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Original Article

Long Time Behavior of the Global Solutions to the Viscous Two-Phase Flow

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Abstract - *In this paper, we mainly investigate the long-time behavior of three-dimensional two-phase flow model under the slip boundary conditions. Our research is constructed from the existence of the global classical solution. It is a new result to the long-time behavior of pressure gradients in 3D bounded smooth domains.*

Keywords - T*wo-phase model, Long-time behavior, Vacuum.*

1. Introduction

This paper is dedicated to considering the viscous two-phase flow model in R^3 , i.e,

$$\begin{cases} \rho_t + \mathrm{div}(\rho u) = 0, \\ m_t + \mathrm{div}(mu) = 0, \\ ((\rho+m)u)_t + \mathrm{div}((\rho+m)u \otimes u) - \mu\Delta u - (\mu+\lambda)\nabla \mathrm{div} u + \nabla P(\rho, m) = 0. \end{cases} \tag{1}$$

Here, $(x,t) \in \Omega \times (0,T]$, Ω is a domain in $\mathbb{R}^3$, $\rho, m \geq 0$, $u = (u^1, u^2, \cdots, u^N)$ and $P(\rho,m) = a\rho^\gamma + bm^\alpha (a,b > 0, \gamma, \alpha \geq 1)$ are the unknown two-fluids densities, velocity and pressure, respectively. And μ is the shear viscosity coefficient, λ is the bulk viscosity coefficient. μ, λ also satisfy the following physical relationship:

$$\mu > 0, 2\mu + 3\lambda \geq 0. \tag{2}$$

We consider the model (1) under the following initial date:

$$\rho(x,0) = \rho_0(x), m(x,0) = m_0(x), (\rho+m)u(x,0) = (\rho_0 + m_0)u_0(x), x \in \Omega, \tag{3}$$

and consider the (Navier-type) slip boundary condition

$$u \cdot n = 0, \mathrm{curl} u \times n = -Au \ , \quad x \in \partial\Omega, \tag{4}$$

where $B = B(x)$ is 3×3 symmetric positive matrix on $\partial\Omega$.

It is very important for hydrodynamics to select appropriate boundary conditions. For these equations, the first thought idea will consider the non-slip (Dirichlet) boundary condition

$$u = 0 \ , \ x \in \partial\Omega. \tag{5}$$

This condition was proposed by G. Stokes in 1845. As early as 1823, Navier proposed another condition, impermeability boundary condition, i.e,

$$u \cdot n = 0 \ , \ x \in \partial\Omega. \tag{6}$$

This article investigates the Navier-type boundary conditions introduced from reference [2] and the boundary conditions derived from gas dynamics theory by Maxwell, which indicate that the tangential slip velocity is directly proportional to the tangential stress, rather than zero, namely:

$$u \cdot n = 0, (D(u)n + \vartheta u)_{\tan} = 0 \ , \ x \in \partial\Omega, \tag{7}$$

where $D(u) = (\nabla u + (\nabla u)^{tr})/2$ is the shear stress, the scalar friction function ϑ measures the tendency to slip on the boundary of the fluid, and v_{tan} represents the tangent projection of the vector v .

Navier initially introduced the slip boundary condition (4) in 1823. Afterwards, this (Navier-type) slip boundary condition has been used in many analysis of various fluid mechanics problems, applications and numerical studies. For details, refer to [3 – 5] and its references.

There are many researches about this model. For the large initial value, Vasseur et al [6] had the existence of global weak solutions to the model (1) with the pressure $P(\rho,m)=a\rho^{\gamma}+bm^{\alpha}(a,b>0,\gamma,\alpha\ge 1)$ and the domination. Later, Novotný et al [7] developed the domination condition to the condition where γ and α all can come into contact $9/5$, taking into account the more normal pressure law that $P(\rho,m)=a\rho^{\gamma}+bm^{\alpha}(a,b>0,\gamma,\alpha\ge 1)$ were considered. Wen [8] studied the existence of global weak solutions of three-dimensional compressible two-phase flow model without control conditions. For the global existence of classical solutions, Li, Liu and Ye [9] first studied the viscous two-phase flow model in a general two-dimensional bounded smooth domain with vacuum, and had the asymptotic behavior of the global classical solutions. In addition, Zhao [10] obtained the long time of the viscous liquid-gas (drift-flux type) two-phase flow in 3D with Cauchy problem. For more relevant studies, refer to references [11-20] please and the references in it.

The results obtained for the CNS (compressible Navier Stokes) equations with isentropic are quite rich. Ding, Wen, and Zhu [21] obtained the global well-posedness of the classical solutions for the initial boundary value problem of CNS in one-dimensional. Specifically, in the case of vacuum, Jiu, Li, and Ye [22] provided the existence of global classical solutions for one-dimensional CNS with large initial values. For the strong solutions, Vaigant and Kazhikhov [23] first considered the global existence for CNS in two-dimension. Later, Jiu, Wang, and Xin discussed the existence of global classical solutions for the two-dimensional compressible Navier Stokes equation system in the periodic region T^2 and the whole space $\mathbb{R}^2$ under conditions of large initial values and possible vacuum in references [24,25], respectively. In the case of the entire space $\mathbb{R}^2$, space weighted energy estimation was introduced. Last for the non-isentropic Navier Stokes system, Huang and Li obtained the existence of global classical solutions in three-dimension in [26], allowing for large oscillations in the initial value and possible vacuum.

Notations: In the following, we give some necessary definitions which will be used later.

We first introduce notation and function spaces which will be employed throughout this paper.

For the integer of $0\le r\le\infty$ and $k\ge 0$, we describe the marks we shall use in this paper. The specific symbol definitions are as below:

$$\begin{cases} L^r=L^r(\Omega), \quad \|u\|_{D^{k,r}}\triangleq\|\nabla^k u\|_{L^r}, \quad D^{k,r}=\{u\in L^1_{loc}(\Omega)\mid \ \|\nabla^k u\|_{L^r}<\infty\}, \\ D^1=\left\{u\in L^6 \mid \ \|\nabla u\|_{L^2}<\infty\right\}, \quad H^k=W^{k,2}, \quad W^{k,r}=L^r\cap D^{k,r}. \end{cases}$$

Next, we set

$$\int f\,dx\triangleq\int_\Omega f\,dx, \quad \bar f\triangleq\frac{1}{|\Omega|}\int_\Omega f\,dx, \quad \dot f\triangleq f_t+(u\cdot\nabla)f. \tag{8}$$

We utilize vorticity ω and effective viscous flux F are defined as the below form:

$$\omega=\nabla\times u,\ F\triangleq(\lambda+2\mu)\operatorname{div}u-(P-\bar P).$$

Eventually, the initial total energy of equation (1) is expressed as

$$C_0\triangleq\int_\Omega\left(\frac12(\rho_0+m_0)|u_0|^2+G(\rho_0,m_0)\right)dx, \tag{9}$$

with

$$G(\rho,m)\triangleq\rho\int_{\bar\rho}^{\rho}\frac{P(s,m)-P(\bar\rho,m)}{s^2}ds+m\int_{\bar m}^{m}\frac{P(\rho,s)-P(\rho,\bar m)}{s^2}ds. \tag{10}$$

Similar to the proof of the result in [1, Theorem 1.1], we will give the following existence of global classical solutions of (1)-(4) in general smooth bounded domain in order to investigate the long-time behavior.

Proposition 1.1. Suppose a simply connected bounded domain $\Omega\subset\mathbb{R}^3$, its smooth boundary $\partial\Omega$ has a finite number of 2-dimensional connected components. The positive constants, M, $\hat\rho,\hat m$, suppose that the 3×3 positive semi-definite symmetric matrix B in (1.4) is enough smooth, and for some $q\in(3,6)$ and $s\in(1/2,1]$, the initial value (ρ_0,m_0,u_0) satisfy

$$\left(\rho_0,m_0,P(\rho_0,m_0)\right)\in W^{2,q}, \quad u_0\in\left\{f\in H^2: f\cdot n=0, \quad \operatorname{curl} f\times n=-Bf \ \text{ on } \partial\Omega\right\},$$
$$0\le\rho_0\le\hat\rho, \quad 0\le m_0\le\hat m, \quad \|u_0\|_{H^s}\le M,$$

and

$$-\mu\Delta u_0-(\mu+\lambda)\nabla\operatorname{div}u_0+\nabla P(\rho_0,m_0)=(\rho_0+m_0)^{1/2}g,$$

for some $g\in L^2$ which called the compatibility condition. Then there exists a positive constant ε depending only on

$\mu,\lambda,\gamma,\alpha,a,b$, $\hat{\rho},\hat{m},s,\Omega,M$, and the matrix A such that the initial-boundary-value problem (1)-(4) has a unique classical solution (ρ,m,u) in $\Omega\times(0,\infty)$ satisfying

$$0\le\rho(x,t)\le 2\hat{\rho},\quad 0\le m(x,t)\le 2\hat{m},\quad (x,t)\in\Omega\times[0,\infty),$$

and for any $0<\tau<T<\infty$,

$$\begin{cases}(\rho,m,P)\in C\left([0,T];W^{2,q}\right),\\ \nabla u\in L^{\infty}\left(\tau,T;W^{2,q}\right)\cap C\left([0,T];H^{1}\right),\\ u_t\in H^{1}\left(\tau,T;H^{1}\right)\cap L^{\infty}_{loc}\left(0,T;H^{2}\right),\\ \sqrt{\rho}u_t,\sqrt{m}u_t\in L^{\infty}\left(0,\infty;L^{2}\right),\end{cases}\tag{11}$$

provided $C_0\le\varepsilon$.

We aim to prove that the long-time behavior of the global classical solution to viscous two-phase flow model. More accurately, we can obtain the below main results.

Theorem 1.2. Satisfying the condition of Proposition 1.1, the below long-time behavior holds

$$\lim_{t\to\infty}\int\left(|P-\overline{P}|^{q}+(\rho+m)^{\frac{1}{2}}|u|^{\nu}+|\nabla u|^{2}\right)dx=0,\tag{12}$$

for any $2\le\nu\le4$ and $2\le q<\infty$.

Our second result is long-time behavior of the gradient pressure.

Theorem 1.3. Satisfying the assumption of Theorem 1.2, assumed that there are some points $x_1\in\Omega$ and satisfies $P_0(x_1)=0$. Then as $t\to\infty$ in the sense, the global classical solution (ρ,m,u) to the problem (1)-(4) obtained in Theorem 1.2 has to blow up, i.e., for any $3<j<\infty$,

$$\lim_{t\to\infty}\|\nabla P(\cdot,t)\|_{L^{j}}=\infty.\tag{13}$$

2. Proofs of Theorems 1.2 and 1.3

By setting

$$D_1(T)\triangleq\sup_{0\le t\le T}\left(\sigma\|\nabla u\|_{L^2}^2\right)+\int_0^T\int\sigma(\rho+m)|\dot{u}|^2\,dxdt,\tag{14}$$

$$D_2(T)\triangleq\sup_{0\le t\le T}\sigma^3\int(\rho+m)|\dot{u}|^2\,dx+\int_0^T\int\sigma^3|\nabla\dot{u}|^2\,dxdt,\tag{15}$$

$$D_3(T)\triangleq\sup_{0\le t\le T}\int(\rho+m)|u|^3\,dx,$$

where $\sigma(t)\triangleq\min\{1,t\}$ and we define $\dot{u}$ in (8).

The following Lemma is very useful to the second section which means the global existence of the classical solution of (1)-(4).

Lemma 2.1. Satisfying the conditions of Proposition 1.1, for $\delta_0\triangleq\frac{2s-1}{4s}\in\left(0,\frac{1}{4}\right]$, there exists a positive constant ε depending on μ , λ , a , b , γ , α , $\hat{\rho}$, $\hat{m}$, s , Ω , M and the matrix B such that if (ρ,m,u) is a classical solution of (1)-(4) on $\Omega\times(0,T]$ satisfying

$$\sup_{\Omega\times[0,T]}\rho\le 2\hat{\rho},\quad \sup_{\Omega\times[0,T]}m\le 2\hat{m},\quad D_1(T)+D_2(T)\le 2C_0^{\frac{1}{3}},\quad D_3(\sigma(T))\le 2C_0^{\delta_0},$$

then the below estimates hold

$$\sup_{\Omega\times[0,T]}\rho\le\frac{7\hat{\rho}}{4},\quad \sup_{\Omega\times[0,T]}m\le\frac{7\hat{m}}{4},\quad D_1(T)+D_2(T)\le C_0^{\frac{1}{3}},\quad D_3(\sigma(T))\le C_0^{\delta_0},\tag{16}$$

provided $C_0\le\varepsilon$.

Proceeding as in the proof in [1, Lemma 2.3], we have the below lemma will be used throughout this paper. The proof of this Lemma is too complicated to be given here.

Lemma 2.2. Assume (ρ,m,u) is the corresponding solution of (1)-(4) on $\Omega\times(0,T]$. Then, $\|\nabla u\|_{L^p}$ obeys the below estimate:

$$\|\nabla u\|_{L^p} \le C\|(\rho+m)\dot{u}\|_{L^2}^{(3p-6)/(2p)}\left(\|P-\overline{P}\|_{L^2}+\|\nabla u\|_{L^2}\right)^{(6-p)/(2p)}+C\left(\|P-\overline{P}\|_{L^2}+\|\nabla u\|_{L^2}\right). \tag{17}$$

where the positive constant C depending only on μ,λ and Ω.

Next, we introduce the below two lemmas which are in [27, Theorem 3.2] and in [28, Propositions 2.6-2.9].

Lemma 2.3. Let h be a non-zero positive integer, $1<g<+\infty$, assume that $\Omega\subset\mathbb{R}^3$ is a simply connected region and satisfy $C^{h+1,1}$ on the boundary $\partial\Omega$. Then for $f\in W^{h+1,g}$ with $f\cdot n=0$ on $\partial\Omega$, there exists a constant $C=C(g,h,\Omega)$ such that

$$\|v\|_{W^{h+1,g}} \le C\left(\|\operatorname{div} v\|_{W^{h,g}}+\|\operatorname{curl} v\|_{W^{h,g}}\right).$$

Especially, for the case of $k=0$, then

$$\|\nabla v\|_{L^g} \le C\left(\|\operatorname{div} v\|_{L^g}+\|\operatorname{curl} v\|_{L^g}\right).$$

Lemma 2.4. Let h be a non-zero positive integer, $1<g<+\infty$, assume a simply connected region $\Omega\subset\mathbb{R}^3$ satisfies $C^{h+1,1}$ on the boundary $\partial\Omega$ which only has a finite connected components in 2-dimension. For $v\in W^{h+1,g}$ with $f\times n=0$ on $\partial\Omega$, then there exists a constant $C=C(h,g,\Omega)$ such that

$$\|v\|_{W^{h+1,g}} \le C\left(\|\operatorname{div} v\|_{W^{h,g}}+\|\operatorname{curl} v\|_{W^{h,g}}+\|v\|_{L^g}\right).$$

Especially, as for the case of Ω is a simply connected region, we have

$$\|v\|_{W^{h+1,g}} \le C\left(\|\operatorname{div} v\|_{W^{h,g}}+\|\operatorname{curl} v\|_{W^{h,g}}\right).$$

Next, we will introduce the following inequalities and provide some given facts that will be often applied later on. We consider the following equation system

$$\begin{cases}\operatorname{div} f = v, \ x\in\Omega,\\ f=0, \quad x\in\partial\Omega.\end{cases} \tag{18}$$

Lemma 2.5. [Theorem III.3.1] The operator $\mathcal{B}=\left[\mathcal{B}_1,\mathcal{B}_2,\mathcal{B}_3\right]$ satisfying the following properties:

(1)

$$\mathcal{B}:\left\{v\in L^p(\Omega):\bar{v}=0\right\}\mapsto (W_0^{1,p}(\Omega))^3$$

is bounded linear, i.e., for any $p\in(1,\infty)$,

$$\|\mathcal{B}[v]\|_{W_0^{1,p}(\Omega)} \le C(p)\|v\|_{L^p(\Omega)}.$$

(2) The function $f=\mathcal{B}[v]$ solve the problem (18).

(3) If v can be written in the form $v=\operatorname{div} j$ for a certain $j\in L^r(\Omega)$, $j\cdot n|_{\partial\Omega}=0$, then

$$\|\mathcal{B}[v]\|_{L^r(\Omega)} \le C(r)\|j\|_{L^r(\Omega)},$$

for any $r\in(1,\infty)$.

In the following, we begin with the standard estimate for (ρ,m,u).

Lemma 2.6. Let (ρ,m,u) be a smooth solution of (1)-(4) on $\Omega\times(0,T]$. Then there is a positive constant C depending only on μ,λ and Ω such that

$$\sup_{0\le t\le T}\int\left((\rho+m)|u|^2+G(\rho,m)\right)dx+\int_0^T\|\nabla u\|_{L^2}^2\,dt \le CC_0. \tag{19}$$

Proof. First of all, integrating $(1)_{1,2}$ over $\Omega\times(0,T)$ and by using of (4), we can have

$$\bar{\rho}=\frac{1}{|\Omega|}\int\rho(x,t)dx=\frac{1}{|\Omega|}\int\rho_0(x)dx,$$

$$\bar{m}=\frac{1}{|\Omega|}\int m(x,t)dx=\frac{1}{|\Omega|}\int m_0(x)dx.$$

Next, we can rewrite $(1)_3$ as the following form:

$$(\rho+m)\dot{u}-(\lambda+2\mu)\nabla\operatorname{div} u+\mu\nabla\times\operatorname{curl} u+\nabla P=0, \tag{20}$$

Where we have used the fact that $-\Delta u=-\nabla\operatorname{div} u+\nabla\times\operatorname{curl} u$.

Multiplying (20) by u and integrating the equality which take two successive operations over Ω, we can get

$$\frac{1}{2}\left(\int(\rho+m)|u|^2\,dx\right)_t+(\lambda+2\mu)\int(\mathrm{div}\,u)^2dx+\mu\int|\mathrm{curl}\,u|^2\,dx+\mu\int_{\partial\Omega}u\cdot B\cdot uds=\int(P-\overline{P})\,\mathrm{div}\,udx, \tag{21}$$

due to the boundary condition (4).

Multiplying $(1)_1$ by $(\int_{\bar\rho}^{\rho}\frac{P(s,m)-P(\bar\rho,m)}{s^2}ds+\frac{P(\rho,m)-P(\bar\rho,m)}{\rho})$, we can arrive at

$$\left(\int\rho\int_{\bar\rho}^{\rho}\frac{P(s,m)-P(\bar\rho,m)}{s^2}dsdx\right)_t+\int\mathrm{div}u(P(\rho,m)-P(\bar\rho,m))dx=0. \tag{22}$$

Using the same method mentioned above, we can obtain

$$\left(\int m\int_{\bar m}^{m}\frac{P(\rho,s)-P(\rho,\bar m)}{s^2}dsdx\right)_t+\int\mathrm{div}u(P(\rho,m)-P(\rho,\bar m))dx=0. \tag{23}$$

Combining (22) and (23) together yields that

$$\left(\int G(\rho,m)dx\right)_t+\int(P-\overline{P})\,\mathrm{div}\,udx=0, \tag{24}$$

which together with the definition of B, (21) and (24) gives

$$\sup_{0\le t\le T}\int\left((\rho+m)|u|^2+G(\rho,m)\right)dx+\int_0^T\|\nabla u\|_{L^2}^2dt\le CC_0. \tag{25}$$

The proof is completed.

Lemma 2.7. Assume (ρ,m,u) is a classical solution of (1)-(4) on $\Omega\times(0,T]$. Then we have

$$\sup_{0\le t\le T}\sigma\|P-\overline{P}\|_{L^2}^2+\int_0^T\|P-\overline{P}\|_{L^2}^2dt\le CC_0^{\frac{1}{2}}, \tag{26}$$

where $C(\lambda,\mu,a,b,\gamma,\alpha,\hat\rho,\hat m,\Omega)$.

Proof. By $(1)_{1,2}$, one can obtain that

$$P_t+\mathrm{div}(Pu)+(a(\gamma-1)\rho^\gamma+b(\alpha-1)m^\alpha)\,\mathrm{div}\,u=0,$$

or

$$P_t+u\cdot\nabla P+(a\gamma\rho^\gamma+b\alpha m^\alpha)\,\mathrm{div}\,u=0,$$

which together with Lemma 2.5 shows that

$$\begin{aligned}\|\mathcal{B}[P_t-\overline{P}_t]\|_{L^2}&\le C\left(\|\mathcal{B}[a\rho^\gamma\mathrm{div}u-\overline{a\rho^\gamma\mathrm{div}u}]\|_{L^2}+\|\mathcal{B}[bm^\alpha\mathrm{div}u-\overline{bm^\alpha\mathrm{div}u}]\|_{L^2}+\|\mathcal{B}[\mathrm{div}(Pu)]\|_{L^2}\right)\\&\le C\left(\|a\rho^\gamma\mathrm{div}u-\overline{a\rho^\gamma\mathrm{div}u}\|_{L^2}+\|bm^\alpha\mathrm{div}u-\overline{bm^\alpha\mathrm{div}u}\|_{L^2}+\|Pu\|_{L^2}\right)\le C\|\nabla u\|_{L^2}.\end{aligned}$$

Multiplying the equality $(1)_3$ by $\mathcal{B}[P-\overline{P}]$ and then integrating over Ω, one can show that

$$\begin{aligned}&\int(P-\overline{P})^2dx=-\int\nabla(P-\overline{P})\cdot\mathcal{B}[P-\overline{P}]dx\\&=\left(\int(\rho+m)u\cdot\mathcal{B}[P-\overline{P}]dx\right)_t-\int(\rho+m)u\cdot\mathcal{B}[P_t-\overline{P}_t]dx+(\lambda+\mu)\int\mathrm{div}u(P-\overline{P})dx\\&+\mu\int\nabla u:\nabla\mathcal{B}[P-\overline{P}]dx-\int(\rho+m)u\cdot\nabla\mathcal{B}[P-\overline{P}]\cdot udx\\&\le\left(\int(\rho+m)u\cdot\mathcal{B}[P-\overline{P}]dx\right)_t+C\left(\|\mathcal{B}[P_t-\overline{P}_t]\|_{L^2}\|u\|_{L^2}+\|u\|_{L^4}^2\|\nabla\mathcal{B}[P-\overline{P}]\|_{L^2}+\|\nabla u\|_{L^2}\|P-\overline{P}\|_{L^2}\right)\\&\le\left(\int(\rho+m)u\cdot\mathcal{B}[P-\overline{P}]dx\right)_t+\varepsilon\|P-\overline{P}\|_{L^2}^2+C\left(\|\mathcal{B}[P_t-\overline{P}_t]\|_{L^2}\|u\|_{L^2}+\|u\|_{L^4}^{\beta}+\|\nabla u\|_{L^2}^{\beta}\right)\\&\le\left(\int(\rho+m)u\cdot\mathcal{B}[P-\overline{P}]dx\right)_t+\varepsilon\|P-\overline{P}\|_{L^2}^2+C\|\nabla u\|_{L^2}^2.\end{aligned}$$

Choosing ε small enough and using $(1)_1$ and the inequality (19) several times, we can obtain

$$
\begin{aligned}
&\int_0^T \| P-\bar{P}\|_{L^2}^2 dt \\
&\le \int (\rho+m)u\cdot \mathcal{B}[P-\bar{P}]dx - \int (\rho_0+m_0)u_0\cdot \mathcal{B}[P_0-\bar{P}_0]dx + C\int_0^T \|\nabla u\|_{L^2}^2 dt \\
&\le C\left(\|(\rho+m)^{\frac{1}{2}}u\|_{L^2}\| \mathcal{B}[P-\bar{P}]\|_{L^2} + \|(\rho_0+m_0)^{\frac{1}{2}}u_0\|_{L^2}\| \mathcal{B}[P_0-\overline{P_0}]\|_{L^2}\right) + CC_0 \\
&\le CC_0^{\frac{1}{2}}.
\end{aligned} \tag{27}
$$

Furthermore, due to $(1)_{1,2}$, we arrive at

$$
\left(P-\bar{P}\right)_t = -\left(a\gamma\rho^\gamma + b\alpha m^\alpha\right)\mathrm{div}u - u\cdot\nabla\left(P-\bar{P}\right) + \overline{a((\gamma-1)\rho^\gamma + b(\alpha-1)m^\alpha)\,\mathrm{div}u}\,. \tag{28}
$$

Multiplying (28) with $2\sigma(P-\bar{P})$, and integrating this result over Ω, we immediately get the below estimate

$$
\left(\sigma\int\left(P-\bar{P}\right)^2 dx\right)_t \le C(\sigma+\sigma')\int (P-\bar{P})^2 dx + C\sigma\int |\nabla u|^2\, dx,
$$

which along with (19) and (28) haves

$$
\sup_{0\le t\le T}\sigma\| P-\bar{P}\|_{L^2}^2 \le CC_0^{\frac{1}{2}}.
$$

The proof is completed.

Proof of Theorem 1.2. To obtain the result (12), multiplying (28) by $2(P-\bar{P})$ and integrating the result over Ω by parts imply

$$
\left(\| P-\bar{P}\|_{L^2}^2\right)_t \le C\|\,\mathrm{div}u\|_{L^2}^2 + C\| P-\bar{P}\|_{L^2}^2,
$$

Then combining with (19) and (26), it leads to

$$
\int_1^\infty \left(\| P-\bar{P}\|_{L^2}^2\right)_t dt \le C, \tag{29}
$$

making use of (29), we derive

$$
\lim_{t\to\infty}\| P-\bar{P}\|_{L^q} = 0, \qquad 2\le q<\infty, \tag{30}
$$

which together with (19) gives

$$
\int (\rho+m)^{\frac{1}{2}}\,|u|^\nu\, dx \le \| u\|_{L^{2(\nu-1)}}^{\nu-1}\left(\int (\rho+m)\,|u|^2\, dx\right)^{\frac{1}{2}} \le C\|\,\nabla u\|_{L^2}^{\nu-1}. \tag{31}
$$

Using the same arguments as (3.24) of Lemma 3.4 in [1], we can easily get

$$
\begin{aligned}
&\left(\int (\lambda+2\mu)\sigma^k(\mathrm{div}u)^2 + \mu\sigma^k\,|\mathrm{curl}\,u|^2\, dx + \mu\int_{\partial\Omega}\sigma^k u\cdot B\cdot u ds\right)_t + \int \sigma^k(\rho+m)\,|\dot{u}|^2\, dx \\
&\le \left(2\int\sigma^k\,\mathrm{div}u(P-\bar{P})dx\right)_t + Ck\sigma^{k-1}\sigma'\| P-\bar{P}\|_{L^2}^2 + C\|\,\nabla u\|_{L^2}^2 + C\sigma^k\|\,\nabla u\|_{L^2}^4 + C\sigma^k\|\,\nabla u\|_{L^3}^3.
\end{aligned} \tag{32}
$$

Let $k=0$ in (32), integrating this in time from 1 to ∞ and making use of (16), (17), (19) and (26), we derive the fact

$$
\begin{aligned}
\int_1^\infty |\phi'(t)|^2\, dt &\le C\int_1^\infty (\|\nabla u\|_{L^2}^2 + \|\,\nabla u\|_{L^2}^4 + \|\,\nabla u\|_{L^3}^3)dt \\
&\le C\int_1^\infty (\|\nabla u\|_{L^2}^2 + \|\,\nabla u\|_{L^2}^4 + \|\,\nabla u\|_{L^2}^3 + \|\,P-\bar{P}\|_{L^3}^3 + \|(\rho+m)\dot{u}\|_{L^2}^3)dt \\
&\le C,
\end{aligned} \tag{33}
$$

which $\phi(t) = (\lambda+2\mu)\|\,\mathrm{div}u\|_{L^2}^2 + \mu\|\,\mathrm{curl}u\|_{L^2}^2$. By (19), we obtain that

$$
\int_1^\infty \|\,\nabla u\|_{L^2}^2 dt \le \int_0^\infty \|\,\nabla u\|_{L^2}^2 dt \le C,
$$

which together with (33) and Lemma 2.2-2.3 implies

$$
\lim_{t\to\infty}\|\,\nabla u\|_{L^2} = 0, \tag{34}
$$

therefore, the proof of (12) is proved.

Proof of Theorem 1.3. If the result of the Theorem 1.2 is not true, then there is a constant $C_1 > 0$ and a subsequence $\left\{t_{n_j}\right\}_{j=1}^{\infty}$, such that $\|\nabla P(\cdot, t_{n_j})\|_{L^r} \le C_1$, when $t_{n_j} \to \infty$. By Gagliardo-Nirenberg's inequality, for $\eta = 3r/\left(3r + 2(r-3)\right) \in (0,1)$, there exists some positive constant C independent of t_{n_j} such that

$$\|P(x,t_{n_j}) - \bar{P}\|_{C(\bar{\Omega})} \le C\|\nabla P(x,t_{n_j})\|_{L^r}^{\eta} \|P(x,t_{n_j}) - \bar{P}\|_{L^2}^{1-\eta} + \|P(x,t_{n_j}) - \bar{P}\|_{L^2}$$
$$\le CC_1^{\eta}\|P(x,t_{n_j}) - \bar{P}\|_{L^2}^{1-\eta} + \|P(x,t_{n_j}) - \bar{P}\|_{L^2},$$

which along with (12) yields

$$\|P(x,t_{n_j}) - \bar{P}\|_{C(\bar{\Omega})} \to 0 \ \ as\, t_{n_j} \to \infty. \tag{35}$$

In addition, for all $t > 0$, there exists a unique particle path $x_0(t)$ satisfying $x_0(t) = x_0$ such that

$$P(x_0(t), t) \equiv 0.$$

Consequently, we can have

$$0 < \bar{P} \equiv |P(x_0(t_{n_j}), t_{n_j}) - \bar{P}| \le \|P(x,t_{n_j}) - \bar{P}\|_{C(\bar{\Omega})},$$

which contradicts (35). Then we can get the result of (13), which completes the proof of Theorem 1.3.

Conflicts of Interest
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