|f_g(0) - f_g(1)| \le 1$. Thus DT_n is the face edge sum divisor cordial graph for $n \equiv 3 \pmod 4$ and $n > 5$.

Therefore, the graph DT_n is a face edge sum divisor cordial graph for all $n \ge 2$ and $n \not\equiv 0 \pmod 4$.

Illustration 2.7 DT_5 and its face edge sum divisor cordial labelling are shown in Figure 2.7

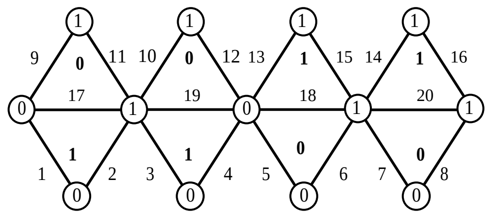

Fig. 2.7

Theorem: 2.8 The graph $S'(K_{1,n})$ is the total face edge sum divisor cordial graph for $n \ge 2$.

Proof.

Let $v, v_1, \ldots, v_n$ be the vertices of $K_{1,n}$. Let $G = S'(K_{1,n})$. Then $v, v_1, \ldots, v_n, v', v_1', \ldots, v_n'$ are the vertices, $e_1, e_2, \ldots, e_{3n}$ are the edges and $f_1, f_2, \ldots, f_{n-1}$ are the interior faces of G, where $e_i = v'v_i$, $e_{n+i} = v_iv$ and $e_{2n+i} = v\,v_i'$ for $1 \le i \le n$ and $f_i = v'v_ivv_{i+1}v'$ for $1 \le i \le n-1$. Then $|V(G)| = 2n+2$, $|E(G)| = 3n$ and $|F(G)| = n-3$.

Define $g : E(G) \rightarrow \{1, 2, \ldots, 3n\}$ as follows

For $n \equiv 0 \pmod 4$

$g(e_i) = 2i$, for $1 \le i \le n$, $g(e_{n+i}) = 2i-1$, for $1 \le i \le n$ and $g(e_i) = i$, for $2n+1 \le i \le 3n$.

Then induced vertex labels are $g^*(v') = 1$, $g^*(v) = 1$, $g^*(v_i) = 0$, for $1 \le i \le n$, $g^*(v_i') = 1$, if i is odd and $g^*(v_i') = 0$, if i is even. Also, the induced face labels are $g^{**}(f_i) = 1$, $1 \le i \le n-1$.

In view of the above defined labeling pattern we have $v_f(1) = \frac{n+4}{2}$, $v_f(0) = \frac{3n}{2}$, $f_g(0) = 0$ and $f_g(1) = n-1$.

Thus $(v_g(0) + f_g(0)) = \frac{3n}{2}$ and $(v_g(1) + f_g(1)) = \frac{3n}{2} + 1$. Then $|(v_g(0)+f_g(0)) - (v_g(1)+f_g(1))| \le 1$.

Thus $S'(K_{1,n})$ is the total face edge sum divisor cordial graph for $n \equiv 0 \pmod 4$.

For $n \equiv 1 \pmod 4$

$g(e_i) = 2i$, for $1 \le i \le n$, $g(e_{n+i}) = 2i-1$, for $1 \le i \le n$ and $g(e_i) = i$, for $2n+1 \le i \le 3n$.

Then induced vertex labels are $g^*(v') = 1$, $g^*(v) = 1$, $g^*(v_i) = 0$, for $1 \le i \le n$, $g^*(v_i') = 0$, if i is odd and $g^*(v_i') = 1$, if i is even. Also, the induced face labels are $g^{**}(f_i) = 1$, $1 \le i \le n-1$.

In view of the above defined labeling pattern we have $v_f(1) = \frac{n+3}{2}$, $v_f(0) = \frac{3n+1}{2}$, $f_g(0) = 0$ and $f_g(1) = n-1$.

Thus $(v_g(0) + f_g(0)) = \frac{3n+1}{2}$ and $(v_g(1) + f_g(1)) = \frac{3n+1}{2}$. Then $|(v_g(0)+f_g(0)) - (v_g(1)+f_g(1))| \le 1$.

Thus $S'(K_{1,n})$ is the total face edge sum divisor cordial graph for $n \equiv 1 \pmod 4$.

For $n \equiv 2 \pmod 4$

$g(e_i) = 2i$, for $1 \le i \le n$, $g(e_{n+i}) = 2i-1$, for $1 \le i \le n$ and $g(e_i) = i$, for $2n+1 \le i \le 3n$.

Then induced vertex labels are $g^*(v') = 1$, $g^*(v) = 0$, $g^*(v_i) = 0$, for $1 \le i \le n$, $g^*(v_i') = 1$, if i is odd and $g^*(v_i') = 0$, if i is even. Also, the induced face labels are $g^{**}(f_i) = 1$, $1 \le i \le n-1$.

In view of the above defined labeling pattern we have $v_f(1) = \frac{n+2}{2}$, $v_f(0) = \frac{3n+2}{2}$, $f_g(0) = 0$ and $f_g(1) = n-1$.

Thus $(v_g(0) + f_g(0)) = \frac{3n}{2} + 1$ and $(v_g(1) + f_g(1)) = \frac{3n}{2}$. Then $|(v_g(0)+f_g(0)) - (v_g(1)+f_g(1))| \le 1$.

Thus $S'(K_{1,n})$ is the total face edge sum divisor cordial graph for $n \equiv 2 \pmod 4$.

For $n \equiv 3 \pmod 4$

$g(e_i) = 2i$, for $1 \le i \le n-1$, $g(e_n) = 2n+1$, $g(e_{n+i}) = 2i-1$, for $1 \le i \le n$ and $g(e_{2n+1}) = 2n$.

$g(e_i) = i$, for $2n+2 \le i \le 3n$.

Then induced vertex labels are $g^*(v') = 0$, $g^*(v) = 1$, $g^*(v_i) = 0$, for $1 \le i \le n-1$, $g^*(v_n) = 1$,

$g^*(v'_1) = 1$, $g^*(v'_{2i+1}) = 0$, for $1 \le i \le \frac{n-1}{2}$ and $g^*(v'_{2i}) = 1$, for $1 \le i \le \frac{n-1}{2}$.

Also, the induced face labels are $g^{**}(f_i) = 1$, $1 \le i \le n-2$ and $g^{**}(f_{n-1}) = 0$.

In view of the above defined labeling pattern we have $v_f(1) = \frac{n+3}{2}$, $v_f(0) = \frac{3n-3}{2}$, $f_g(0) = 1$ and $f_g(1) = n-2$.

Thus $(v_g(0) + f_g(0)) = \frac{3n-1}{2}$ and $(v_g(1) + f_g(1)) = \frac{3n-1}{2}$. Then $|(v_g(0)+f_g(0)) - (v_g(1)+f_g(1))| \le 1$.

Thus $S'(K_{1,n})$ is the total face edge sum divisor cordial graph for $n \equiv 3 \pmod 4$.

Hence the, graph $S'(K_{1,n})$ is the total face edge sum divisor cordial graph for $n \ge 2$.

Illustration 2.8 The graph $S'(K_{1,4})$ and its total face edge sum divisor cordial labelling is shown in Figure 2.8.

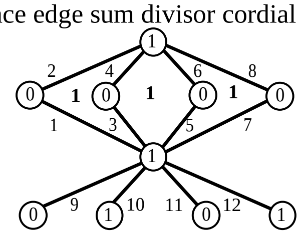

Fig. 2.8

Theorem: 2.9 The graph $S'(P_n)$ is the total face edge sum divisor cordial graph for $n \ge 2$.

Proof

Let $v_1, v_2, \ldots, v_n$ be the vertices of P_n. Let $G = S'(P_n)$. Then $v_1, v_2, \ldots, v_n, v'_1, v'_2, \ldots, v'_n$ are the vertices, $e_1, e_2, \ldots, e_{3n-3}$ are the edges and $f_1, f_2, \ldots, f_{n-2}$ are the interior faces of G, where $e_i = v'v_i$, $e_{n+i} = v_iv$ and $e_{2n+i} = v\,v'_i$ for $1 \le i \le n$ and $f_i = v'v_ivv_{i+1}v'$ for $1 \le i \le n-1$. Then $|V(G)| = 2n$, $|E(G)| = 3n-3$ and $|F(G)| = n-2$. Define $g : E(G) \to \{1, 2, \ldots, 3n-3\}$ as follows

For $n \equiv 0 \pmod 4$

$g(e_i) = 2i$, for $1 \le i \le n$, $g(e_{n+i}) = 2i-1$, for $1 \le i \le n$ and $g(e_i) = i$, for $2n+1 \le i \le 3n$.

Then induced vertex labels are $g^*(v') = 1$, $g^*(v) = 1$, $g^*(v_i) = 0$, for $1 \le i \le n$, $g^*(v'_i) = 1$, if i is odd and $g^*(v'_i) = 0$, if i is even. Also, the induced face labels are $g^{**}(f_i) = 1$, $1 \le i \le n-1$.

In view of the above defined labeling pattern we have $v_f(1) = \frac{n+4}{2}$, $v_f(0) = \frac{3n}{2}$, $f_g(0) = 0$ and $f_g(1) = n-1$.

Thus $(v_g(0) + f_g(0)) = \frac{3n}{2}$ and $(v_g(1) + f_g(1)) = \frac{3n}{2}+1$. Then $|(v_g(0)+f_g(0)) - (v_g(1)+f_g(1))| \le 1$.

Thus $S'(K_{1,n})$ is the total face edge sum divisor cordial graph for $n \equiv 0 \pmod 4$.

For $n \equiv 1 \pmod 4$

$g(e_i) = 2i$, for $1 \le i \le n$, $g(e_{n+i}) = 2i-1$, for $1 \le i \le n$ and $g(e_i) = i$, for $2n+1 \le i \le 3n$.

Then induced vertex labels are $g^*(v') = 1$, $g^*(v) = 1$, $g^*(v_i) = 0$, for $1 \le i \le n$, $g^*(v'_i) = 0$, if i is odd and $g^*(v'_i) = 1$, if i is even. Also, the induced face labels are $g^{**}(f_i) = 1$, $1 \le i \le n-1$.

In view of the above defined labeling pattern we have $v_f(1) = \frac{n+3}{2}$, $v_f(0) = \frac{3n+1}{2}$, $f_g(0) = 0$ and $f_g(1) = n-1$.

Thus $(v_g(0) + f_g(0)) = \frac{3n+1}{2}$ and $(v_g(1) + f_g(1)) = \frac{3n+1}{2}$. Then $|(v_g(0)+f_g(0)) - (v_g(1)+f_g(1))| \le 1$.

Thus $S'(K_{1,n})$ is the total face edge sum divisor cordial graph for $n \equiv 1 \pmod 4$.

For $n \equiv 2 \pmod 4$

$g(e_i) = 2i$, for $1 \le i \le n$, $g(e_{n+i}) = 2i-1$, for $1 \le i \le n$ and $g(e_i) = i$, for $2n+1 \le i \le 3n$.

Then induced vertex labels are $g^*(v') = 1$, $g^*(v) = 0$, $g^*(v_i) = 0$, for $1 \le i \le n$, $g^*(v'_i) = 1$, if i is odd and $g^*(v'_i) = 0$, if i is even. Also, the induced face labels are $g^{**}(f_i) = 1$, $1 \le i \le n-1$.

In view of the above defined labeling pattern we have $v_f(1) = \frac{n+2}{2}$, $v_f(0) = \frac{3n+2}{2}$, $f_g(0) = 0$ and $f_g(1) = n-1$.

Thus $(v_g(0) + f_g(0)) = \frac{3n}{2}+1$ and $(v_g(1) + f_g(1)) = \frac{3n}{2}$. Then $|(v_g(0)+f_g(0)) - (v_g(1)+f_g(1))| \le 1$.

Thus $S'(K_{1,n})$ is the total face edge sum divisor cordial graph for $n \equiv 2 \pmod 4$.

For $n \equiv 3 \pmod 4$

$g(e_i) = 2i$, for $1 \le i \le n-1$, $g(e_n) = 2n+1$, $g(e_{n+i}) = 2i-1$, for $1 \le i \le n$, $g(e_{2n+1}) = 2n$ and $g(e_i) = i$, for $2n+2 \le i \le 3n$.

Then induced vertex labels are $g^*(v') = 0$, $g^*(v) = 1$, $g^*(v_i) = 0$, for $1 \le i \le n-1$, $g^*(v_n) = 1$,

$g^*(v'_1) = 1$, $g^*(v'_{2i+1}) = 0$, for $1 \le i \le \frac{n-1}{2}$ and $g^*(v'_{2i}) = 1$, for $1 \le i \le \frac{n-1}{2}$.

Also, the induced face labels are $g^{**}(f_i) = 1$, $1 \le i \le n-2$ and $g^{**}(f_{n-1}) = 0$.

In view of the above defined labeling pattern we have $v_f(1) = \frac{n+3}{2}$, $v_f(0) = \frac{3n-3}{2}$, $f_g(0) = 1$ and $f_g(1) = n-2$.

Thus $(v_g(0) + f_g(0)) = \frac{3n-1}{2}$ and $(v_g(1) + f_g(1)) = \frac{3n-1}{2}$. Then $|(v_g(0)+f_g(0)) - (v_g(1)+f_g(1))| \le 1$.

Thus $S'(P_n)$ is the total face edge sum divisor cordial graph for $n \equiv 3 \pmod 4$.

Hence graph $S'(P_n)$ is the total face edge sum divisor cordial graph for $n \ge 2$.

Illustration 2.9 The graph $S'(P_5)$ and its total face edge sum divisor cordial labelling is shown in Figure 2.9

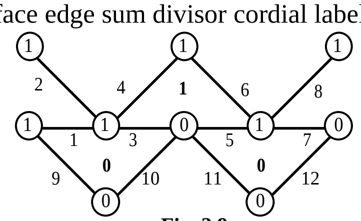

Fig. 2.9

Theorem: 2.10 The graph DT_n is total face edge sum divisor cordial labelling for $n \ge 2$.

Proof.

Let $v_1,v_2,\ldots,v_n$ be the successive vertices of P_n. Let G be a graph DT_n. Then $v_1,v_2,\ldots,v_n$, $u_1,u_2,\ldots,u_{n-1}$ and $w_1,w_2,\ldots,w_{n-1}$ are vertices, $e_1,e_2,\ldots,e_{5n-5}$ are edges and $f_1,f_2,\ldots,f_{2n-2}$ are the interior faces of G. $e_i = v_iv_{i+1}$, for $1 \le i \le n-1$, $e_{n-1+2i-1} = v_iu_i$, for $1 \le i \le n-1$, $e_{n-1+2i} = u_iv_{i+1}$, for $1 \le i \le n-1$, $e_{3n-3+2i-1} = v_iw_i$, for $1 \le i \le n-1$, $e_{3n-3+2i} = w_iv_{i+1}$, for $1 \le i \le n-1$ and $f_i = e_ie_{n+i}e_{n+i-1}$ for $1 \le i \le n-1$ and $f_{n-1+i} = e_ie_{2n+i-1}e_{2n+i-2}$ for $1 \le i \le n-1$. Then $|V(G)| = 3n-2$, $|E(G)| = 5n-5$ and $|F(G)| = 2n-2$.

Define $g : E(G) \to \{1, 2, \ldots, 5n-5\}$ as follows.

Case 1: n = 2

$g(e_1) = 4$, $g(e_2) = 1$, $g(e_3) = 2$, $g(e_4) = 3$ and $g(e_5) = 5$.

Then induced vertex labels are $g^*(v_1) = 1$, $g^*(v_2) = 0$, $g^*(u_1) = 0$ and $g^*(w_1) = 1$.

Also, the induced face labels are $g^{**}(f_1) = 0$ and $g^{**}(f_2) = 1$.

In view of the above defined labeling pattern we have $v_g(1) = v_g(0) = 2$ and $f_g(1) = f_g(0) = 1$.

Thus $(v_g(0) + f_g(0)) = 3$ and $(v_g(1) + f_g(1)) = 3$. Then $|(v_g(0)+f_g(0)) - (v_g(1)+f_g(1))| \le 1$.

Therefore graph DT_2 is the total face edge sum divisor cordial graph.

Case 2: n = 3

$g(e_1) = 1$, $g(e_2) = 3$, $g(e_3) = 2$, $g(e_{i+1}) = g(e_i)+2$, for i = 3,4,5, $g(e_7) = 5$, $g(e_{i+1}) = g(e_i)+2$, for i = 7,8 and $g(e_{10}) = 10$.

Then induced vertex labels are $g^*(v_i) = 1$, for i = 1,2, $g^*(v_3) = 0$, $g^*(u_i) = 1$, for i = 1,2, $g^*(w_1) = 1$ and $g^*(w_2) = 0$.

Also the induced face labels are $g^{**}(f_i) = 0$, for i = 1,2,3 and $g^{**}(f_4) = 1$.

In view of the above defined labeling pattern we have $v_g(1) = 5$, $v_g(0) = 2$ and $f_g(1) = 1$ and $f_g(0) = 3$.

Thus $(v_g(0) + f_g(0)) = 5$ and $(v_g(1) + f_g(1)) = 6$. Then $|(v_g(0)+f_g(0)) - (v_g(1)+f_g(1))| \le 1$.

Thus graph DT_3 is the total face edge sum divisor cordial graph.

Case 3: n = 4

$g(e_1) = 1$, $g(e_2) = 2$, $g(e_3) = 3$, $g(e_4) = 6$, $g(e_{i+1}) = g(e_i)+2$, for i = 4,5, $g(e_7) = 5$, $g(e_8) = 12$, $g(e_9) = 14$, $g(e_{10}) = 7$, $g(e_{11}) = 9$, $g(e_{12}) = 4$, $g(e_{13}) = 11$, $g(e_{i+1}) = g(e_i)+2$, for 13,14.

Then induced vertex labels are $g^*(v_i) = 1$, for $1 \le i \le 4$, $g^*(u_1) = g^*(w_1) = 1$, $g^*(u_2) = g^*(w_2) = 0$ and $g^*(u_3) = g^*(w_3) = 1$.

Also, the induced face labels are $g^{**}(f_i) = 0$, for $1 \le i \le 6$.

In view of the above defined labeling pattern we have $v_g(1) = 8$, $v_g(0) = 2$, $f_g(1) = 0$ and $f_g(0) = 6$.

Thus $(v_g(0) + f_g(0)) = 8$ and $(v_g(1) + f_g(1)) = 8$. Then $|(v_g(0)+f_g(0)) - (v_g(1)+f_g(1))| \leq 1$.

Thus graph DT_4 is the total face edge sum divisor cordial graph.

Case 4: $n > 4$.

Subcase 4.1: $n \equiv 0 \pmod 4$

$g(e_i) = 2i-1$, for $1 \leq i \leq \frac{n-2}{2}$, $g(e_{\frac{n}{2}}) = 2$, $g(e_{i+1}) = 2i-1$, for $\frac{n}{2} \leq i \leq n-2$, $g(e_{n-1+i}) = 2i+4$, for $1 \leq i \leq n-1$,

$g(e_{2n-1}) = 2n-3$, $g(e_{n+i}) = 2i+4$, for $n \leq i \leq 2n-3$, $g(e_{3n-3+i}) = 2n-3+2i$, for $1 \leq i \leq n-2$, $g(e_{4n-4}) = 4$,

$g(e_{2n-2+i}) = 2i-3$, for $2n-1 \leq i \leq \frac{5n-6}{2}$, $g(e_i) = i$, for $\frac{9n-8}{2} \leq i \leq 5n-5$.

Then induced vertex labels are $g^*(v_i) = 1$, for $1 \leq i \leq \frac{n+2}{2}$, $g^*(v_i) = 0$, for $\frac{n+4}{2} \leq i \leq n-1$, $g^*(v_n) = 1$,

$g^*(u_i) = 1$, for $1 \leq i \leq \frac{n-2}{2}$, $g^*(u_i) = 0$, for $i = \frac{n}{2}$, $g^*(u_i) = 1$, for $\frac{n+2}{2} \leq i \leq n-1$,

$g^*(w_i) = 1$, for $1 \leq i \leq \frac{n-2}{2}$, $g^*(w_i) = 0$, for $i = \frac{n}{2}$, $g^*(w_i) = 1$, for $i = \frac{n+2}{2}$, $g^*(w_i) = 0$, for $\frac{n+4}{2} \leq i \leq n-1$.

Also the induced face labels are $g^{**}(f_i) = 0$, for $1 \leq i \leq \frac{3n}{2}$ and $g^{**}(f_i) = 1$, for $\frac{3n+2}{2} \leq i \leq 2n-2$.

In view of the above defined labeling pattern we have $v_g(1) = 2n$, $v_g(0) = n-2$, $f_g(0) = \frac{3n}{2}$ and $f_g(1) = \frac{n-4}{2}$.

Thus $(v_g(0) + f_g(0)) = \frac{5n-4}{2}$ and $(v_g(1) + f_g(1)) = \frac{5n-4}{2}$. Then $|(v_g(0)+f_g(0)) - (v_g(1)+f_g(1))| \leq 1$.

Thus the graph DT_n is the total face edge sum divisor cordial graph for $n \equiv 0 \pmod 4$ and $n > 4$.

Subcase 4.2: $n \equiv 1,3 \pmod 4$

$g(e_i) = 2i-1$, for $1 \leq i \leq n-1$, $g(e_{n-1+i}) = 2i$, for $1 \leq i \leq 2n-2$, $g(e_{3n-3+i}) = 2n-3+2i$, for $1 \leq i \leq n-1$, $g(e_i) = i$, for $4n-3 \leq i \leq 5n-5$.

Then induced vertex labels are $g^*(v_i) = 1$, for $1 \leq i \leq \frac{n+1}{2}$, $g^*(v_i) = 0$, for $\frac{n+3}{2} \leq i \leq n$,

$g^*(u_i) = 1$, for $1 \leq i \leq n-1$, $g^*(w_i) = 1$, for $1 \leq i \leq \frac{n-1}{2}$, $g^*(w_i) = 0$, for $\frac{n+1}{2} \leq i \leq n-1$.

Also the induced face labels are $g^{**}(f_i) = 0$, for $1 \leq i \leq \frac{3n-3}{2}$ and $g^{**}(f_i) = 1$, for $\frac{3n-1}{2} \leq i \leq 2n-2$.

In view of the above defined labeling pattern we have $v_g(1) = 2n-1$, $v_g(0) = n-1$, $f_g(0) = \frac{3n-3}{2}$ and $f_g(1) = \frac{n-1}{2}$.

Thus $(v_g(0) + f_g(0)) = \frac{5n-5}{2}$ and $(v_g(1) + f_g(1)) = \frac{5n-3}{2}$. Then $|(v_g(0)+f_g(0)) - (v_g(1)+f_g(1))| \leq 1$.

Thus the graph DT_n is the total face edge sum divisor cordial graph for $n \equiv 1,3 \pmod 4$ and $n > 4$.

Sub Case 4.3: $n \equiv 2 \pmod 4$

$g(e_i) = 2i-1$, for $1 \leq i \leq n-2$, $g(e_{n-1}) = 5n-6$, $g(e_{n-1+i}) = 2i$, for $1 \leq i \leq 2n-4$, $g(e_{3n-4}) = 5n-9$, $g(e_{3n-3}) = 5n-8$,
$g(e_{3n-3+i}) = 2n-5+2i$, for $1 \leq i \leq n-2$, $g(e_i) = i-3$, for $4n-4 \leq i \leq 5n-7$. $g(e_{5n-6}) = 5n-7$, $g(e_{5n-5}) = 5n-5$.

Then induced vertex labels are $g^*(v_i) = 1$, for $1 \leq i \leq \frac{n}{2}$, $g^*(v_i) = 0$, for $\frac{n+2}{2} \leq i \leq n$,

$g^*(u_i) = 1$, for $1 \leq i \leq n-2$, $g^*(u_{n-1}) = 0$, $g^*(w_i) = 1$, for $1 \leq i \leq \frac{n-2}{2}$, $g^*(w_i) = 0$, for $\frac{n}{2} \leq i \leq n-2$, $g^*(w_{n-1}) = 1$.

Also the induced face labels are $g^{**}(f_i) = 0$, for $1 \leq i \leq \frac{3n}{2}$ and $g^{**}(f_i) = 1$, for $\frac{3n+2}{2} \leq i \leq 2n-2$.

In view of the above defined labeling pattern we have $v_g(1) = 2n-2$, $v_g(0) = n$, $f_g(0) = \frac{3n-4}{2}$ and $f_g(1) = \frac{n}{2}$.

Thus $(v_g(0) + f_g(0)) = \frac{5n-4}{2}$ and $(v_g(1) + f_g(1)) = \frac{5n-4}{2}$. Then $|(v_g(0)+f_g(0)) - (v_g(1)+f_g(1))| \leq 1$.

Thus the graph DT_n is the total face edge sum divisor cordial graph for $n \equiv 2 \pmod 4$ and $n > 4$. Therefore, the graph DT_n is the total face edge sum divisor cordial graph for all $n \geq 2$.

Illustration 2.10 The graph DT_4 and its total face edge sum divisor cordial labelling are shown in Figure 2.10.

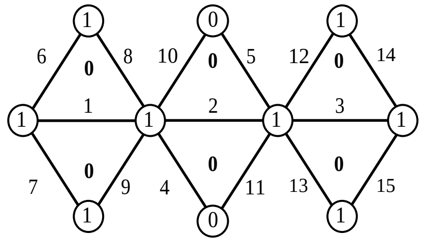

Fig. 2.10

3. Conclusion

This paper discusses some new families of edge sum divisor cordial graphs, face edge sum divisor cordial graphs and total face edge sum divisor cordial graphs.

References

[1] J.A. Bondy, and U.S.R. Murthy, Graph Theory and Application (North Holland), New York, 1976.

[2] D.M. Burton, *Elementary Number Theory*, Second Edition, Wm. C. Brown Company Publishers, 1980.

[3] Ibrahim Cahit, "Cordial Graphs: A Weaker Version of Graceful and Harmonious Graphs," *Ars Combinatoria*, vol. 23, pp. 201-208, 1987. [Google Scholar] [Publisher link]

[4] G. Chartrand, and L. Lesniak, *Graphs and Digraphs*, Third edition, Chapman and Hall/CRC, Florida, 2000. [Google Scholar]

[5] A. Farhana Abbas, and M. Mohamed Sheriff, "Face Integer Cordial, Face Integer Edge Cordial and Total Face Integer Edge Cordial Graphs," *Journal of Applied Science and Computations*, vol. 6, no. 3, pp. 1426-1438, 2019.

[6] Joseph A. Gallian, "A Dynamic Survey of Graph Labeling," *The Electronic Journal of Combinatorics*, 2022, #DS6.

[7] S.W. Golomb, "How to Number a Graph," *Graph Theory and Computing*, pp. 23-37, 1972. [CrossRef][Google Scholar] [Publisher link]

[8] F. Harary, *Graph Theory*, Addison Wesley, Reading, Massachusetts, 1972.

[9] P. Lawrence Rozario Raj, and S. Koilraj, "Cordial Labeling for the Splitting Graph of Some Standard Graphs," *International Journal of Mathematics and Soft Computing*, vol. 1, no. 1, pp. 105-114, 2011. [CrossRef][Google Scholar] [Publisher link]

[10] P. Lawrence Rozario Raj, and R. Lawrence Joseph Manoharan, "Face and Total Face Edge Product Cordial Graphs," *International Journal of Mathematics Trends and Technology*, vol. 19, no. 2, pp. 136-149, 2015. [CrossRef][Google Scholar] [Publisher link]

[11] P. Lawrence Rozario Raj, and R. Lawrence Joseph Manoharan, "Face and Total Face Product Cordial Labeling of Graphs," *International Journal of Innovative Science, Engineering & Technology*, vol. 2, no. 9, pp. 93-102, 2015. [Google Scholar]

[12] A. Lourdusamy, and F. Patrick, "Sum Divisor Cordial Graphs," *Proyecciones Journal of Mathematics*, vol. 35, no. 1, pp. 119-136, 2016. [CrossRef][Google Scholar] [Publisher link]

[13] M. Mohamed Sheriff, A. Farhana Abbas, and P. Lawrence Rozario Raj, "Face Integer Cordial Labeling of Graphs," *International Journal of Mathematics Trends and Technology*, vol. 41, no. 2, pp. 177-185, 2017. [CrossRef][Google Scholar] [Publisher link]

[14] M. Mohamed Sheriff, and A. Farhana Abbas, "A Study on Face Integer Edge Cordial Graphs," *International Journal of Fuzzy Mathematical Archive*, vol. 15, no. 2, pp. 209-216, 2018. [CrossRef][Google Scholar]

[15] M. Mohamed Sheriff, and G. Vijayalakshmi, "Face Sum Divisor Cordial Graphs," *International Journal of Fuzzy Mathematical Archive*, vol. 15, no. 2, pp. 197-204, 2018. [CrossRef][Google Scholar]

[16] M. Mohamed Sheriff, and G. Vijayalakshmi, "Face Sum Divisor Cordial Labeling of Wheel Related Graphs," *The International Journal of Analytical and Experimental Modal Analysis*, vol. 11, no. 10, pp. 44-48, 2019.

[17] M. Mohamed Sheriff, and G. Vijayalakshmi, "Face Sum Divisor Cordial Labeling of Cycle Related Graphs," *Journal of Applied Science and Computations*, vol. 6, no. 2, pp. 875-881, 2019.

[18] A. Rosa, On certain valuations of the vertices of a graph, Theory of Graphs (Internat. Sympos., Rome, 1966), Gorden and Breach, (1967), N.Y. and Dunod Paris, pp. 349-355, 1967.

[19] E. Sampathkumar, and H.B. Walikar, "On Splitting Graph of a Graph," J. Karnatak Univ. Sci., 25 and 26 (Combined) (1980-81), pp. 13-16.

[20] Samir K. Vaidya, and Chirag M. Barasara, "Total Edge Product Cordial Labeling of Graphs," *Malaya Journal of Matematik*, vol. 3, no. 1, pp. 55-63, 2013. [Google Scholar]

[21] R. Varatharajan, S. Navanaeethakrishnan, and K. Nagarajan, "Divisor Cordial Graphs," *International Journal of Mathematical Combinatories*, vol. 4, pp. 15-25, 2011. [Google Scholar]

[22] G. Vijayalakshmi, and M. Mohamed Sheriff, "Total Face Edge Sum Divisor Cordial Graphs," *International Journal of Food and Nutritional Sciences*, vol. 11, no. 11, pp. 170-178, 2022.

[23] G. Vijayalakshmi, M. Mohamed Sheriff, and P. Lawrence Rozario Raj, "Edge Sum Divisor Cordial Graphs," *Journal of Otorhinolaryngology and Head and Neck Surgery*, vol. 26, no. 2, pp. 407-417, 2022.

[24] G. Vijayalakshmi, and M. Mohamed Sheriff, "Face Edge Sum Divisor Cordial Graphs," *Indian Journal of Natural Sciences*, vol. 14, no. 77, pp. 54615-54620, 2023.

[25] Rahime Yilmaz, and Ibrahim Cahit, "E-Cordial Graphs," *Ars Combinatoria*, vol. 46, 1997.