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Original Article

Exact Angle Trisection with Straightedge and Compass by Secondary Geometry

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Abstract - *Angle trisection is a classical problem of straightedge and compass construction from the ancient Greek mathematics. It concerns construction of an angle equal to one third of a given arbitrary angle, using only two tools: an unmarked straightedge and a compass. There are three classical problems in the ancient Greek mathematics which were extremely influential in the development of geometry. These problems were those of squaring the circle, doubling the cube and trisecting an angle. This Thesis focuses on the problem of trisecting an arbitrary angle. It is possible to trisect certain angles, e.g. a right angle. It is difficult to give an accurate date as to when the problem of trisecting an angle first appeared. Result of this research paper is an exact solution for the thousand-year challenge "Trisecting the Angle" by a construction with only a straightedge and a compass by means of the secondary Geometry.*

Keywords - *Angle trisection, Divide angle into 3 equally small angle, Divide angle with straightedge and compass, Divide angle by secondary geometry, Trisecting the angle.*

1. Introduction

Angle trisection is a classical problem of straightedge and compass construction from the ancient Greek mathematics. It concerns construction of an angle equal to one third of a given arbitrary angle, using only two tools: an unmarked straightedge and a compass.

There are three classical problems in the ancient Greek mathematics which were extremely influential in the development of geometry. These problems were those of squaring the circle, doubling the cube and trisecting an angle. This Thesis focuses on the problem of trisecting an arbitrary angle. It is possible to trisect certain angles, e.g. a right angle. It is difficult to give an accurate date as to when the problem of trisecting an angle first appeared. Result of this research paper is an exact solution for the thousand-year challenge "Trisecting the Angle" by a construction wit only a straightedge and a compass by means of the secondary Geometry.

However we do know that Hippocrates, who made the first major contribution to the problems of squaring a circle and doubling a cube, also studied the problem of trisecting an angle. And also he did use a mark in the straightedge to make the ruler no longer an unmarked straightedge.

Around 387 BC, Plato said: "*In proceeding in* [*a mechanical*] *way, did not one lose irredeemably the best of geometry...*". Most historians of Mathematics believe that many of the results given in the *Book of Lemmas* are indeed due to Archimedes and the *on spirals* result given on trisecting an angle is so much in the spirit of the work. However this trisecting is not an exact method and does not use a straightedge. Nicomedes method used the *conchoids curve*, which is not always actually drawn and was more theoretical rather than practical. In practice the conchoid was not always actually drawn. Obviously, the ways of trisecting an angle by Hippocrates, Archimedes, or using the conchoid of Nicomedes (around 200 BC) are correct but do not obey the "rules of the game" i.e. using a straightedge and a compass. Maybe they have thought of all the ways but can't do it, so they have to invent their own way to solve this problem. Later, there were many more attempts of successive generations of Mathematicians who failed to do so, therefore they devised many different ways, and thus Mathematics had a chance to develop.

Pierre Wantzel proved in 1837 that the problem, as stated, is impossible to solve for arbitrary angles. In 1837, Wantzel published proofs in Liouville's Journal of "the means of ascertaining whether a geometric *problem can be solved* *with ruler and compasses",* and he was the first to prove trisecting an angle could not be solved with a straightedge and a compass.

This method uses a straightedge & a compass to construct and prove it is possible to trisect an arbitrary angle simply, without the use of any curve. Mathematics tools I used are some geometry theorems in secondary level. The results of my method will give a proof which is a counter-proof to Wantzel's.

At the end of 2021, after 10 years, I discovered my own solution for trisecting any arbitrary angle. My research results for trisecting an angle also opens a simple route to construct a Morley Equilateral Triangle with only a straightedge and a compass.

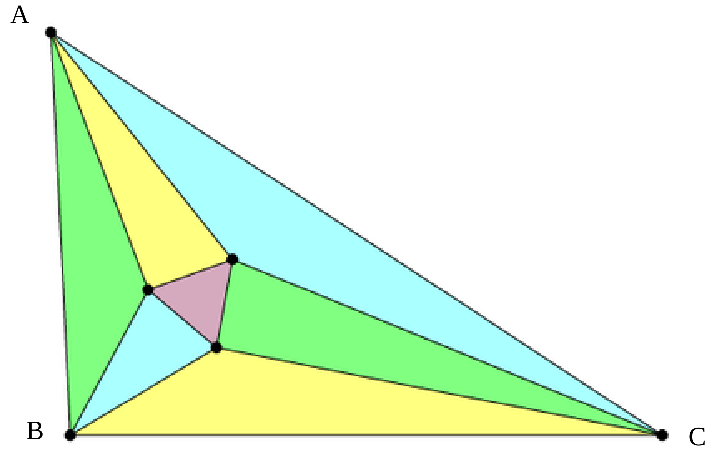


Figure of a Morley triangle ABC, where the lines at each angle of the triangle are the trisectors (named temporarily) of the angle

2. Trisection Ruler for an Arbitrarily given Angle

This part consists of some simple theoretical mathematics at the secondary geometry level.

2.1. Theorem 1: Given an arbitrary angle $\widehat{MPN} = \alpha < 60°$, $\alpha \subset$ r, then we can construct, with a compass & a straightedge, an angle $\widehat{APB} = \beta = 3\alpha$, $\beta < 180°$, where the two side lines of the unit angle α are placed in β and divide β into 3 equal angles α. These side lines are optionally defined as "TRISECTORS" (plural) of the angle β.

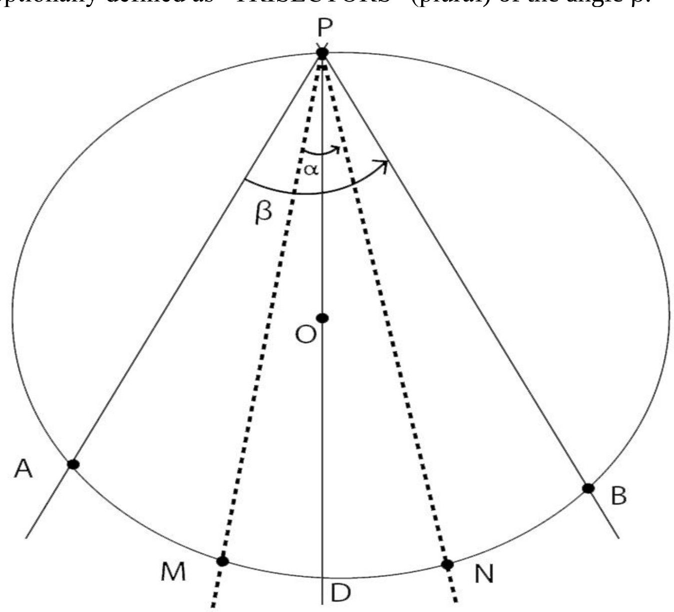


Fig. 1 The two "trisectors" OM & ON of the angle $\widehat{APB} = \beta = 3\alpha$, α is the given angle $\widehat{MON}$.

Proof: Aim to prove it is possible to use a straightedge & a compass to construct the angle $\widehat{APB} = \beta < 180°$ from the given angle $\alpha < 60°$, $\alpha \subset r$, as follows:

Let P be the vertex of the given angle $\widehat{APB}$ and PD is the bisector of α. Choose point O in PD so that OP = r, r is an arbitrary length and $r \subset r$. Then draw a circle (O, r), which meets 2 sides of α at points M & N, where $\widehat{MPN} = \alpha$. Use a compass to mark 2 points A & B on the circle where arcs $\widehat{MN} = \widehat{AM} = \widehat{NB}$. Because arc $\widehat{AB}$ is measured equally to 3 times arc $\widehat{MN}$, then the angle $\widehat{APB} = 3\alpha$, say $\beta = 3\alpha$.

Thus, we obtain the angle $\widehat{APB} = \beta = 3\alpha$, $\beta = 3\alpha < 180°$, as required.

2.2. Theorem 2: For a given straight-line segment AB, AB = 2r, $r \subset r$, then there only exists a unique regular upside-down semi-hexagon ABCD, where AC = CD = DB = r = AM = MB.

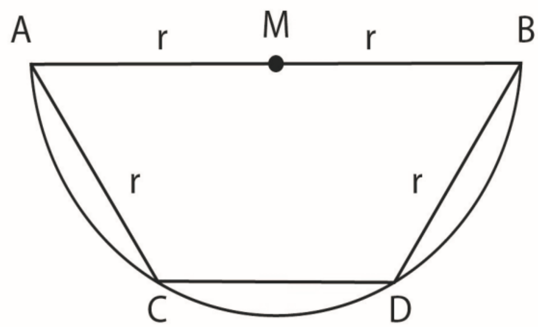


Fig. 2 The unique semi-hexagon upside down for a given r

Proof: Let M be the mid-point of a given straight line segment AB, AM = MB = r, $r \subset r$. Use a compass to draw a semi-circle (M, r) upside-down with diameter AB. Then with the compass, mark points C & D on the semi-circle, where the circle arcs $\widehat{AC} = \widehat{CD} = \widehat{DB}$ have their arc chords AC = CD = DB = r, to get an inscribed semi-hexagon ABCD (regular) in the semi-circle. Because the given length r is unique, the semi-circle (M, r) is unique. Then the inscribed semi-hexagon ABCD in the semi-circle is also unique for a given r.

2.3. Definition 1: With an arbitrary length p, $p \subset r$, Given an angle $\widehat{AOB} = \beta$, $\beta \subset r$ and $\beta < 180°$ then the regular semi-hexagon ABCD inscribed in the angle $\widehat{AOB}$, where AB = 2r & CD = r, $r \subset r$, are perpendicular to the bisector of the angle, described by the following Figure 3, is called the Trisection Ruler of the given angle $\widehat{AOB} = \beta$, $\beta \subset r$ and $\beta < 180°$.

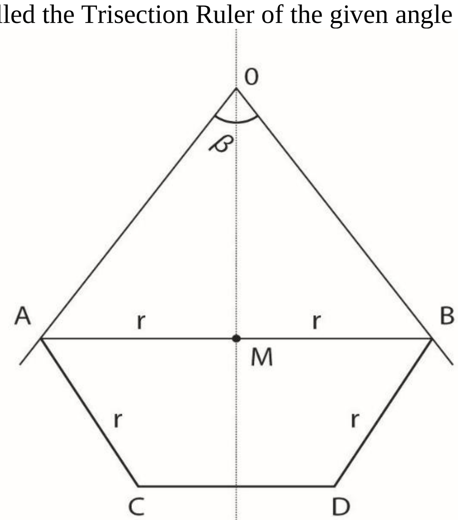


Fig. 3 Trisection Ruler ABCD of a given angle $\widehat{AOB} = \beta$, $\beta < 180°$

Discussion: Use a compass to draw a circle (O, p), p is an arbitrary length and p ⊂ r, O is the vertex of the given angle $\widehat{AOB}$ = β, β ⊂ r. Then the circle cuts 2 sides of $\widehat{AOB}$ at A & B. Let M be the mid-point of AB and AM = MB = r, r ⊂ r, then AB is obviously perpendicular to the bisector of the angle $\widehat{AOB}$ at M. Use the compass to draw a semi-circle (M, r) with the diameter AB. And then, with the compass, mark points C & D on the semi-circle circumference where the arc chords AC = CD = DB = r ($\widehat{AC}$ = $\widehat{CD}$ = $\widehat{DB}$) to get an inscribed regular semi-hexagon ABCD (regular) in the semi-circle. Therefore, for any given arbitrary p, p ⊂ r, in both sides of the given angle $\widehat{AOB}$ = β, β <180°, we can get a Trisection Ruler of the angle $\widehat{AOB}$, which locates at AB, OA = OB = p. Because there are an infinite number of p ⊂ r then we can construct an infinite number Trisection Rulers for any given angle $\widehat{AOB}$ = β, β <180°.

The above discussion is also a guide to construct the Trisection Ruler of any given angle which is less than 180°.

2.4. Corollary 1: Given an angle $\widehat{UOV}$ = β, β <180° , and its Trisection Ruler ABCD with sides r, r ⊂ r, then there exist a unique arc chord XO of the arc $\widehat{XO}$ of the circle (C, CO), which has properties:

- XO = r
- XO is horizontal

and

- the arc chord XO above is unique.

Proof: By the Definition 1 and its Discussion, the given angle $\widehat{UOV}$ = β, β <180°, has a Trisection Ruler ABCD, OA = OP = p, p (arbitrary) ⊂ r, described in Figure 4 below. Let AC = CD = DB = r and AB = 2r (by Definition 1), r ⊂ r. By the Definition 1 and its Discussion, CD is perpendicular to the bisector of the angle $\widehat{UOV}$ at the mid-point M of CD and CM = MD = r. From C draw a circle (C, CO) and a horizontal straight line from point O to the left to meet the circle circumference at X. Then the vertical straight line from C cuts XO at I and CI ⊥ XO. Because CI is prolonged vertically and will be a radius of (C, CO), I is the mid-point of XO or XI = IO. Consider the rectangular shape ICMO with 2 vertical sides IC & OM and 2 horizontal sides IO and CM to get IO = CM = ½ r = XI. From XI = IO we get the length of the horizontal chord XO equal to r. Of course, from O in the circumference of the circle (C, CO) and above AB, only one horizontal chord OX has the proved unique length r. In the other words, any other horizontal chord from any other point located in the circumference of (C, CO) above AB has its length different from r.

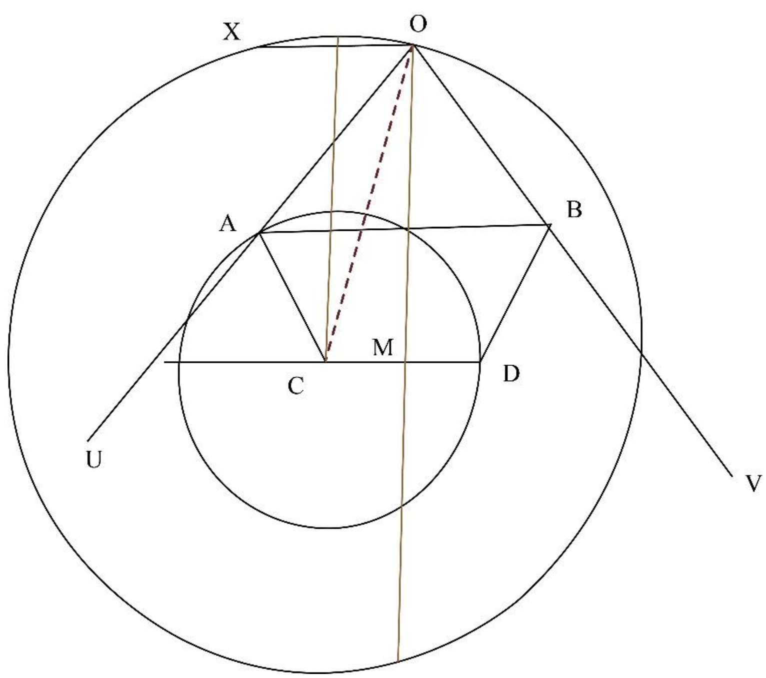


Fig. 4 Draw for the Proof of Corollary 1.

3. Method to Trisect an Angle by Secondary Geometry

3.1. Trisection Theorem (Core Theorem)

Assume a given angle $\widehat{UOV} = \beta$, $\beta \subset r$ & $\beta < 180°$, are divided into 3 equal angles α, $\alpha < 60°$, by 2 trisectors OS & OT, then there always exist a Trisection Ruler ABCD:

- ➢ of which vertices A & B are located in OU & OV so that OA = OB = p, p is an arbitrary length belongs to r, and
- ➢ the vertices C & D are located in the 2 trisectors OS & OT.

Proof:

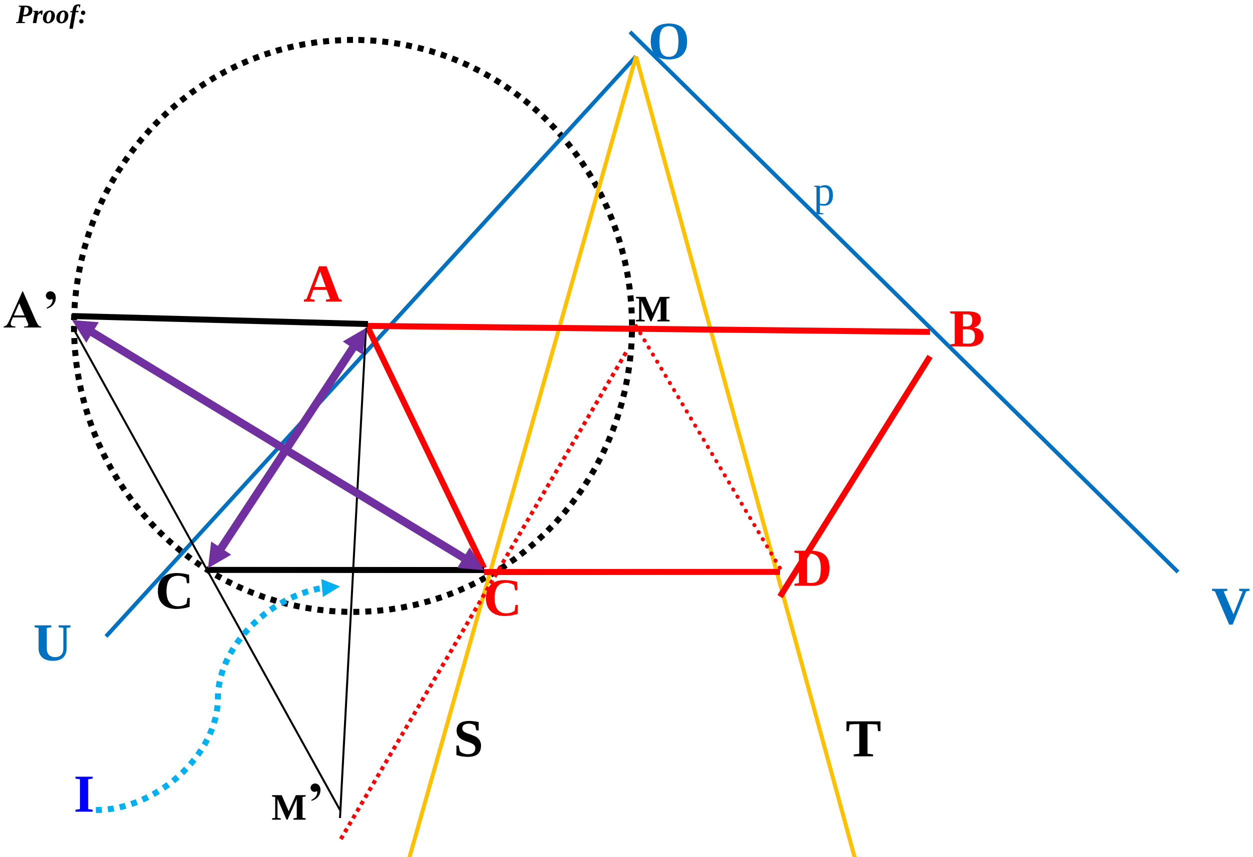


Fig. 5 Drawn shape for the Proof of the Trisection Theorem (Core Theorem)

By the assumption of this Theorem, Figure 5 above describes the given angle $\widehat{UOV} = \beta$, $\beta < 180°$ & $\beta \subset r$, has its trisectors OS & OV (Angle $\widehat{UOS}$ = Angle $\widehat{SOT}$ = Angle $\widehat{TOV} = \alpha$, $\alpha < 60°$ & $3\alpha = \beta$).

Let $p \subset r$ be an arbitrary length to mark OA = OB = p, then draw the horizontal line segment AB. Obviously, this straight-line segment cuts the bisector of $\widehat{UOV}$ at the mid-point M of AB. Let AM = r = MB, then circle (A, r) cuts the left trisector OS at C to make AC = r. Connect CM and prolong it to M' so that C is the mid-point of MM'. Let A'A be the horizontal radius of the circle (A, r) then connect A'M (the horizontal diameter of the circle).

Consider triangle A'M'M:

- ➢ Two points A & C are the mid-points of A'M & M'M to result A'M' // AC & A'M' = 2r, because AC = r . (a)
- ➢ Let C' be the mid-point of A'M', then C'C // A' M and C'C = ½A'M = r because A'M is the horizontal diameter of the circle (A, AC = r) & C is the mid-point of MM'. (b)

From (a) & (b), and C is located in the circumference of the circle (A, r), we get C'C is an arc chord length which is equal to r and parallel to the diameter A'M of (A, AC = r). This result shows that the arc chord MC = CC' = A'C' = r and ACM is an equilateral triangle with AM = MC = CA = r.

By symmetry, triangle MBD is equal to the triangle AMC via the symmetric axe OM (bisector of the given angle $\widehat{UOV}$) and MBD is also an equilateral triangle with sides r. Then the isosceles triangle CMD has its angle $\widehat{CMD}$ = 60° to be an equilateral triangle too.

Thus, these 3 triangles AMC, MDC &MBD are all equilateral then ABDC is an upside down regular semi-hexagon (Trisection Ruler) as required.

Alternative Proof 1:

In the Figure 5 below, let OA = OB = p, arbitrary p ⊂ r, locate in the two sides of the given angle $\widehat{AOB}$ = β, β < 180° and β ⊂ r. Let M be the mid-point of AB, then let AM = MB = r and AB = 2r, r ⊂ r. Use a compass to draw the circle (A, AC = r) which cuts AB at M and the trisector OS at C, then AC = r = AM. Draw another circle (C, r), which cuts AB at A and M'. Then,

CM' = r {radius of (C, CM' = r)} (1).

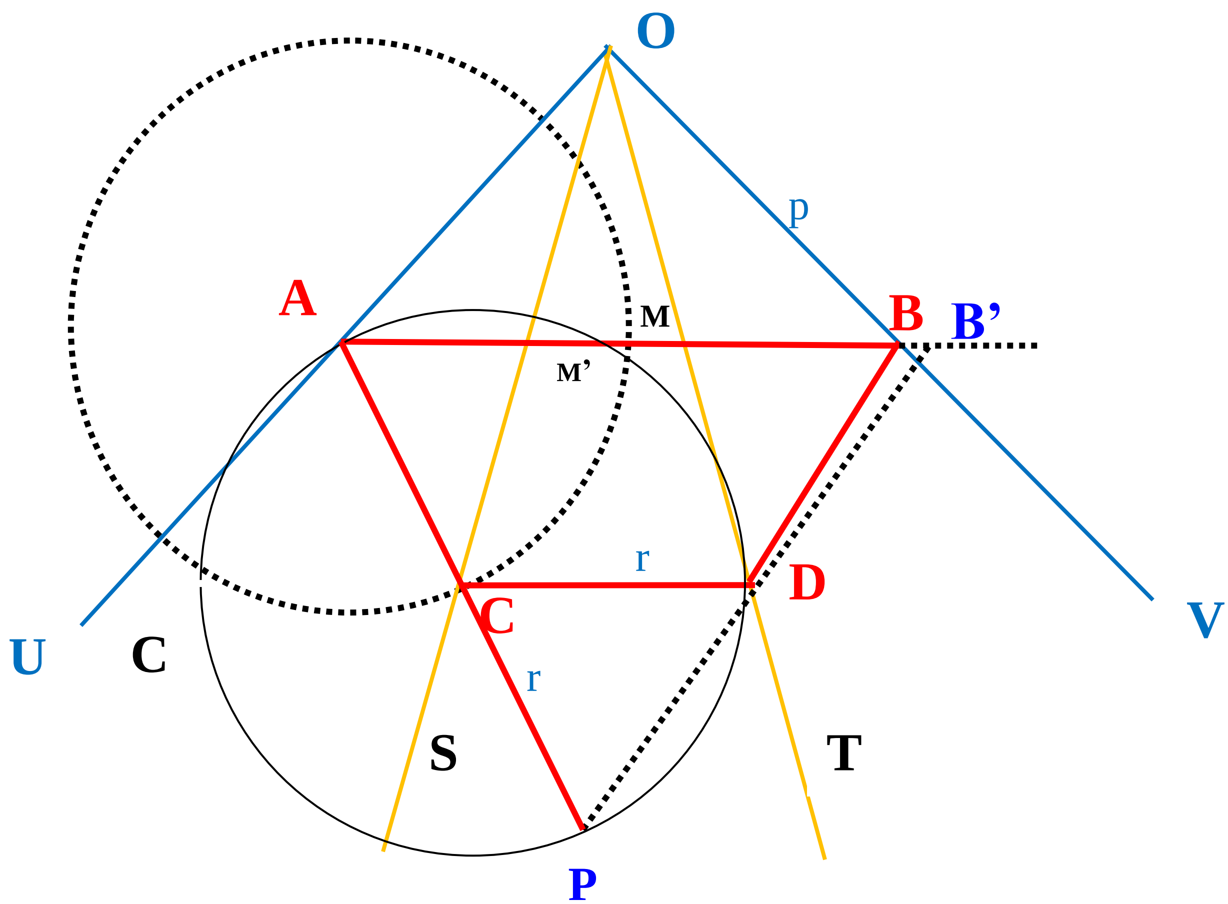


Fig. 6 Drawn shape for this Alternative Proof 1 of the above Core Theorem.

Use a straightedge to draw a horizontal radius from C, which cuts the circle (C, CM' = r) at D to get:

CD = r (2).

The radius CA of (C, CM' = r) is lengthened to meet its circumference at P to have the diameter AP with the mid-point C, then connect PM' to get AB ⊥ PM' at M', because the angle $\widehat{AM'P}$ is inscribed in the upper semi-circle (C, CM' = r).

Then use the straightedge to connect PD and lengthen PD to meet AB at B'. Consider the triangle APB' which has the properties as follows:

AB // CD (3)

AC = CP (4),

By (3) & (4), point D is the mid-point of the other side PB' of the triangle APB'. Then by (2), AB' = 2xCD = 2r = AB. In the other words B' is located at B and the triangle APB' is exactly triangle APB (B' overlaps B). And also M' be M {by (1), CM' = CP and ½AM' = ½CD => AM' = r = AM}. Then the Figure 6 above should be draw exactly into Figure 6.1 below.

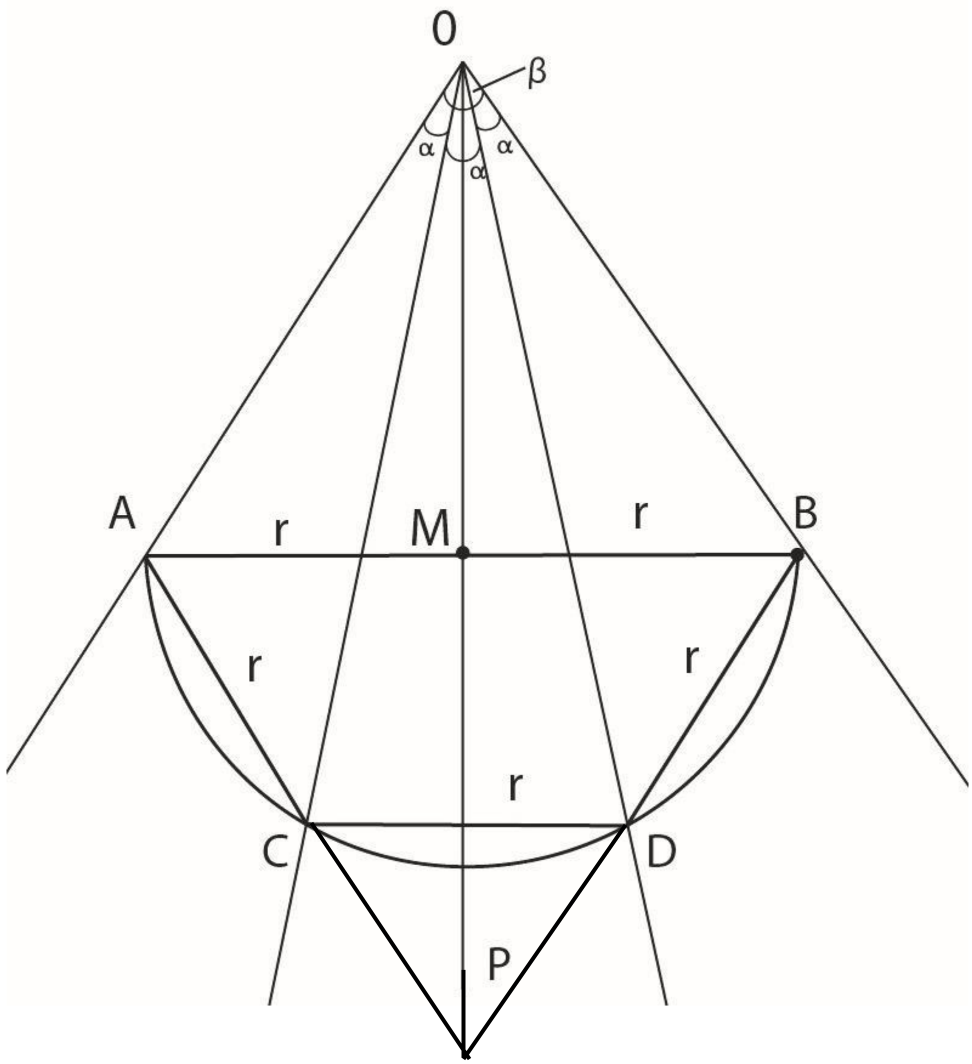


Fig. 6.1

Therefore, PA = 2r = PB => triangle APB is isosceles. From AB = 2r, we have now an equilateral triangle APB and P must be in the bisector of the given angle $\widehat{AOB} = \beta$ and D is the symmetrical point of C through the symmetry axis OP (bisector of angle $\widehat{AOB}$).

This proof show us the Trisection Ruler ABCD (upside down semi regular hexagon) of the given angle $\widehat{AOB} = \beta$, which is constructed by a straightedge & a compass, is described as follows:

- 2 vertices of the horizontal large base are located in the 2 sides of the angle AOB.
- 2 vertices of the small base are located in the 2 trisectors of the angle AOB.

Alternative Proof 2

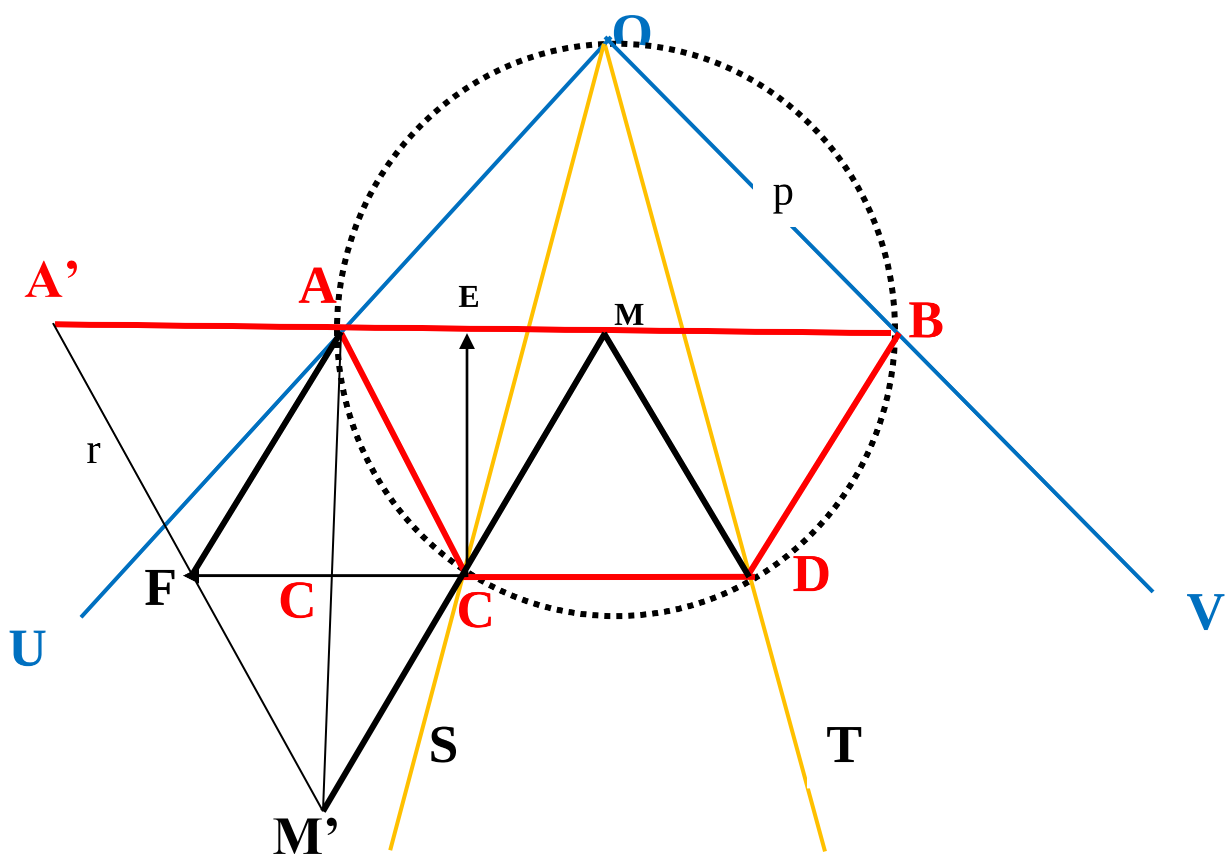


Fig. 7 Drawn shape for this Alternative Proof 2 of the above Core Theorem.

In Figure 7 above, let OA = OB = p, p $\subset$ r, be an arbitrary length on the two sides of the given angle $\widehat{UOV} = \beta$, $\beta \subset$ r & $\beta < 180°$, which has two trisectors OS & OT. Let M be the mid-point of AB, then let r $\subset$ r be equal to MA or MB and AB = 2r. Use a compass to draw the circle (M, MA = r = MB) which cuts the trisectors OS & OT at C & D, then MC = r = MD.

Let E be the mid-point of AM. (5)

Let M' be a symmetric point of M by the axe point C, then connect AM' to get point C' which is the intersect point of AM' and CD prolonged. The special properties of the triangle AMM' and (5) give us the following results:

C'C = ½ AM = ½ r = AE = EM

From A, draw a parallel straight line to MC, which meets the prolonged CD at F then AF = MC = MD = r = AM = FC. (6)

Connect M'F then lengthen it to meet the prolonged MA at A'. And then consider the triangle A'MM' and (6) to get the following results:

- F is the mid-point of A'M' as C is the mid-point of MM' and FC // A'M.
- FC = r = ½ A'M and this make A'M = 2r and A is the mid-point of A'M.
- A'F = MC = r & A'M' = 2r, as A' is symmetric point of M by the point axe A and F is symmetric point of C by the axe point C'.

By the 3 results above and MM' = 2r, triangle A'MM' is equilateral and $\widehat{AMC} = 60°$ and ABCD is the Trisection Ruler of the given angle $\widehat{UOV}$ at AB, OA = OB = p and AB = 2r, as required.

Thus, the above Alternative Proof 2 show us that ABCD is a Trisection Ruler (semi regular hexagon) of the given angle $\widehat{UOV} = \beta$, $\beta \subset r$ & $\beta < 180°$, where OA = OB = p, $p \subset r$ & p is an arbitrary length, as required.

3.2. Trisecting Theorem

Given an angle $\widehat{UOV} = \beta$, $\beta < 180°$ & $\beta \subset r$, then its Trisection Ruler ABCD, which is constructed by a straightedge & a compass, identifies the two <u>Trisectors</u> OC & OD of the angle $\widehat{UOV}$.
$\widehat{UOV}$.

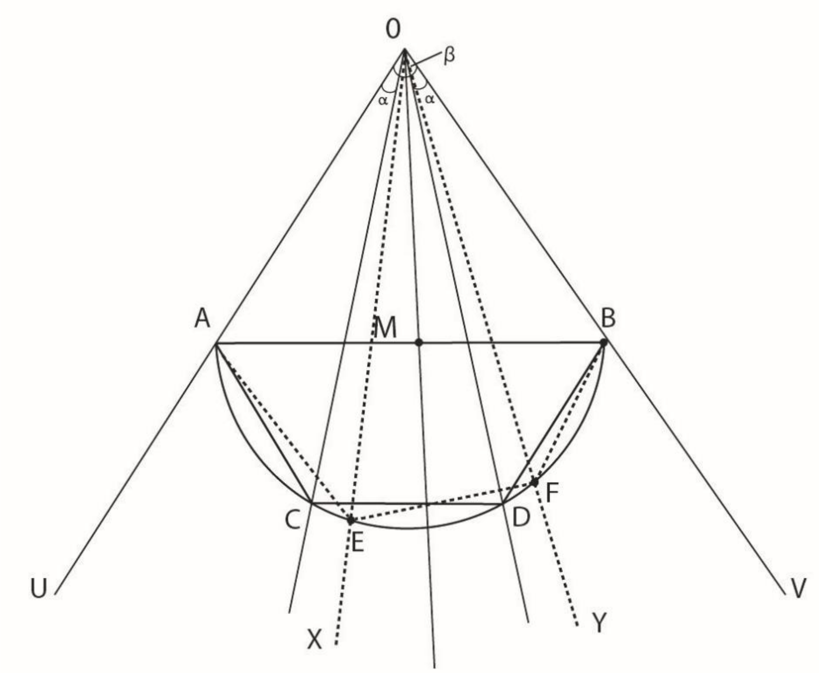


Fig. 8 Drawn for the proof of the Trisection Theorem above

Proof: By Theorem 2 and Definition 1, for a any given angle $\widehat{UOV} = \beta < 180°$ & $\beta \subset r$, we can always construct the angle's Trisection Ruler ABCD, OA = p = OB, $p \subset r$, arbitrarily. This Trisection Ruler is inscribed in the semi-circle (M, r), r = ½ AB.

Aim to prove OC and OD are the 2 trisectors of the given angle $\widehat{UOV} = \beta$, by contradiction.

Assume OC & OD are NOT the 2 trisectors, then there exists OX & OY which are the two trisectors of the given angle. These assumed trisectors OX & OY cut the semicircle at E & F, then by this assumption and Definition 1 and the CORE THEOREM, ABEF is a regular semi-hexagon, upside down, which has the large base AB. Because the arc chord AC = CD = DB = r, and ABCD is the unique regular semi-hexagon inscribed in the semi-circle (M, r), by Theorem 2, then ABEF must NOT be a regular semi-hexagon inscribed in the same semi-circle. Therefore, OX & OY are not the trisectors of the given angle $\widehat{UOV}$.

Thus, the claim of the two trisectors OC & OD is exact, accurate and correct for the given angle $\widehat{UOV} = \beta < 180°$.

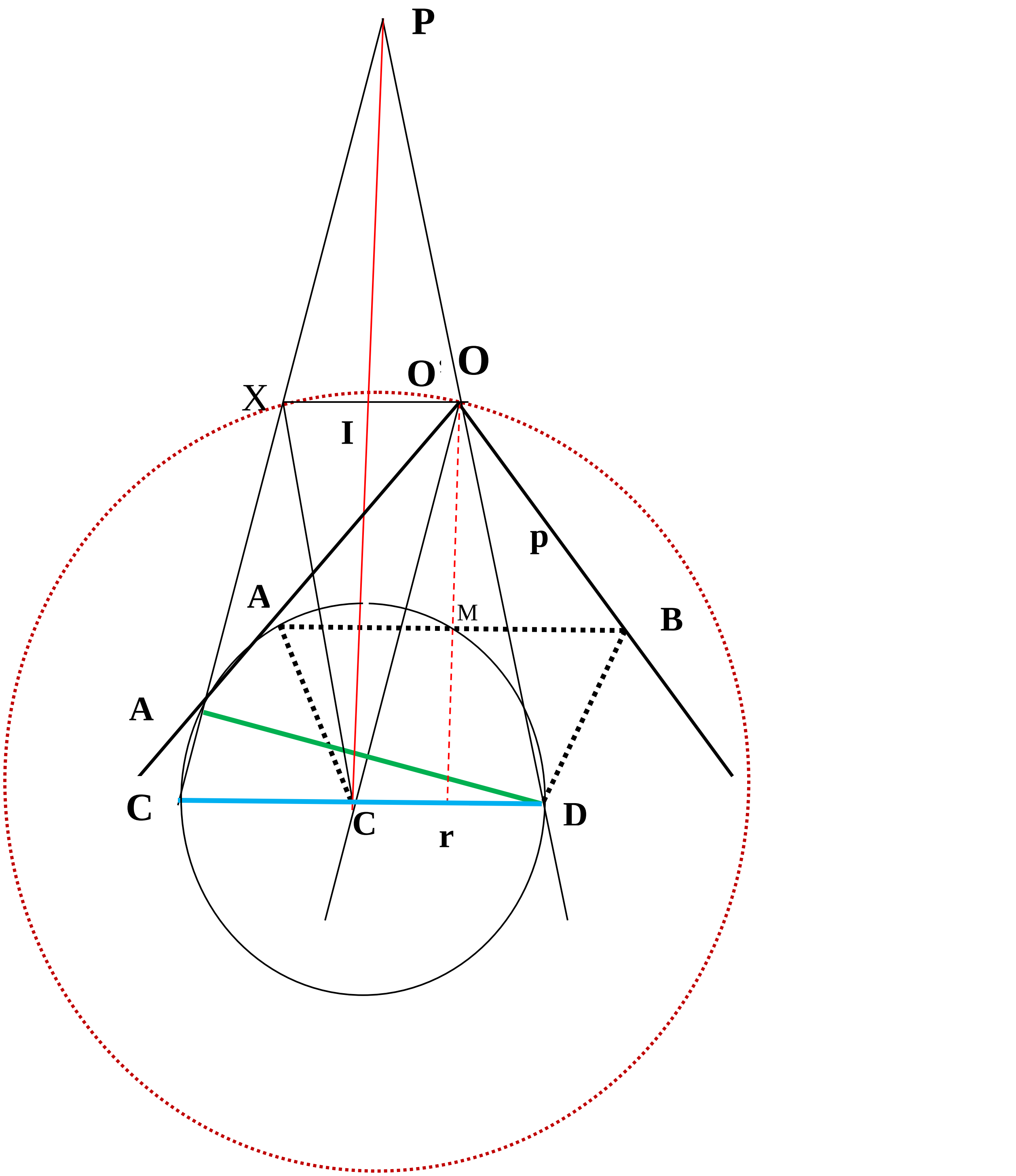


Fig. 9 Draw for Alternative Proof of the Trisection Theorem

Alternative Proof: From point C in the Trisection Ruler ABCD, OA = p = OB, p ⊂ r, and AC = CD = DB = BM = MA = r, r ⊂ r, of the given angle $\widehat{AOB} = \beta$, $\beta < 180°$ & $\beta \subset r$, we draw the circle (C, CO). Then, lengthen DC to the left so that C is the mid-point of C'D and draw the circle (C, CA = r) which cuts the side OA of the angle AOB at A' and meets ABCD at D. Obviously, C' is the symmetric point of D via point C then C'C = CD = r .

Connect DA' and C'A' to get C'A' ⊥ DA', as this angle $\widehat{C'A'D}$ cuts the diameter of the circle (C, r).
Prolong C'A' to meet (C, CO) at X then choose point P to satisfy XP = XC', and then connect CX to get CX = radius of the circle (C, CO), CX = CO. Connect CP then draw a horizontal line from X to meet CP at the mid-point of CP, to get XI = ½r because X & I are two mid-points in sides PC & PC' of the triangle PC'C. (8)
From the mid-point X of PC', we consider the triangle PC'D to get the distance from X to the mid-point O' of PD is equal to ½C'D or { XO' = r = C'C = CD and XO'// C'C}. These results give us:

- From (8), IC is the symmetric axe of the two line segments XC' & O'D and IO' = ½ r. (9)
- XO'DC' is an isosceles trapezoid with two horizontal bases XO' & C'D. (10)
- From (10), we get the triangle C'PD is isosceles and PC is the symmetric axe of C'PD. (11)

From (11) and {C is the centre of circle (C, CO) and X is located in the circumference of circle (C, CO)}, we get:
- point O' must be located in the circumference of the circle (C, CO) with XO' = ½C'D = r. (12)

By (12) and the Corollary 1 (see the Corollary 1 in page 7 above), O' must be located at point O, exactly and correctly. This derives CO // C'X and CO ⊥ DA' at the mid-point of DA' (because C is the mid-point of C'D).
From the above proof results, we consider triangle A'OD to get OC is a perpendicular bisector from vertex O. In the other words, angle $\widehat{A'OC} = \widehat{COD}$ or $\widehat{AOC} = \widehat{COD}$. By the Definition 1 (page 6), $\widehat{AOC} = \widehat{DOB}$, therefore:

$$\widehat{AOB} = \widehat{COD} = \widehat{DOB}$$

Thus, OC and OD are the two trisectors of the given angle AOB as required.

At the end of this alternative proof, we can change Figure 9 above into Figure 10 as follow:

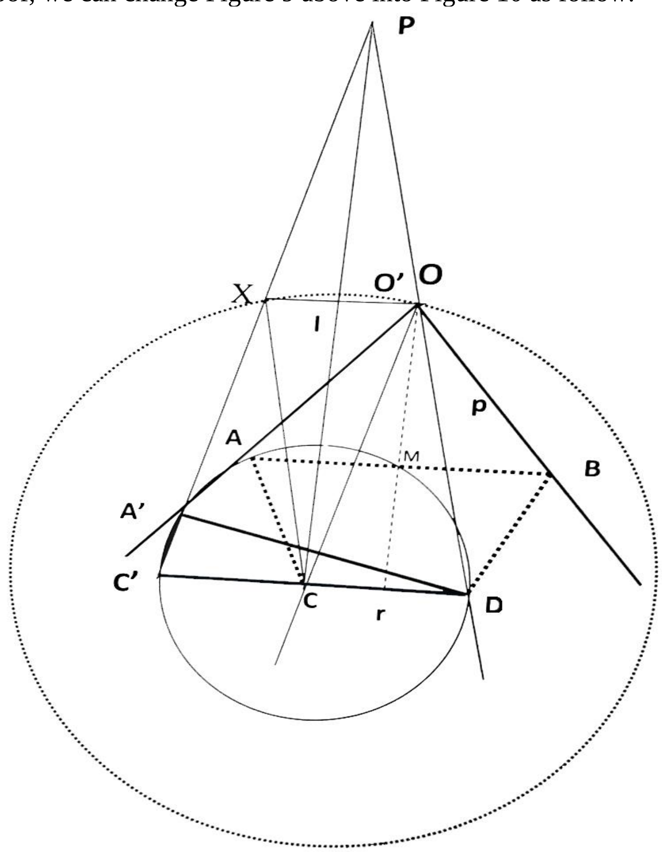


Fig. 10 Draw of the Alternative Proof for the Trisections Theorem

4. Special Cases

4.1. Right Angle

It is very easy to divide a given right angle $\widehat{UOV}$ into 3 equal angles, 30° each. Choose an arbitrary length p, p ⊂ r, in the vertical side OU of $\widehat{UOV}$ to mark a point A, OA = p, p is an arbitrary length. Then take a compass & a straightedge to draw the circle (A, p) and an equilateral triangle AOQ located in the right of the circles (A, p). And then, OQ is one trisector of the angle UÔV, by the expression $\{\widehat{UOV} - \widehat{AOQ} = 90° - 60° = 30° = \widehat{QOV}\}$. At the end, draw the bisector of the 60-degree $\widehat{AOQ}$ to get the other Trisector of the given angle $\widehat{UOV} = 90°$. This method uses only a compass & a straightedge.

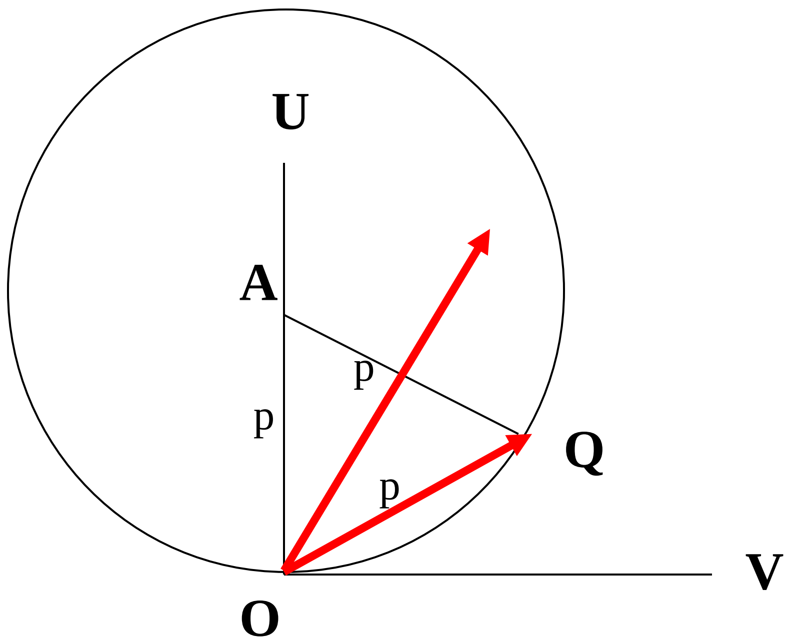


Fig. 11 Two trisectors of a given right angle

4.2. Flat angle (Straight-line angle/ Angle = 180°)

For an angle $\widehat{UOV}$ = 180°, with a compass & a straightedge, draw a equilateral triangle OAB, A is located in the left side OU of the given $\widehat{UOV}$ and B is located under the horizon line OA, then OB is one trisector of the given flat angle UÔV. The other trisector of $\widehat{UOV}$ is just the bisector of angle $\widehat{BOV}$, say OC, as the following Figure 12. This method uses only a compass & a straightedge.

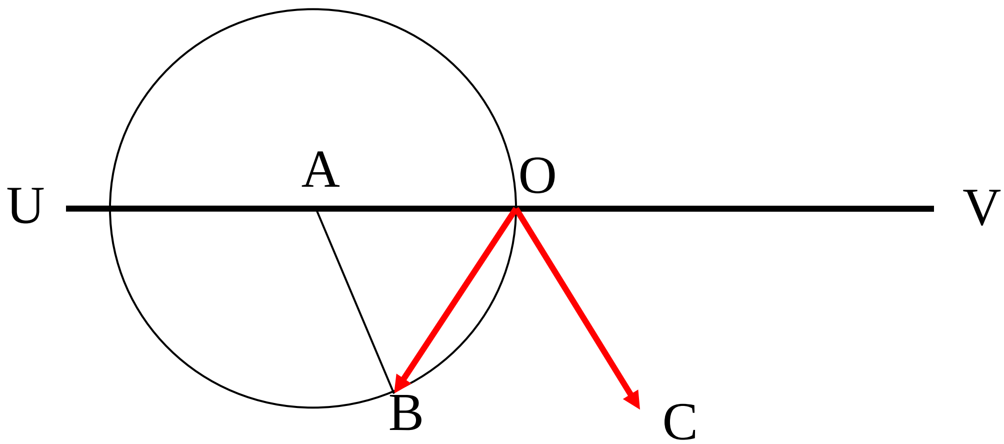


Figure 12: Two Trisectors OB & OC of a flat angle.

4.3. 180°<Angle < 270°

For a given angle $\widehat{UOV} = \beta$, 180° < β < 270°, β ⊂ r, let OA be an extensive line of side OU of $\widehat{UOV}$ to the right-hand side (Figure 13 below). This straight line divides $\widehat{UOV} = \beta$ into 2 angles, which are a flat angle $\widehat{UOA}$ = 180° and an angle $\widehat{VOA}$ = α < 90°, σ ⊂ r.

Then apply Section 2 above and use a compass & a straightedge to draw the 2 trisectors (2 Trisecting Lines) OM & ON of the flat angle $\widehat{UOA}$ (these two trisectors divide this flat angle into three 60-degree angles). And then apply Part III above to draw two trisectors of the angle α, using a compass & a straightedge, where these two trisectors divide σ into 3 equal angles γ, γ < 30°, γ ⊂ r.

The remain work is to place consecutively, one angle 60° and γ to the left side OU in the given angle $\widehat{UOV}$ to get one trisector OP. And then draw the bisector of the angle $\widehat{POV}$ to get the other trisector OQ of the given angle $\widehat{UOV}$.

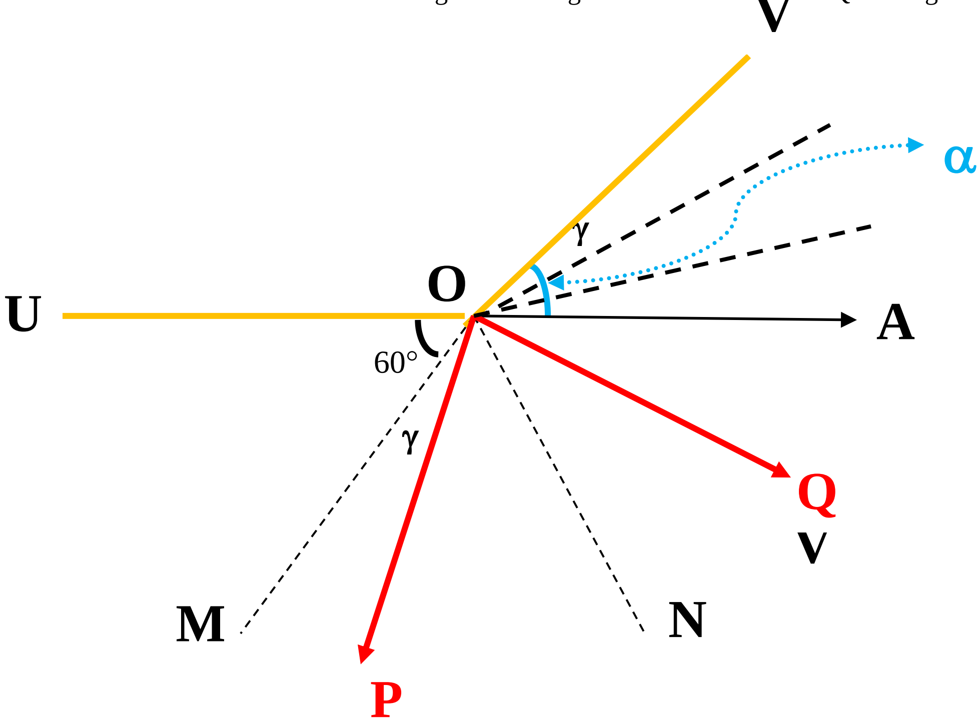


Fig. 13 The two Trisectors of a given angle β, 180°< β < 270°.

4.4. 270°- Angle

For any angle $\widehat{UOV}$ = 270° which is equal to 3x90° (3 right angles), we only need to lengthen the angle sides from the angle vertex to get its 2 trisectors OA & OB, as the described Figure 14 below:

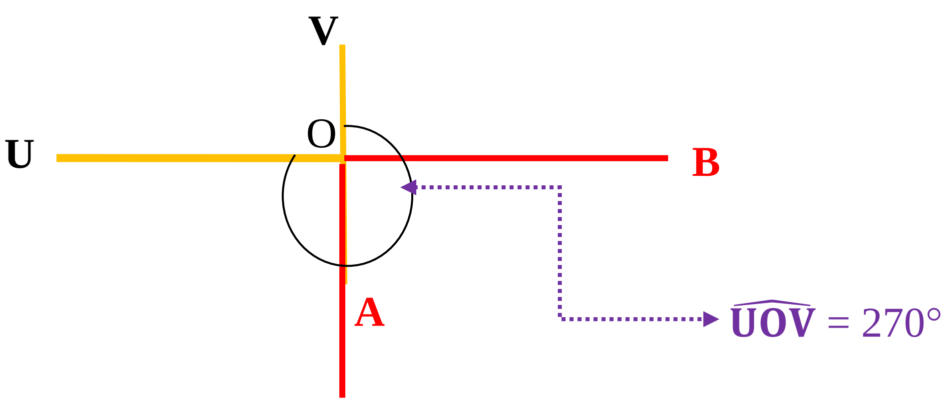


Fig. 14 The two Trisectors of a given angle β = 270°

4.5. 270°<Angle < 360°

For a given angle $\widehat{UOV}$ = β, β ⊂ r, 270° < β < 360°, lengthen the side OU of $\widehat{UOV}$ to the right, then draw a perpendicular straight line to OU (Which is MN) through vertex O to get 3 right angles (named 1, 2 & 3 in Figure 15 below), containing in

$\widehat{UOV}$. This work only uses a straightedge & a compass. And then, the remaining part of $\widehat{UOV}$, which is the angle $\widehat{MOV} = \alpha < 90°$ & $\alpha \subset r$. Now, we apply Part III above to draw two trisectors of the angle α, using a compass & a straightedge, where these two trisectors divide σ into 3 equal angles γ, $\gamma < 30°$, $\gamma \subset r$. The remain work is to attach one right angle 90° and γ to the horizontal side OU in the given angle $\widehat{UOV}$ to get one trisector OP. And then draw the bisector of the angle $\widehat{POV}$ to get the other trisectors OQ of the given angle $U\hat{O}V = \beta$, $270° < \beta < 360°$.

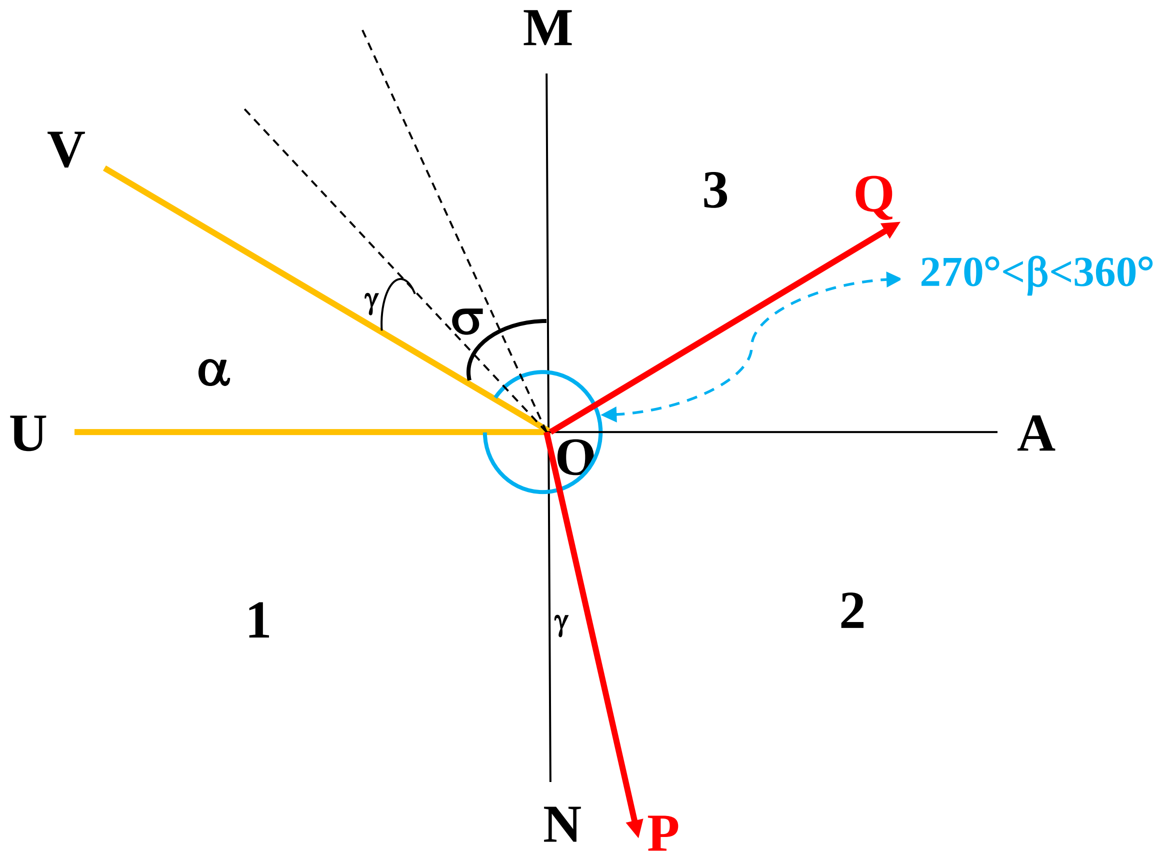


Fig. 15 The two Trisectors of a given angle β, 270°< β < 360°

4.6. 360° - Angle (Full corner / Perigon)

For a given 360-degree angle $\widehat{UOV}$, from vertex O we can use a straightedge & a compass to construct a regular hexagon inscribed in a circle (O, r), r is an arbitrary length, $r \subset r$. Then connect the 3 vertices, which are not consecutive, to obtain an equilateral triangle inscribed in the circle (O, r). And then connect O to two vertices of the triangle to obtain the 2 trisectors OP & OQ of the given 360-degree angle $\widehat{UOV}$ (the other vertex of the triangle is connect to O to make a duplicated side of the given 360-degree angle $\widehat{UOV}$).

5. Discussion and Conclusion

Method: For any arbitrarily given angle we can draw two straight lines from the angle vertex to divide the angle into 3 equal smaller angles by using a compass & a straightedge, as follows:

Choose arbitrarily 2 equal lengths p, $p \subset r$, in 2 sides of the given angle, then draw a semi-hexagon, which is a Trisecting Ruler of the angle, at the length p. Two vertices of the longer base of the Ruler are located in 2 sides of the given angle. The straight lines connect the angle vertex and the other 2 vertices of the Ruler are the two Trisectors of the given angle.

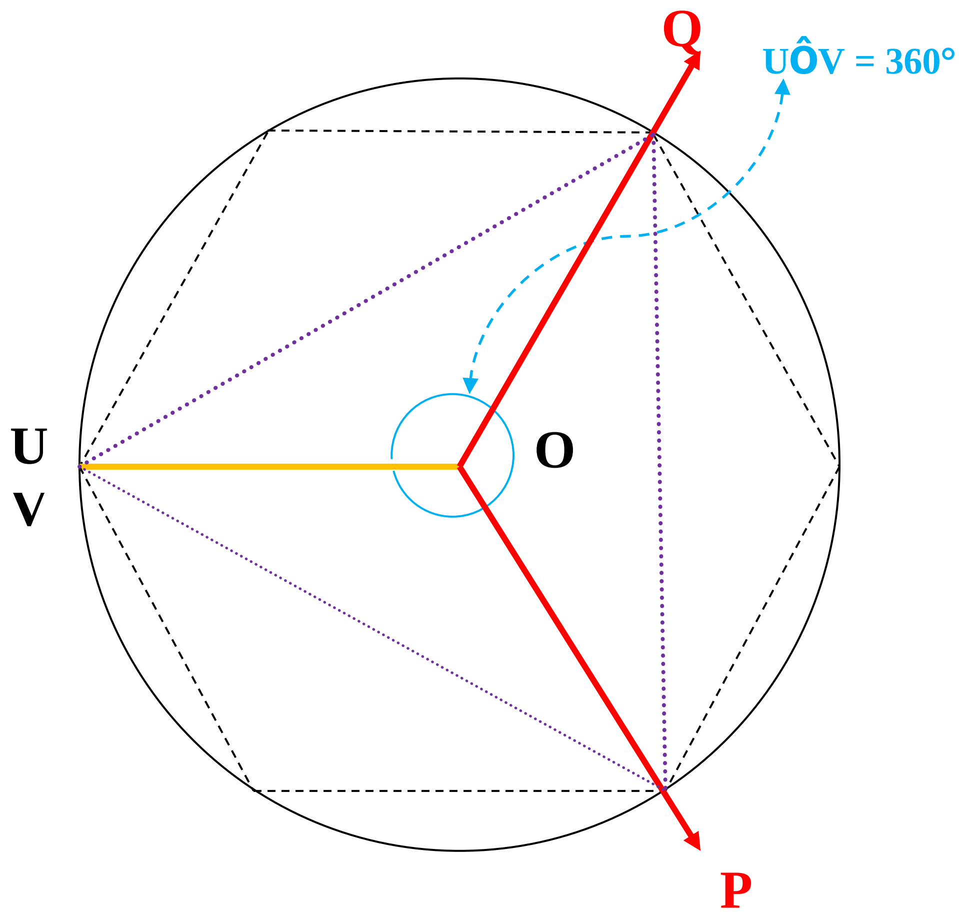


Fig. 16 The two Trisectors OQ and OP of a given 360-degre

This result of my research includes properties as follows:

- It is possible to construct two trisectors of any given angle exactly, using a compass & a straightedge with simple Geometry (secondary level) only.

- The advantage of this Trisection Method for any given angle is based on the invention of my TRISECTION RULER so that we can apply the simple Geometry to solve the "unsolved angle trisection problem".

- The TRISECTION RULER is actually similar to a geometric parameter which

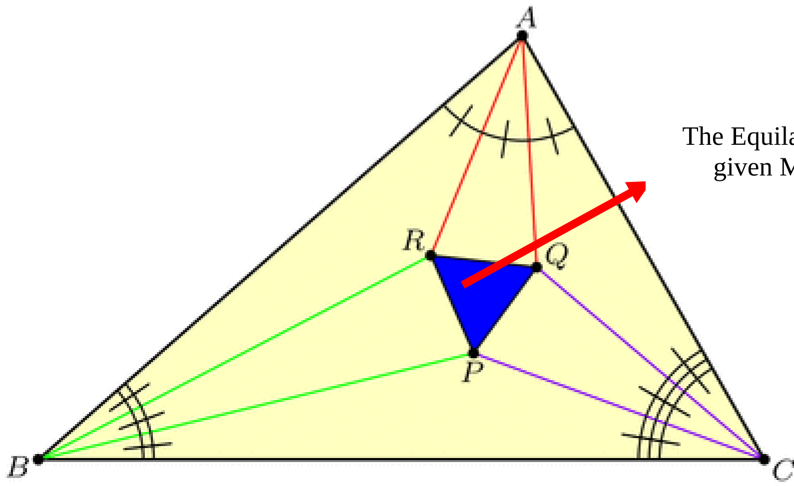
  ➢ depends on the size of a given angle

  and

  ➢ an arbitrary parameter p for two lengths located in 2 sides from the vertex of the angle.

- It is a counter proof to the Wantzel's complicated proof for the impossibility of any solution for the "Trisecting an Angle" problem.

This success of the trisectors construction results an easy method to draw a Morley Triangle, that anyone can do. Firstly, construct the trisectors of the angles A, B & C of a given Morley Triangle. Secondly, connect three intersect points of 2 consecutive trisectors of {A & B}, of {B & C} and of {C & A} to construct the Equilateral Triangle of the given triangle as the following image.



Last but not least, I would like to summarize this contribution into the following THEOREM:

Angle Trisection Theorem: Any given angle can be divided into 3 equal sub-angles by a compass and a straightedge, exactly and accurately.

Acknowledgements

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References

[1] Andrew M. Gleason, "Angle Trisection, the Heptagon, and the Triskaidecagon," *The American Mathematical Monthly*, vol. 95, no. 3, pp. 185-194, 2018. [CrossRef] [Google Scholar] [Publisher Link]
[2] M.L. Wantzel, "Research on the Means of Knowing Whether a Problem in Geometry can be Solved with Ruler and Compass," pp. 366–372, 1837. [Publisher Link]
[3] Underwood Dudley, *The Trisectors*, The Mathematical Association of America, 1994. [Google Scholar]
[4] Robert C.Yates, *The Trisection Problem*, First Published in 1942. [Google Scholar] [Publisher Link]
[5] Harold Scott Macdonald Coxeter, *Introduction to Geometry (2nd Ed.)*, John Wiley & Sons, 1967.
[6] J.L. Heiberg, *Euclid's Elements*, Cambridge University Press, 1883.
[7] S. Lang, Geometry: Euclid and Beyond, Springer Science & Business Media, 1997.
[8] R.B. Nelsen, *Proofs without Words II*, Mathematical Association of America, 2001.
[9] W. Dunham, *The Mathematical Universe*, John Wiley & Sons, 2007.
[10] M. Field, and J. Jones, "Impossible Constructions," *Mathematical Gazette*, vol. 56, no. 397, 15-22, 1972.
[11] J.B. Kennedy, "Angle Trisection, The Heptagon, and The Triskaidecagon," *American Mathematical Monthly*, vol. 77, no. 8, pp. 851-869, 1970.
[12] Mario Livio, *The Golden Ratio: The Story of Phi, The World's Most Astonishing Number*, Broadway Books, 2003.
[13] S.A. Stahl, "The Impossible Trisection Problem," *The College Mathematics Journal*, vol. 24, no. 2, pp. 100-110, 1993.
[14] W. Szmielew, *The Problem of Trisecting an Angle*, Mathematica Slovaca, vol. 44, no. 3, pp. 321-342, 1994.
[15] Jean-Pierre Tignol, *Galois' Theory of Algebraic Equations*, World Scientific, 2001.
[16] B.L. Van der Waerden, *A History of Algebra*, Springer Science & Business Media, 1991.
[17] I.M. Yaglom, and V.G. Boltyanskii, *Geometric Transformations II*, Random House, 1962.
[18] J.K. Yu, *Golden Section: Nature's Greatest Secret*, World Scientific, 2005.
[19] H.J.M. Bos, and J. Smit, "Euclides Triangles," *New Archives for Mathematics*, vol. 9, no. 4, pp. 227-235, 2008.
[20] Florian Cajori, *History of Elementary Mathematics*, The Macmillan Company, 1898. [Google Scholar]
[21] H.W. Guggenheimer, *Differential Geometry*, Dover Publications, 1997.
[22] M. Hazewinkel, *Encyclopedia of Mathematics*, Springer Science & Business Media, 2001.
[23] W.R. Knorr, *The Ancient Tradition of Geometric Problems*, Dover Publications, 1986.
[24] Eli Maor, *Trigonometric Delights*, Princeton University Press, 1998.