

Original Article

Radial Basis Function Techniques for Addressing Partial Differential Equations

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Abstract - In this paper, we have defined radial functions and mentioned their importance and uses in various applications that fall under different fields. In particular, we talked about the use of radial functions in data interpolation in various dimensions, and several methods were discussed in this regard, and the advantages and disadvantages of using radial functions in interpolation. The use of radial functions in solving partial differential equations was also discussed, focusing on their distinctive properties in the solution, as they are considered mesh-free techniques. Several methods of solving partial differential equations depending on their radial functions have been mentioned; such as, Kansa method, LRBFCM, RBF-DQ, and RBF-PUM.

Keywords - Radial Basis Functions (RBFs), Data Interpolation, Partial Differential Equations (PDEs), Mesh-free Methods.

1. Introduction

Partial differential equations PDEs can be employed for the representation of a wide variety of science problems, such as mechanical, electrical, biological, chemical. Currently, researchers are investing numerical methods for solving various types of PDEs since obtaining the analytical solution for PDEs is cumbersome[1]. When applying numerical methods, all data that related to the problem must be in a certain format (predetermined pattern), and included in a region. Moreover, in the current time numerical methods are frequently used for solving various types of PDEs since finding their precise solution is cumbersome analytically. Some problems with regard to employing numerical techniques cannot be solved in a traditional manner due their complexity. Therefore, in such cases and for overcoming these difficulties mesh free approaches are invested, and RBF alignment technique is one of mesh free methods which is used in this regard[2].

Hardy 1971 was the first who introduced the RBF methodology. As, he used it with a topological application regarding quadratic surfaces. Multicultural (MQ) approximation scheme was presented by Hardy. In 1982, Richard Frank, evaluated blending methods of rectangles, the methods of inverse distance weighted, finite elements methods, Foley's methods, and other methods on the interpolation of scattered data. Franke assessed the techniques relying on diverse parameters such as accuracy, storage and the method's processing. It was found that the multiquadric method (belongs to RBF) is among the best ones[3]. Furthermore, in 1990, Kansa used the Multiquadric method[4] for solving a PDE (globally supported interpolant), and that's known as the method of Kansa, and it was used for addressing different applications. However, Kansa method possesses a number of drawbacks; such as the nature of the interpolation matrix as it is unsymmetrical, and that leads to problems in case of considerable number of nodes. Subsequently, Fasshauer in 1996[5], proposed an approach based on the modifications of Kansa method namely "Hermite approach". On one hand, the resulted matrices from Hermite technique are naturally symmetric and have got a smaller number of conditions. On the other hand, Hermite method has certain shortcomings. Since, its implementation is more difficult than the unsymmetrical approach.

However, applied to various applications, Kansa method has some disadvantage like unsymmetrical nature of interpolation matrix which leads to ill conditioned matrix for large number of nodes. In 1996, Fasshauer proposed a hermite based approach as modification of Kansa method. The collocation matrices from this approach are symmetric in nature and generally have smaller condition number[6]. However, the symmetric RBF collocation approach has its own limitations. Symmetric Collocation approach is difficult to implement as compared to unsymmetrical approach. In 2004, Chantasiriwan[7] solved the diffusion problem by employing the local RBF. Continuing this approach, several researchers used localized RBF methods for solving various PDEs. As a rule, there is a parameter that decides the RBF shape known as the parameter of shape, and it will be discussed later in this paper. When choosing small shape parameter in some RBFs, that leads to improved accuracy, but result in the matrix ill conditioning. In 2003, Shu[8] proposed a method to integrate RBFs' feature of meshfree with the method of Differential



Quadrature DQ that is characterized by its elevated accuracy and simplicity by introducing a hybrid technique called RBF-DQ. This method has been invested by researchers to address PDEs in fluids problems (for example as Shallow water problems, Navier-stokes). In addition, in 2003, Tolstykh [9] utilized local group of nodes to produce the method of radial basis finite difference, namely RBF-FD. Furthermore, RBF-PUM is a promising technique. It is used for solving PDEs, and it involves unity method segmentation along with RBF. The RBF-PUM method mainly revolves around portioning the domain into interfering subdomains. It was found that, RBF-PUM decreases the computational cost while preserving high accuracy. This paper introduces a review of the RBF approaches. The basic concepts and definition of RBFs are discussed in the second section. In the third section, a review of the development of the methods is presented. Last section represents the concluding part of the paper. We tried to give the mathematical formulation of the methods wherever possible. In authors' knowledge, there is no such survey available in which all the methods related to RBF are presented. A chronological scheme of various RBF methods in association with their related researcher is demonstrate din Table 1. Also, a summary of the development of RBF approaches is shown in Figure 1.

2. Radial Basis Functions (RBFs)

A function $\Phi: R^t \rightarrow R$ can be radial if there is a single function $\varphi: [0, \infty) \rightarrow R$ such that $\Phi(x) = \varphi(\|x\|)$ where $\|x\|$ is the Euclidean factor. A radial basis function $\varphi(r)$ is a uni variate continuous real valued function which relies on the distance from the origin (or any other fixed center point).

In math, the function smoothness can be defined as a property characterized by the continuous derivatives number that the function has along some domain[10]. RBFs are generally identified based on the smoothens. Some functions are infinitely smooth and other are piece-wise smooth. The infinitely smooth RBFs are associated with a parameter called the shape parameter as it can be defined as follows $\epsilon > 0$. As, this parameter controls the shape of the RBF; for example, the RBF shape becomes flat if $\epsilon \rightarrow 0$. Some commonly used RBFs are shown in table 2.

Table 1. Commonly used RBFs

Name of the RBF	Equation ($r = \ x\ $)
Gaussian Function (GS)	$\varphi(r) = e^{-(\epsilon r)^2}$
Linear radial function (LR)	$\varphi(r) = r$
Multiquadric (MQ)	$\varphi(r) = \sqrt{1 + (\epsilon r)^2}$
Inverse quadric (IQ)	$\varphi(r) = \frac{1}{1 + (\epsilon r)^2}$
Polyharmonic Spline (PHS)	$\varphi(r) = \begin{cases} r^{2k-1}; k \in N \\ r^{2k} In(r); k \in N \end{cases}$
Thin Plate Spline (TPS)	$\varphi(r) = r^2 In(r)$
Inverse Multiquadric (IMQ)	$\varphi(r) = \frac{1}{\sqrt{1 + (\epsilon r)^2}}$

2.1. Using of RBFs for the Interpolation and Approximation of Scattered Data

For the interpolation of the multi-dimensional data, RBFs is one of the adequate methods used for that objective. Where, RBFs produce smooth and less fluctuating interpolation compared to the inverse distance weighting. The interpolation of multi-dimensional scattered data involves several applications regarding the Computer Graphics. For instance, reconstruction of surfaces[11], animation blending [12], facial expression re targeting, color calibration [13], and etc. Accordingly, the utilization of efficient methods in this regard is highly demanded.

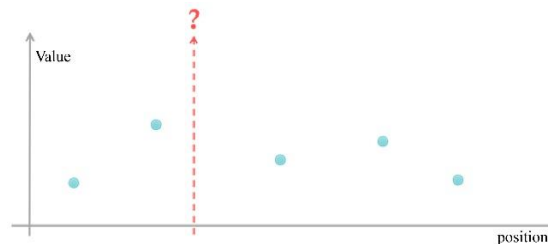


Fig. 1 The representation of scattered data

The scattered data points have no specified structure, as they are simply scattered in p-dimensional space. Piece-wise linear interpolation can be used in 1D space. However, for 2D space the given data spatial domain is tessellated into triangular mesh. While, in 3D a tetrahedral mesh is needed to be constructed for interpolation.

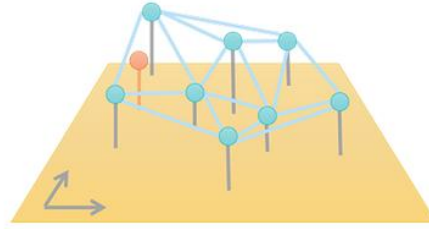


Fig. 2 2D surface interpolation of data with triangular mesh

Mostly, the interpolation results rely on the intermediate structure (e.g., the mesh type). In addition to this, as dimension becomes greater, the interpolation cost gets excessively higher. Hence, researchers always pursue a methodology for which the interpolation of data points is isolated from the manifest tessellation in the spatial domain.

To find the smooth function based on the interpolation problem, it is expressed as follows:

$$s(x_{1_i}) = f_1(i) \text{ for } i = 1, 2, 3 \dots, N \dots (1)$$

for the given data (x_i, f_i) with $x_i = 1, 2, 3 \dots, N, x_i \in R^t$ and $f_i \in R$
 A RBF interpolant takes the form:

$$S(x) = \sum_{i=1}^n \alpha_i \varphi(\|x - x_i\|) \tag{1}$$

α_i can be obtained by requiring the condition presented in equation 1. As, a result:

$$A\alpha = f \tag{2}$$

Where:

$$A_{ij} = \|x_i - x_j\| \quad i, j = 1, 2, 3 \dots, N \tag{3}$$

$$f = [f(x_1), f(x_2), f(x_3) \dots, f(x_N)]^T \tag{4}$$

$$\alpha = [\alpha(x_1), \alpha(x_2), \alpha(x_3), \dots, \alpha(x_N)]^T \tag{5}$$

The matrix A is defined as the interpolation matrix.

RBF methods possess several appealing features, and one of these features regarding interpolation problems is that a unique interpolate is usually ensured under rather the centers mild conditions.

When getting closer to a particular data point x_i within multi-dimensional space, an anticipation of the value getting closer to f_i occurs. Typically, it might be blending weight assignment in accordance with the reciprocal of the difference between a query location x and x_i (the data point):

$$\tilde{f}(x) = \sum_{i=1}^n \frac{w_i(x)}{\sum_j w_j(x)} f_i, \quad w_i(x) = \frac{1}{\|x - x_i\|} \tag{6}$$

This interpolation cannot be considered as smooth for the data points. For boosting the smoothies of interpolation, blending weights can be increased as follows:

$$\|x - x_i\|^{-p}, p > 0 \tag{7}$$

This technique is known as Shepard method[12].

When $p > 1$, \tilde{f} becomes smooth at the points of interpolation. However, that also generates flat spots near the data points due to the approaching of the first derivative of \tilde{f} to zero around the entire data points[14]. Such flat spots arouse unnecessary fluctuation regarding interpolation, as shown in the following figure. Therefore, that urges the need of a less oscilaitng and smooth interpolant.

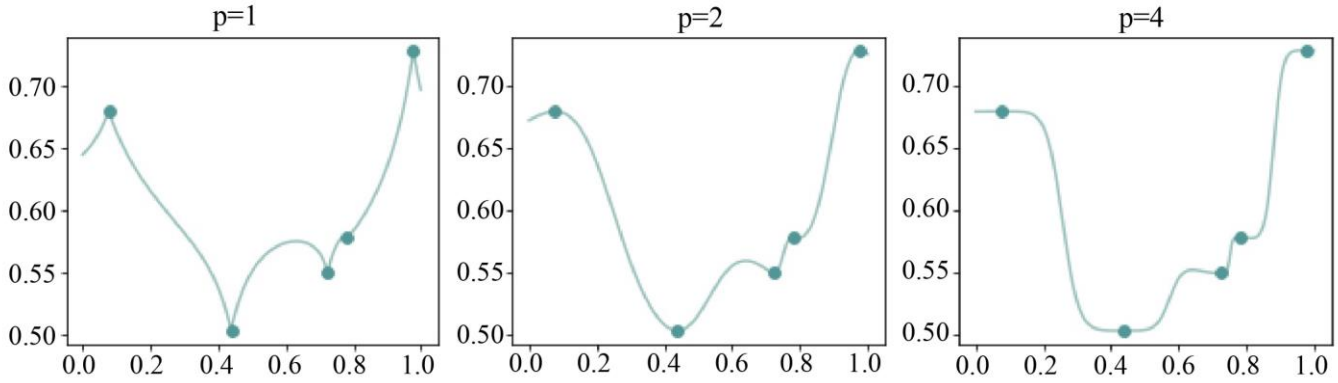


Fig. 3 Shepard interpolation at a number of p values.

In order to make the concept of inverse distance weighting as a rule, one can invest a uni variate function $\Phi: [0, \infty] \rightarrow \mathbb{R}$ to define weights in accordance with distances. However, investing those weights alone would not help the composed function passing through the entire data points, and that because of the effects' overlapping ranges among the data points.

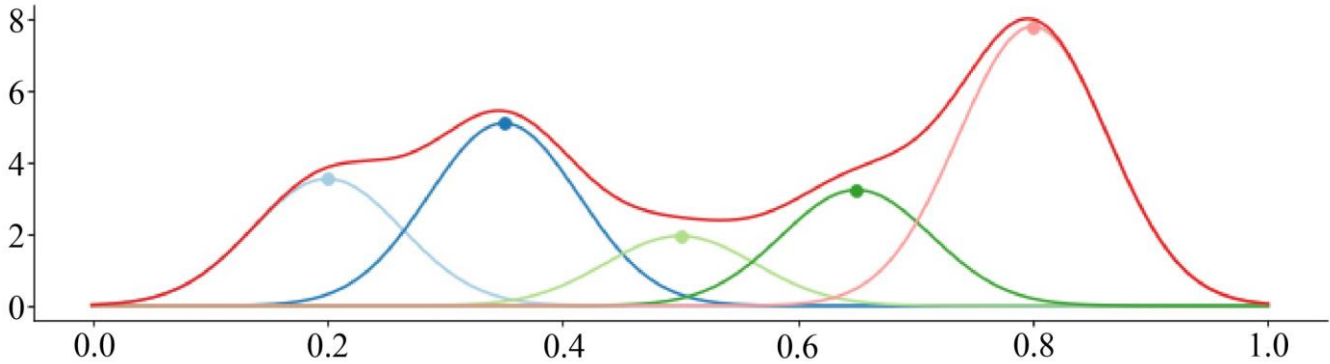


Fig. 4 The summation of Gaussian kernels alone is not able to pass through all data points

Consequently, it is necessary to add a scale factor w_i to guarantee the value f_i at x_i :

$$\tilde{f}(x) = \sum_{i=1}^n w_i \Phi(\|x - x_i\|), \tilde{f}(x_i) = f_i, \text{ for } 1 \leq i \leq n \quad (8)$$

The function Φ (Kernel function) is considered a radial function, as it only relies on distances $\|x - x_i\|$, thus the same value is assigned to the entire regions on the hyper sphere. Letting $\Phi_{i,j} = \Phi(\|x_i - x_j\|)$, the system of linear equations is as follows:

$$\begin{bmatrix} \Phi_{1,1} & \Phi_{1,2} & \dots & \Phi_{1,n} \\ \Phi_{1,2} & \Phi_{2,2} & \dots & \Phi_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{n,1} & \dots & \dots & \Phi_{n,n} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} \Rightarrow \Phi w = f$$

The matrix Φ is similar to the previous

$$\begin{bmatrix} \Phi_{1,1} & \Phi_{1,2} & \dots & \Phi_{1,n} \\ \Phi_{2,1} & \Phi_{2,2} & \dots & \Phi_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{n,1} & \dots & \dots & \Phi_{n,n} \end{bmatrix} \begin{bmatrix} \omega_{1,1} & \omega_{1,2} \\ \omega_{2,1} & \omega_{2,2} \\ \vdots & \vdots \\ \omega_{n,1} & \omega_{n,2} \end{bmatrix} = \begin{bmatrix} f_{1,1} & f_{1,2} \\ f_{2,1} & f_{2,2} \\ \vdots & \vdots \\ f_{n,1} & f_{n,2} \end{bmatrix}$$

case when f is multi-dimensional, but solving the weights is done for each dimension.

For the computation of \tilde{f} , a distance metric along with kernel function must be selected. In 1D interpolation, the effects of kernel selections are shown in the following figure[5]:

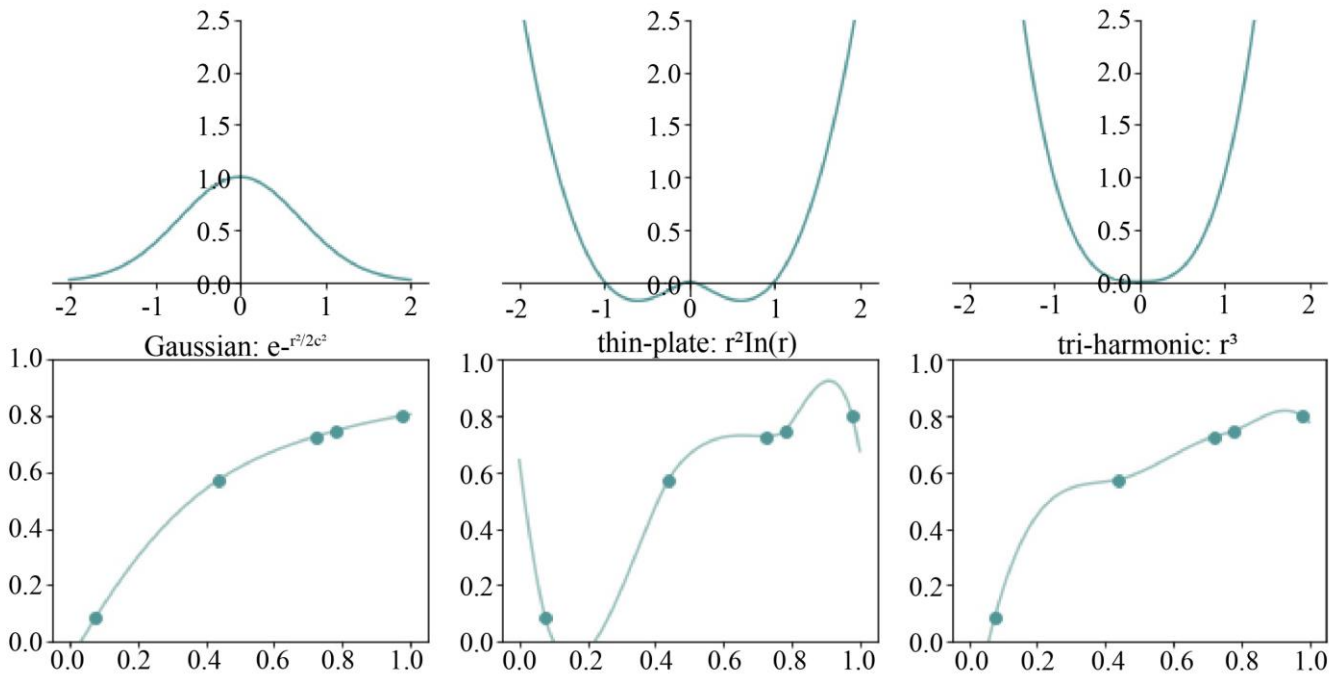


Fig. 5 Different RBF kernels for Interpolation

Number of kernels cannot be considered as inversely proportional to the distances, and sometimes they become zero at the data point. For example, taking tri-harmonic in consideration, there is a proportional relation between its value and the distance between the data point and the query position. It is shown in Figure (6) that it performs well due to the occurrence of extra weights w_1, \dots, w_n to correspond to the variations.

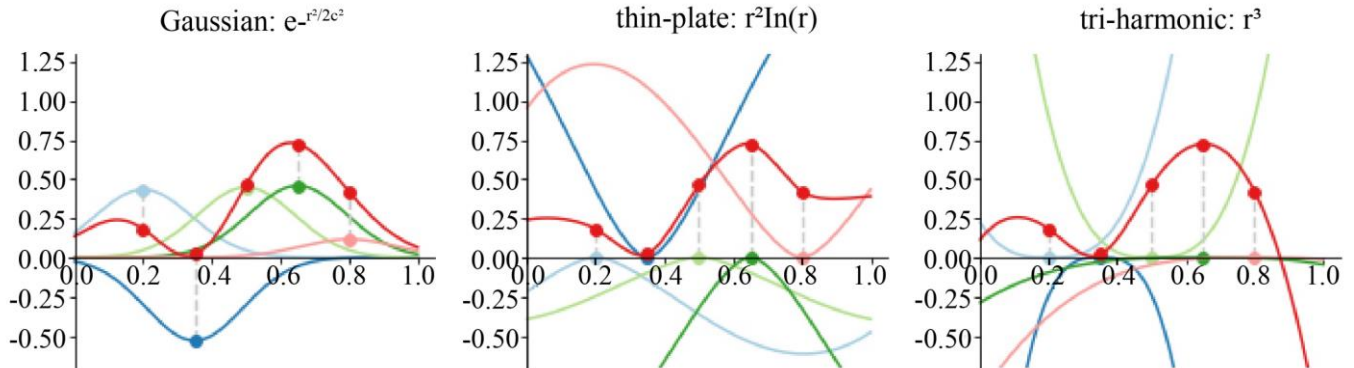


Fig. 6 RBF composition with various kernels: given data points shown in red, extra weights w_i could reconcile the differences and make the interpolant pass through all the data points.

To calculate the weights w_1, w_2, \dots, w_n , a system of n equations with n unknowns. If the rows in Φ are not similar, the solution of the system is unique. Conversely, the system gets ill-conditioned. To cope address the numerical problem a regularize $w^T w$ is added for the diffuse weights' preferences to the objective function instead of solving $\Phi w = f$ in direct manner. Furthermore, \hat{f} becomes an approximation rather than data points interpolation as demonstrated in Figure (7).

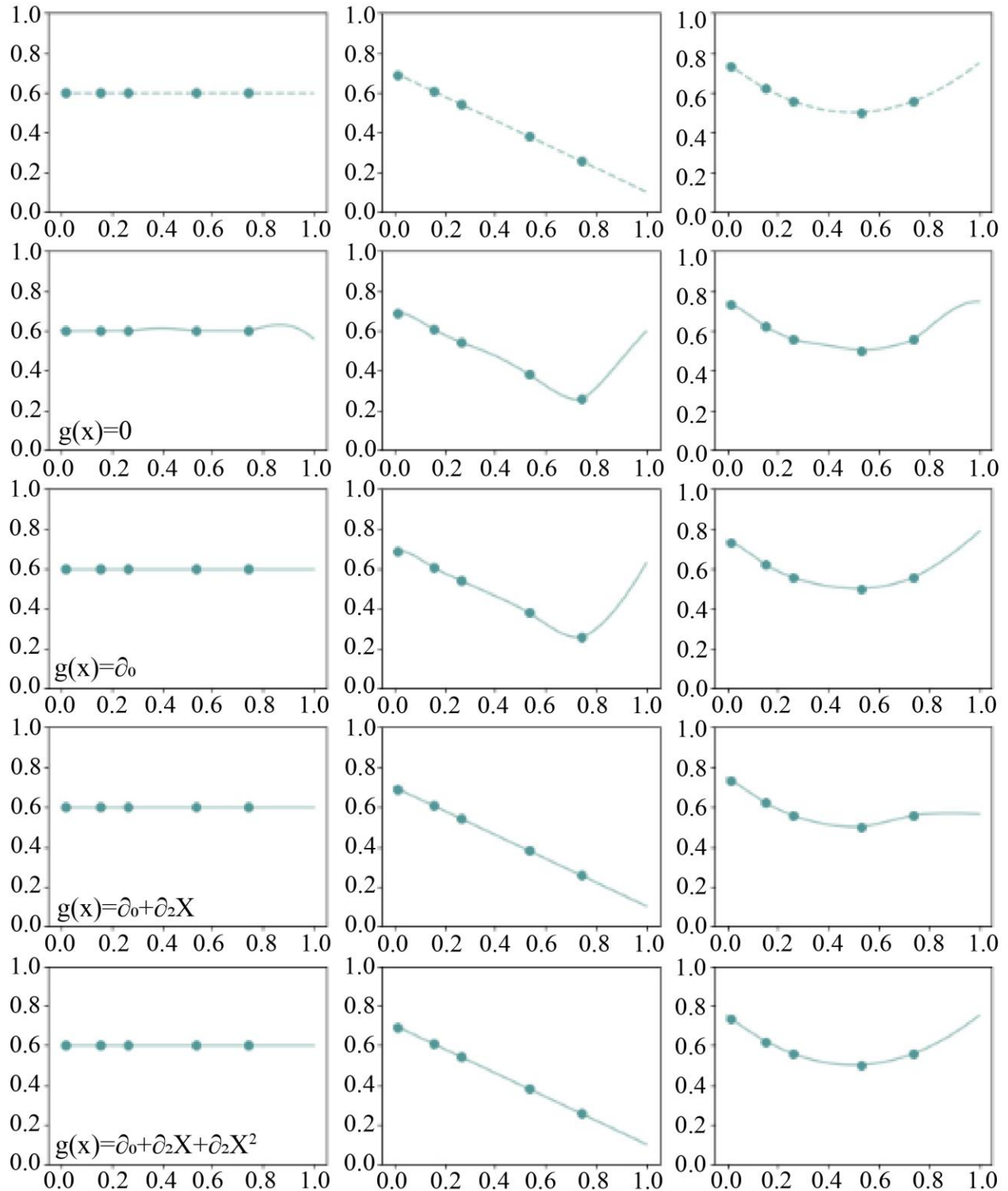


Fig. 7 Thin-plate RBF interpolation with regularization weights: ($\lambda = 1e-6, 1e-4$ and $1e-2$).

$$E(w) = \|\Phi w - f\|^2 + \lambda \|w\|^2 \quad (9)$$

For the determination of wights which essentially reduces the objective function $E(x)$, its derivative with regard to should be zero:

$$\frac{\partial E}{\partial w} = 2\Phi^T(\Phi w - f) + 2\lambda w = 0 \quad (10)$$

$$(\Phi^T \Phi + \lambda I)w = \Phi^T f \quad (11)$$

$$w = (\Phi^T \Phi + \lambda I)^{-1} \Phi^T f \quad (12)$$

Based on the Figure (8), \tilde{f} becomes less fitted according to the data points when an increase in regularization weight λ occurs. In addition, regularization can help avoiding the noise of data over fitting, and thus would improve the system stability.

3. Polynomials Reproducing

The interpolation of data points x_1, x_2, \dots, x_n relying RBFs is possible. However, the formulated function \tilde{f} may be unable to express a polynomial function assigned at different locations. The following figure demonstrates the data interpolation produced from different types of functions (constant, linear, and quadratic) using thin-plate kernel r^2 :

In order to make the interpolant capable of representing the polynomial function, monomials is appended $g(x) = b(x)^T c$ to the interpolant. Assuming Π_m^p is the polynomials set with p variables of total degree of m . Hence, a number of unknown coefficients c_j is generated, and the number of unknowns is bigger than the equations number (under determined system). Consequently, other constrains should be added to obtain a unique solution. The linear system becomes as follows:

$$\Phi w + Pc = f \quad (13)$$

Where P the polynomial basis, and c involves the polynomial basis coefficients.

The search of the additional relation involves that the terms Pc precisely corresponds to the data points, then the weights w_i of RBFs approach zero. The representation of f with other coefficients d .

$$\Phi w + Pc = Pd \quad (14)$$

After multiplying each term by w^T , it is obtained:

$$w^T \Phi w + w^T Pc = w^T Pd \quad (15)$$

$$w^T \Phi w = w^T P(d - c) = 0 \quad (16)$$

When $w^T P = 0$ is required, then $w = 0$. Also, this means that Pc and Pd are identical when $w = 0$, and the reproduction of polynomial is occurred exactly.

4. Truncated Exponential Radial Basis Function (TERBF)

The modeling of surfaces is closely connected to approximation and interpolation based on methods of level set, RBFs, and least squares methods. RBFs accompanied with global support possess an adequate approximation influence, but that is often associated with discrete matrix which has extremely large condition number. As a result, the numerical calculations become time consuming. Consequently, in 2019, Xu et al[15] introduced a radial truncated exponential function on n -dimensional space which is arbitrary \mathbb{R}^n , and its support is compact.

The form of the truncated exponential function is:

$$\varphi(r) = (e^{1-r} - 1)^l \quad (17)$$

Where l is an integer as $l > 0$, and $r \in \mathbb{R}$. When $r = \|x\|$ and $x \in \mathbb{R}^n$, $\Phi(x) = \varphi(r)$ becomes radial function and its center is the origin (on \mathbb{R}^n).

The authors discussed the interpolation of the scattered data relying on RBFs (compactly supported) $\Phi(x, x_k) = (e^{1-\|x-x_k\|} - 1)^l, x, x_k \in \mathbb{R}^n$. Assuming a set of scattered points $X = \{x_1, x_2, \dots, x_N\} \subset \mathbb{R}^n$, the target function interpolation can be presented as follows:

$$P_f(x) = \sum_{j=1}^N c_j \Phi(x, x_j), \quad x \in \mathbb{R}^n \tag{18}$$

By solving the problem of interpolation a system of linear equations is obtained as follows:

$$Ac = y \tag{19}$$

[] found that the $D_\Phi(x)$ can be introduced for C^0 as follows:

$$D_\Phi(x) = \|\Phi\|_{L^\infty(B(0,2ch_{x,\Omega}))} \tag{20}$$

And relying on the theorem of Lagrange’s mean value, they had:

$$\|\Phi\|_{L^\infty(B(0,2ch_{x,\Omega}))} = \max_{r \in (0,4ch_{x,\Omega})} |e^{1-r} - 1|^l \tag{21}$$

$$\leq C \max_{r \in (0,4ch_{x,\Omega})} |1 - r|^l \tag{22}$$

$$= C \|\Psi\|_{L^\infty(B(0,2ch_{x,\Omega}))} \tag{23}$$

As the truncated power radial basis function is denoted by Ψ , and that corresponds to:

$$\|\Psi\|_{L^\infty(B(0,2ch_{x,\Omega}))} \leq Ch_{x,\Omega}^{\frac{1}{2}} \tag{22}$$

The native space methodology was used by[16] for the estimation of errors of the interpolated scattered data as discussed previously. For examining the approximation of the proposed radial function, the researchers employed both the single level and multilevel interpolation for surface modeling.

By using the single-level approximation, the RMS error is calculated as follow:

$$RMS - error = \sqrt{\frac{1}{M} \sum_{k=1}^M [f(\xi_k) - P_f(\xi_k)]^2} = \frac{1}{\sqrt{M}} \|f - P_f\|_2 \tag{23}$$

Where ξ_k are the points of evaluation. It is observed from the tables () and () that TERBF (truncated exponential radial basis function) interpolation can retain better accuracy of approximation, and also generates an interpolation matrix which is well-conditioned in comparison with other supported function (Gaussian, MQ, IMQ). The proposed method’s condition numbers relatively lesser (around 105) even though N equals to 4225 and $\epsilon = 0.7$.

Table 2. TERBF interpolation to the 2D Franke’s function with $\epsilon=1$

N	RMS-Error	Rate	Cond (A)
9	1.951235×10^{-1}	-	$6.639719 \times 10^{+0}$
25	5.018953×10^{-2}	1.958929	$2.405994 \times 10^{+1}$
81	1.628459×10^{-2}	1.623879	$1.669026 \times 10^{+2}$
289	6.727682×10^{-3}	1.275326	$1.250365 \times 10^{+3}$
1089	2.402630×10^{-3}	1.485495	$1.058555 \times 10^{+4}$
4225	9.728457×10^{-4}	1.304332	$9.410946 \times 10^{+4}$

Table 3. TERBF interpolation to the 2D Franke’s function with $\epsilon=0.7$

N	RMS-Error	Rate	Cond (A)
9	1.728785×10^{-1}		$1.275042 \times 10^{+0}$
25	4.535991×10^{-2}	1.930269	5.066809×10^{-1}
81	1.335521×10^{-2}	1.764015	3.608813×10^{-2}
289	5.013012×10^{-3}	1.413653	2.719227×10^{-3}
1089	1.773595×10^{-3}	1.499001	$2.305630 \times 10^{+4}$
4225	7.107796×10^{-4}	1.319203	$2.050036 \times 10^{+5}$

In addition, it was evident that the error curves that belong to the interpolation of Gaussian and MQ for the most substantial datasets became erratic. On the other hand, the interpolation curves of TERBF and IMQ are relatively smooth. In specific, TERBF considerably improves the interpolation matrix with respect to the condition number. Moreover, Numerical calculation indicated that the interpolation TERBF -based is remarkably faster compared to the use of globally supported radial basis functions for the interpolation of scattered data. .

With regard to the multi-level approximation, it was first conducted by [17] and investigated by several researchers [5], [18-24]. The multilevel interpolation based on compactly supported-radial basis functions with gradually smaller support has an advantage that is characterized by recursive property, while the need of memory by allocation is considered as a disadvantage. The Numerical results indicated that multilevel interpolation based on TERBF is quite efficient for the approximation of 3D explicit surface. These finding sareillustrated

d)) However, to ameliorate the requirements of allocation memory of the multilevel approach, one can take advantage of the method of the hierarchical collocation studied in [22].

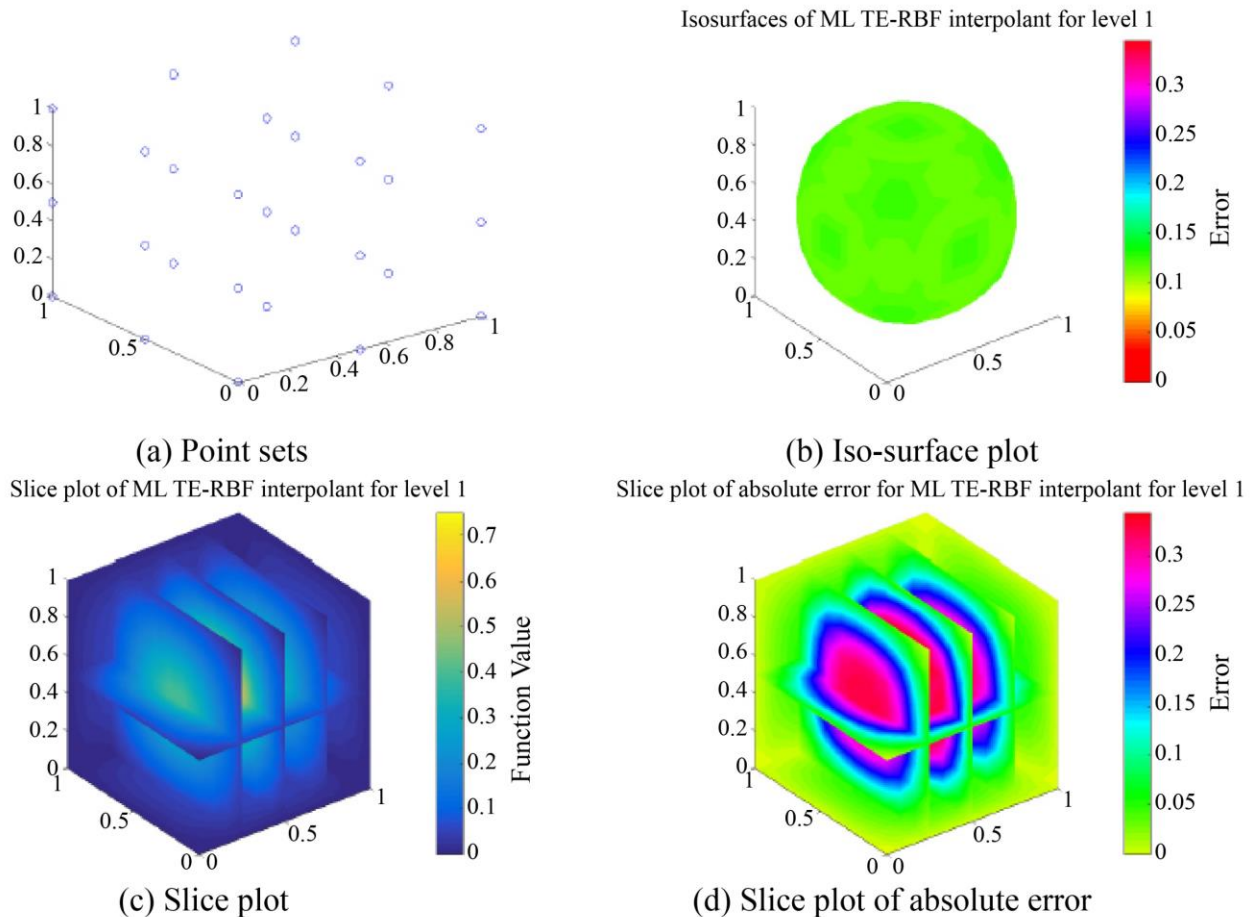
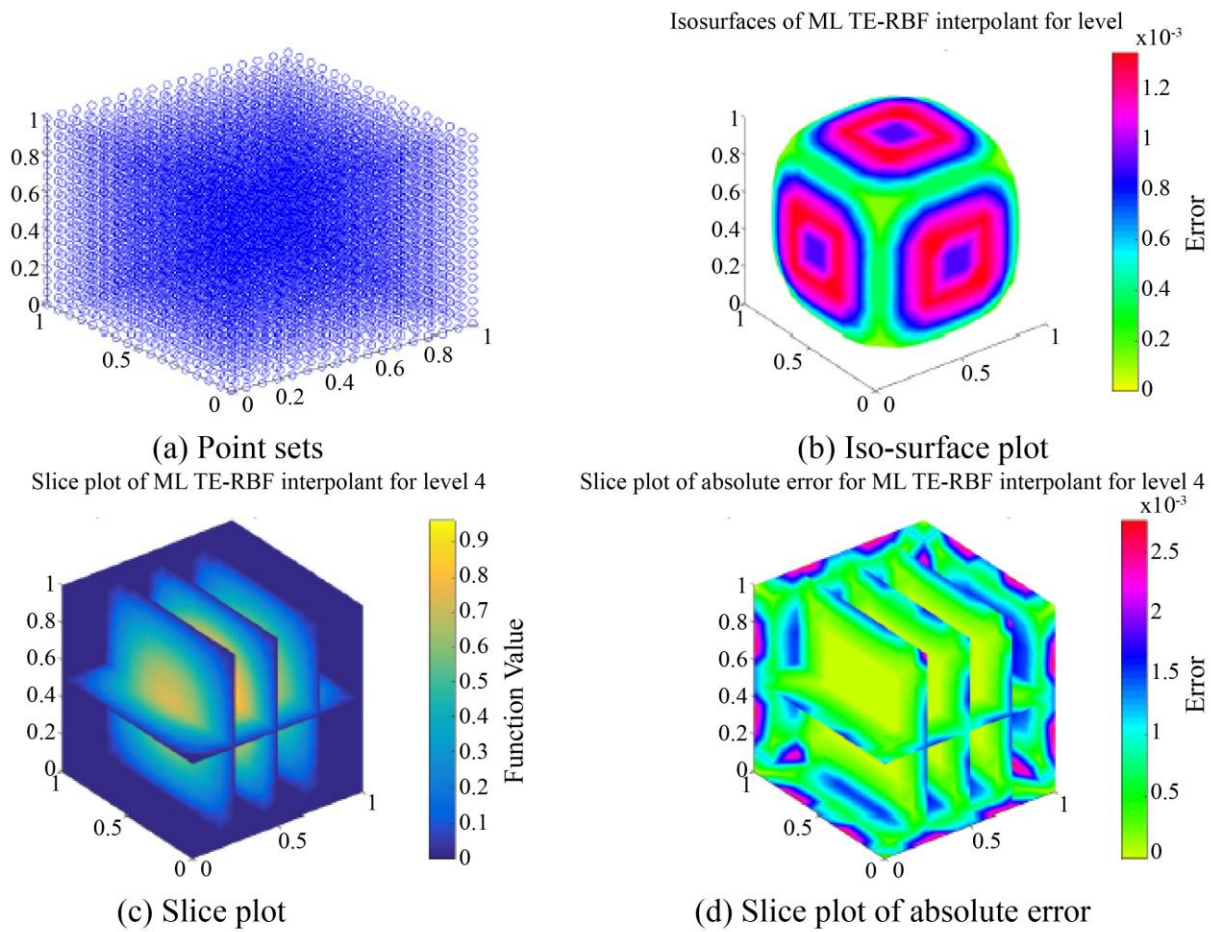


Fig. 8 Fits and errors at Level 1.



5. Conclusion

It is found based on previous studies that the RBFs is an adequate approach for producing smooth interpolation with reduced fluctuations when addressing high dimensional data interpolation. Moreover, it is possible to raise the interpolant capacity by adding polynomial terms, and also executing regularization to avert the over fitting of data.

The dependence of numerical methods on radial basis functions is growing sharply throughout the years because of their mesh free nature. Several approaches rely on RBFs formulations have been presented in this review, and these approaches are introduced with the mathematical modeling for making them more understandable. The unsymmetrical method (Kansa method) considered the simplest method for addressing PDEs. RBF collocation methods are associated with high computational cost due to the dense matrices; thus, alternative methods are always needed for tackling such difficulties. For example, the methods of RBF local approximation are common because of their capability of local adaptation. The high efficiency of scalability of the methods of RBF for solving PDEs (high dimensional) remains under consideration.

References

- [1] Rolland L. Hardy, "Multiquadric Equations of Topography and Other Irregular Surfaces," *Journal of Geophysical Research*, vol. 76, no. 8, pp. 1905-1971, 1971. [[CrossRef](#)] [[Google Scholar](#)] [[Publisher Link](#)]
- [2] Richard Franke, "Scattered Data Interpolation: Tests of Some Methods," *Mathematics of Computation*, vol. 38, pp. 181-200, 1982. [[Google Scholar](#)] [[Publisher Link](#)]
- [3] Charles A. Micchelli, "Interpolation of Scattered Data: Distance Matrices and Conditionally Positive Definite Functions," *Constructive Approximation*, 1986.
- [4] E.J. Kansa, "Multiquadrics – A Scattered Data Approximation Scheme with Applications to Computational Fluid Dynamics I: Surface Approximations and Partial Derivative Estimates," *Computers and Mathematics with Applications*, vol. 19, no. 8-9, pp. 127-145, 1990. [[CrossRef](#)] [[Google Scholar](#)] [[Publisher Link](#)]

- [5] E.J. Kansa, "Multiquadrics -A Scattered Data Approximation Scheme with Applications to Computational Fluid-Dynamics II: Solutions to Parabolic, Hyperbolic and Elliptic Partial Differential Equations," *Computers and Mathematics with Applications*, vol. 19, no. 8-9, pp. 147–161, 1990. [[CrossRef](#)] [[Google Scholar](#)] [[Publisher Link](#)]
- [6] Gregory E. Fasshauer, "Solving Partial Differential Equations by Collocation with Radial Basis Functions," *Mehaute A, Rabut C, Schumaker L.L. Editors, Surface Fitting and Multiresolution Methods*, Vanderbilt University Press, Nashville, 1997 . [[Google Scholar](#)] [[Publisher Link](#)]
- [7] H. Power, and V. Barraco, "A Comparison Analysis Between Unsymmetric and Symmetric Radial Basis Function Collocation Methods for the Numerical Solution of Partial Differential Equations," *Computers and Mathematics with Applications*, vol. 43, no. 3-5, pp. 551–583, 2002. [[CrossRef](#)] [[Google Scholar](#)] [[Publisher Link](#)]
- [8] E. Larsson, and B. Fornberg, "A Numerical Study of Some Radial Basis Function-based Solution Methods for Elliptic PDEs," *Computers and Mathematics with Applications*, vol. 46, no. 5-6, pp. 891–902, 2003. [[CrossRef](#)] [[Google Scholar](#)] [[Publisher Link](#)]
- [9] Leevan Ling, and Edward J. Kansa, "A Least-squares Preconditioner for Radial basis Functions Collocation Methods," *Advances in Computational Mathematics*, vol. 23, pp. 31–54, 2005. [[CrossRef](#)] [[Google Scholar](#)] [[Publisher Link](#)]
- [10] Smooth Functions or Continuous. [Online]. Available: <https://math.stackexchange.com/questions/472148/smooth-functions-or-continuous>
- [11] Daniel Cohen-Or et al., *A Sampler of Useful Computational Tools for Applied Geometry, Computer Graphics, and Image Processing*, CRC Press, 2015. [[Google Scholar](#)]
- [12] Ken Anjyo, J.P. Lewis, and Frédéric Pighin, "Scattered Data Interpolation for Computer Graphics," *SIGGRAPH Course Notes*, pp. 1-69, 2014. [[CrossRef](#)] [[Google Scholar](#)] [[Publisher Link](#)]
- [13] Colour Manipulation with the Colour Checker Lut Module, 2016. [Online]. Available: <https://www.darktable.org/2016/05/colour-manipulation-with-the-colour-checker-lut-module/>
- [14] Donald Shepard, "A Two-dimensional Interpolation Function for Irregularly-Spaced Data," *Proceedings of the 1968 ACM National Conference*, pp. 517–524, 1968. [[CrossRef](#)] [[Google Scholar](#)] [[Publisher Link](#)]
- [15] Qiuyan Xu, and Zhiyong Liu, "Scattered Data Interpolation and Approximation with Truncated Exponential Radial Basis Function," *Mathematics*, vol. 7, no. 11, p. 1101, 2019. [[CrossRef](#)] [[Google Scholar](#)] [[Publisher Link](#)]
- [16] Robert Schaback, "A Unified Theory of Radial Basis Functions: Native Hilbert Spaces for Radial Basis Functions II," *Journal of Computational and Applied Mathematics*, vol. 121, no. 1-2, pp. 165–177, 2000. [[CrossRef](#)] [[Google Scholar](#)] [[Publisher Link](#)]
- [17] Michael S. Floater, and Armin Iske, "Multistep Scattered Data Interpolation Using Compactly Supported Radial Basis Functions," *Journal of Computational and Applied Mathematics*, vol. 73, no. 1-2, pp. 65–78, 1996. [[CrossRef](#)] [[Google Scholar](#)] [[Publisher Link](#)]
- [18] C.S. Chen et al., "Multilevel Compact Radial Functions based Computational Schemes for Some Elliptic Problems," *Computers and Mathematics with Application*, vol. 43, no. 3-5, pp. 359–378, 2002. [[CrossRef](#)] [[Google Scholar](#)] [[Publisher Link](#)]
- [19] A. Chernih, and Q.T. Le Gia, "Multiscale Methods with Compactly Supported Radial basis Functions for the Stokes Problem on Bounded Domains," *Advances in Computational Mathematics*, vol. 42, pp. 1187–1208, 2016. [[CrossRef](#)] [[Google Scholar](#)] [[Publisher Link](#)]
- [20] Patricio Farrell, and Holger Wendland, "RBF Multi Scale Collocation for Second Order Elliptic Boundary Value Problems," *SIAM Journal on Numerical Analysis*, vol. 51, no. 4, 2013. [[CrossRef](#)] [[Google Scholar](#)] [[Publisher Link](#)]
- [21] Gregory E. Fasshauer, and Joseph W. Jerome, "Multistep Approximation Algorithms: Improved Convergence Rates Through Postconditioning with Smoothing Kernels," *Advances in Computational Mathematics*, vol. 10, pp. 1–27, 1999. [[CrossRef](#)] [[Google Scholar](#)] [[Publisher Link](#)]
- [22] Zhiyong Liu, and Qiuyan Xu, "Multiscale RBF Collocation Method for the Numerical Solution of Partial Differential Equations," *Mathematics*, vol. 7, no. 10, p. 964, 2019. [[CrossRef](#)] [[Google Scholar](#)] [[Publisher Link](#)]
- [23] Jichun Li, and C.S. Chen, "Some Observations on Unsymmetric Tricubic Radial Basis Function Collocation Methods for Convection Diffusion Problems," *International Journal for Numerical Methods in Engineering*, vol. 57, no. 8, pp. 1085–1094, 2003. [[CrossRef](#)] [[Google Scholar](#)] [[Publisher Link](#)]
- [24] B. Šarler, and R. Vertnik, "Meshfree Explicit Local Radial Basis Function Collocation Method for Diffusion Problems," *Computers and Mathematics with Applications*, vol. 51, no. 8, pp. 1269–1282, 2006. [[CrossRef](#)] [[Google Scholar](#)] [[Publisher Link](#)]