

Original Article

Quasi-Convolution Properties of a New Family of Close-to-Convex Functions Involving a q - p -Opoola Differential Operator

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Abstract -

In this paper, by means of a q -difference operator, we first introduce a q - p -Opoola differential operator.

$$D_{p,q}^N F(z) = z^p - \sum_{k=p+r}^{\infty} \left(\frac{[K]_q}{[P]_q} \right)^N [1 + (k + \varphi - \gamma - 1)\lambda]^n a_k z^k$$

($p, r, \lambda \in N := \{1, 2, \dots\}$; $n, t \in N \cup 0 := \{0, 1, 2, \dots\}$), which is related to $D^n(\gamma, \varphi, t)f(z) = z + \sum_{k=2}^{\infty} [1 + (k + \varphi - \gamma - 1)t]^n a_k z^k$, $z \in U$ $0 \leq \gamma \leq \varphi$, $t \geq 0$ when $p = r = 1$ in $F(z) \in T(p, r)$. Via employing the q - p -Opoola differential operator, we define a new family of close-to-convex functions and obtain a coefficient estimate theorem. Furthermore, we also introduced several modified-Hadamard products for the family and finally gave some corollaries.

Keywords - Quasi-Convolution Properties, Close-to-Convex Functions, q -difference operator, q - p -Opoola differential operator, Modified-Hadamard product.

1. Introduction

In 2017, Opoola [19] introduced and studied the generalized differential operator $D^n(\gamma, \varphi, t)f(z)$:

$$D^n(\gamma, \varphi, t)f(z) = z + \sum_{k=2}^{\infty} [1 + (k + \varphi - \gamma - 1)t]^n a_k z^k, \quad 0 \leq \gamma \leq \varphi, \quad t \geq 0.$$

for $f(z) \in A$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

with normalization $f(0) = f'(0) - 1 = 0$. As usual, A denotes the class of analytic functions in the unit disk $E = \{z \in C : |z| < 1\}$ where C is the set of complex numbers. Several works involving the differential operator $D^n(\gamma, \varphi, t)f(z)$ have appeared in the Literature, for example ([15], [[20] - [22]]). Eventually, the differential operator $D^n(\gamma, \varphi, t)f(z)$ becomes Sălăgean and Al-Oboudi differential operators when the parameters involved are and by letting $t = \lambda$ for the Al-Oboudi differential operator.

We note that $T(1, 1) = T$ such that



$$D^n(\gamma, \varphi, t)f(z) = z + \sum_{k=2}^{\infty} [1 + (k + \varphi - \gamma - 1)t]^n a_k z^k, \quad a_k \geq 0, z \in U \quad 0 \leq \gamma \leq \varphi, \quad t \geq 0.$$

for $f(z) \in T$ where T denote the subclass of A consisting of functions with negative coefficients onward. Now, let $T(p, r)$ be the class of functions $F(z)$ of the form.

$$(1.1) \quad f(z) = z + \sum_{k=2}^{\infty} [1 + (k + \varphi - \gamma - 1)t]^n a_k z^k,$$

(where $a_k \geq 0$; $\lambda \geq 0$; $0 \leq \varphi \leq \gamma$; $p, r, n \in N = \{1, 2, \dots\}$) be defined and analytic in $E = \{z \in C : |z| < 1\}$.

Definition 1.1. The Jackson’s q-difference operator introduced in [13] is defined as follows

$$\partial_q F(z) = \begin{cases} \frac{F(z) - F(qz)}{(1-q)z} & \text{if } z \neq 0, \\ F'(0) & \text{if } z = 0. \end{cases}$$

and

$$(1.2) \quad \partial_q F(z) = [P]_q z^{p-1} - \sum_{k=p+r}^{\infty} [K]_q [1 + (k-1)\lambda]^n a_k z^{k-1},$$

where

$$(1.3) \quad [K]_q = \frac{1 - q^k}{1 - q} \text{ and, as } q \rightarrow 1^- \Rightarrow [K]_q \rightarrow k.$$

Some other relevant works on (1.2) include ([1]-[3]), [5], [6], [9], [10], [16], [27], [28]).

Now we define the p-q - Opoola differential operator by

$$D_{p,q}^0 F(z) = F(z)$$

$$D_{p,q}^1 F(z) = \frac{z}{[P]_q} \partial_q F(z)$$

$$= z^p - \sum_{k=p+r}^{\infty} \frac{[K]_q}{[P]_q} [1 + (k + \varphi - \gamma - 1)\lambda]^n a_k z^k$$

$$D_{p,q}^2 F(z) = \frac{z}{[P]_q} \partial_q (D_{p,q}^{N-1} F(z)),$$

Hence,

$$(1.4) \quad \begin{aligned} D_{p,q}^N F(z) &= \frac{z}{[P]_q} \partial_q (D_{p,q}^{N-1} F(z)) \\ &= z^p - \sum_{k=p+r}^{\infty} \left(\frac{[K]_q}{[P]_q} \right)^N [1 + (k + \varphi - \gamma - 1)\lambda]^n a_k z^k \end{aligned}$$

Where $N \in N \cup 0$.

Note that:

- (i) Putting $p = 1, \varphi = \gamma, \lambda = 1, q \rightarrow 1^-$ in (1.4), we have the Salagean differential operator D^N studied in [25];
- (ii) Putting $p = 1, \varphi = \gamma, q \rightarrow 1^-$ in (1.4), we have the Al-Oboudi differential operator;
- (iii) Putting $n = 0$ in (1.4), we the q-p-Salagean differential operator studied in [18];
- (iv) Letting $q \rightarrow 1-, \varphi = \gamma, \lambda = 1$ in (1.4), we have the operator DN p introduced and studied by Shenan et al. [26], Kamali and Orhan [14], Aouf and Mostafa [7] and Aouf et al. [8].

Definition 1.2. For some $\mu(0 \leq \mu < [p]_q)$ and $\xi(0 < \xi \leq 1)$, a function $F(z) \in T(p, r)$ is in the class $T_{p,q}^{\varphi, \gamma, \lambda}(N, r, \mu, \xi)$ if

it satisfies

$$(1.5) \quad \left| \frac{\frac{\partial_q(D_{p,q}^N F(z))}{z^{p-1}} - [P]_q}{\frac{\partial_q(D_{p,q}^N F(z))}{z^{p-1}} - [P]_q - 2\mu} \right| < \xi$$

In what follows, we shall state and prove our results for the class $T_{p,q}^{\varphi,\gamma,\lambda}(N, r, \mu, \xi)$. These results are the coefficient estimate theorem of several modified-Hadamard products for the class.

Coefficients problems and modified-Hadamard product problems have appeared very significantly in the Literature, and for a few examples, interested readers are referred to ([11], [12], [23], [24]).

2. Main Result

Properties of the class $T_{p,q}(N, r, \mu, \xi)$ such as

Theorem 2.1. Let $F(z)$ be given by (1.1), then $F(z) \in T_{p,q}(N, r, \mu, \xi)$ if and only if

$$(2.1) \quad \sum_{k=p+r}^{\infty} \left(\frac{[K]_q}{[P]_q} \right)^N [1 + (k + \varphi - \gamma - 1)\lambda]^n a_k \leq 2\xi([p]_q - \mu).$$

Proof. Assume that (2.1) holds, we find from (1.1) and (1.5) that

$$\begin{aligned} & \left| \frac{\partial_q(D_{p,q}^N F(z))}{z^{p-1}} - [p]_q z^{p-1} \right| - \xi \left| \frac{\partial_q(D_{p,q}^N F(z))}{z^{p-1}} + [p]_q z^{p-1} - 2\mu z^{p-1} \right| \\ &= \left| \left([p]_q z^{p-1} - \sum_{k=p+r}^{\infty} [K]_q \left(\frac{[K]_q}{[P]_q} \right)^N [1 + (k + \varphi - \gamma - 1)\lambda]^n a_k z^{k-1} \right) - [p]_q z^{p-1} \right| - \\ & \xi \left| \left([p]_q z^{p-1} - \sum_{k=p+r}^{\infty} [K]_q \left(\frac{[K]_q}{[P]_q} \right)^N [1 + (k + \varphi - \gamma - 1)\lambda]^n a_k z^{k-1} \right) + [p]_q z^{p-1} - 2\mu z^{p-1} \right| \\ &= \left| \sum_{k=p+r}^{\infty} [K]_q \left(\frac{[K]_q}{[P]_q} \right)^N [1 + (k + \varphi - \gamma - 1)\lambda]^n a_k z^{k-1} \right| - \\ & \xi \left| 2([p]_q - \mu) z^{p-1} - \sum_{k=p+r}^{\infty} [K]_q \left(\frac{[K]_q}{[P]_q} \right)^N [1 + (k + \varphi - \gamma - 1)\lambda]^n a_k z^{k-1} \right| \\ &\leq -2\xi([p]_q - \mu) |z|^{p-1} + (1 + \xi) \sum_{k=p+r}^{\infty} [K]_q \left(\frac{[K]_q}{[P]_q} \right)^N [1 + (k + \varphi - \gamma - 1)\lambda]^n a_k |z|^{k-1} \\ &< -2\xi([p]_q - \mu) + (1 + \xi) \sum_{k=p+r}^{\infty} [K]_q \left(\frac{[K]_q}{[P]_q} \right)^N [1 + (k + \varphi - \gamma - 1)\lambda]^n a_k \leq 0. \end{aligned}$$

Hence, by the maximum modulus theorem, we have

$$\begin{aligned} & \left| \frac{\frac{\partial_q(D_{p,q}^N F(z))}{z^{p-1}} - [P]_q}{\frac{\partial_q(D_{p,q}^N F(z))}{z^{p-1}} + \xi([p]_q - \mu)} \right| = \\ & \left| \frac{\sum_{k=p+r}^{\infty} [K]_q \left(\frac{[K]_q}{[P]_q} \right)^N [1 + (k + \varphi - \gamma - 1)\lambda]^n a_k z^{k-1}}{2\xi([p]_q - \mu) z^{p-1} - \xi \sum_{k=p+r}^{\infty} [K]_q \left(\frac{[K]_q}{[P]_q} \right)^N [1 + (k + \varphi - \gamma - 1)\lambda]^n a_k z^{k-1}} \right| < 1 \end{aligned}$$

Now since $Re(z) \leq |z|$ for all z , we have

$$Re \left\{ \frac{\sum_{k=p+r}^{\infty} [K]_q \left(\frac{[K]_q}{[P]_q} \right)^N [1 + (k + \varphi - \gamma - 1)\lambda]^n a_k z^{k-1}}{2\xi([p]_q - \mu) z^{p-1} - \xi \sum_{k=p+r}^{\infty} [K]_q \left(\frac{[K]_q}{[P]_q} \right)^N [1 + (k + \varphi - \gamma - 1)\lambda]^n a_k z^{k-1}} \right\} < 1$$

Now choose values of z on the real axis so that $\frac{\partial_q(D_{p,q}^N F(z))}{z^{p-1}}$ is real and letting $q \rightarrow 1^-$ through real values, we have

$$\sum_{k=p+r}^{\infty} [K]_q \left(\frac{[K]_q}{[P]_q}\right)^N [1 + (k + \varphi - \gamma - 1)\lambda]^n a_k 2\xi([p]_q - \mu) - \xi \sum_{k=p+r}^{\infty} [K]_q \left(\frac{[K]_q}{[P]_q}\right)^N [1 + (k + \varphi - \gamma - 1)\lambda]^n a_k.$$

This gives the required condition.

3. Modified-Hadamard Product Theorems for $T_{p,q}(N, r, \mu, \xi)$

In this section, we will introduce new theorems for modified Hadamard products for the functions that belong to the class $T_{p,q}(N, r, \mu, \xi)$.

Let $F_w (w = 1, 2, \dots, s)$ be defined by

$$z^p - \sum_{k=p+r}^{\infty} [1 + (k + \varphi - \gamma - 1)\lambda]^n a_{k,w} z^k \tag{3.1}$$

where $(a_{k,w} \geq 0, \lambda \geq 0, p, r, n \in N = \{1, 2, \dots\})$.

The modified Hadamard product of F_1 and F_2 is defined by

$$(F_1 * F_2)(z) = z^p - \sum_{k=p+r}^{\infty} [1 + (k + \varphi - \gamma - 1)\lambda]^n a_{k,1} a_{k,2} z^k. \tag{3.2}$$

Theorem 3.1. Let $F_w(z) \in T_{p,q}(N, r, \mu, \xi) (w = 1, 2, \dots, s)$ defined by (3.1), then $((F_1 * F_2 * \dots * F_s)(z))(z) \in T_{p,q}(N, r, \mu, \xi)$, where

$$\theta = [P]_q - \frac{\prod_{w=1}^s ([P]_q - \mu_w)}{\left[[p+r]_q (1+\xi) \left(\frac{[p+r]_q}{[P]_q}\right)^N [1+(k+\varphi-\gamma-1)\lambda]^n \right]^{s-1}} \tag{3.3}$$

The result is sharp for the functions

$$F_w(z) = z^p - \frac{2\xi([P]_q - \mu_w)}{[p+r]_q (1+\xi) \left(\frac{[p+r]_q}{[P]_q}\right)^N [1+(k+\varphi-\gamma-1)\lambda]^n} z^{p+r} \tag{3.4}$$

Proof. To prove the Theorem, we will use the mathematical induction.

For $s = 1$, we see that $\eta = [P]_q - \mu_1$. For $s = 2$, we have

$$\sum_{k=p+r}^{\infty} \frac{[k]_q (1+\xi) \left(\frac{[K]_q}{[P]_q}\right)^N [1+(k+\varphi-\gamma-1)\lambda]^n}{2\xi([P]_q - \mu_w)} a_{k,w} \leq 1 (w = 1, 2). \tag{3.5}$$

This gives that

$$\sum_{k=p+r}^{\infty} \frac{[k]_q (1+\xi) \left(\frac{[K]_q}{[P]_q}\right)^N [1+(k+\varphi-\gamma-1)\lambda]^n}{\sqrt{\prod_{w=1}^2 2\xi([P]_q - \mu_w)}} \sqrt{a_{k,1} a_{k,2}} \leq 1. \tag{3.6}$$

To prove the case where $s = 2$, we have to find the largest θ such that

$$\sum_{k=p+r}^{\infty} \frac{[k]_q (1+\xi) \left(\frac{[K]_q}{[P]_q}\right)^N [1+(k+\varphi-\gamma-1)\lambda]^n}{2\xi([P]_q - \theta)} a_{k,1} a_{k,2} \leq 1. \tag{3.7}$$

such that

$$\frac{\sqrt{a_{k,1} a_{k,2}}}{2\xi([P]_q - \theta)} \leq \frac{1}{\sqrt{\prod_{w=1}^2 2\xi([P]_q - \mu_w)}}. \tag{3.8}$$

Then using (3.6), we need to find the largest ω such that

$$\frac{1}{2\xi([P]_q - \theta)} \leq \frac{[k]_q(1 + \xi) \left(\frac{[k]_q}{[p]_q}\right)^N [1 + (k + \varphi - \gamma - 1)\lambda]^n}{\prod_{w=1}^2 2\xi([P]_q - \mu_w)} \quad (3.9)$$

that is

$$\theta \leq [P]_q - \frac{\prod_{w=1}^2 2\xi([P]_q - \mu_w)}{2\xi[k]_q(1 + \xi) \left(\frac{[k]_q}{[p]_q}\right)^N [1 + (k + \varphi - \gamma - 1)\lambda]^n} \quad (3.10)$$

Defining the function $\Phi(k)$ by

$$\Phi(k) = [p]_q - \frac{\prod_{w=1}^2 2\xi([p]_q - \mu_w)}{2\xi[k]_q(1 + \xi) \left(\frac{[k]_q}{[p]_q}\right)^N [1 + (k + \varphi - \gamma - 1)\lambda]^n} \quad (3.11)$$

we see that $\Phi(k) \geq 0$ for $k \geq p + r$. This implies that

$$\theta \leq \Phi(p + r) = [p]_q - \frac{\prod_{w=1}^2 2\xi([p]_q - \mu_w)}{2\xi[p + r]_q(1 + \xi) \left(\frac{[p + r]_q}{[p]_q}\right)^N [1 + (k + \varphi - \gamma - 1)\lambda]^n} \quad (3.12)$$

Therefore, the result is true for $s = 2$.

Suppose that the result is true for any positive integer s . Then we have $(F_1 * F_2 * \dots * F_s)(z) \in T_{p,q}(N, r, \mu, \xi)$, where

$$\Psi = [P]_q - \frac{([p]_q - \theta)2\xi([p]_q - \mu_{s+1})}{2\xi[p + r]_q(1 + \xi) \left(\frac{[p + r]_q}{[p]_q}\right)^N [1 + (k + \varphi - \gamma - 1)\lambda]^n} \quad (3.13)$$

After simple calculation, we have

$$\Psi = [P]_q - \frac{\prod_{w=1}^{s+1} 2\xi([p]_q - \mu_w)}{2\xi[p + r]_q(1 + \xi) \left(\frac{[p + r]_q}{[p]_q}\right)^N [1 + (k + \varphi - \gamma - 1)\lambda]^n} \quad (3.14)$$

Thus, the result is true for $s + 1$; the result is true for any positive integer s .

Putting $\mu_w = \mu(w = 1, 2, \dots, s)$ in Theorem 3.1, we have:

Corollary 3.2. If $F_w(z) \in T_{p,q}(N, r, \mu, \xi)$ ($w = 1, 2, \dots, s$), then $(F_1 * F_2 * \dots * F_s)(z) \in T_{p,q}(N, r, \mu, \xi)$, where

$$\theta_1 = [P]_q - \frac{2[\xi]^{s-1}([p]_q - \mu)^s}{\left[[p + r]_q(1 + \xi) \left(\frac{[p + r]_q}{[p]_q}\right)^N [1 + (k + \varphi - \gamma - 1)\lambda]^n \right]^s} \quad (3.15)$$

The result is sharp for the functions

$$F_w(z) = z^p - \frac{2\xi([p]_q - \mu)}{[p + r]_q(1 + \xi) \left(\frac{[p + r]_q}{[p]_q}\right)^N [1 + (k + \varphi - \gamma - 1)\lambda]^n} z^{p+r} \quad (3.16)$$

Putting $\xi = 1$ in Theorem 3.1, we have:

Corollary 3.3. If $F_w(z) \in T_{p,q}(N, r, \mu_w, 1) = T_{p,q}(N, r, \mu_w)$ ($w = 1, 2, \dots, s$), then $(F_1 * F_2 * \dots * F_s)(z) \in T_{p,q}(N, r, \mu, \xi)$, where

$$\eta = [p]_q - \frac{\prod_{w=1}^s ([p]_q - \mu_w)}{\left[[p + r]_q \left(\frac{[p + r]_q}{[p]_q}\right)^N [1 + (k + \varphi - \gamma - 1)\lambda]^n \right]^{s-1}} \quad (3.17)$$

The result is sharp for the functions

$$\eta = z^p - \frac{([p]_q - \mu_w)}{[p+r]_q \left(\frac{[p+r]_q}{[p]_q}\right)^N [1 + (k + \varphi - \gamma - 1)\lambda]^n} z^{p+r} \quad (3.18)$$

where $(w = 1, 2, \dots, s)$.

Theorem 3.4. If $F_w(z) \in T_{p,q}(N, r, \mu_w, \xi) (w = 1, 2, \dots, s)$, then

$$g(z) = z^p - \sum_{k=p+r}^{\infty} \left(\sum_{k=p+r}^{\infty} a_{k,w}^2 \right) z^k \quad (3.19)$$

belongs to $T_{p,q}(N, r, \mu_w, \xi)$ where

$$\theta_2 \leq [p]_q - \frac{s[2\xi([p]_q - \mu_w)]^2}{2\xi(1 + \xi)[p+r]_q \left(\frac{[p+r]_q}{[p]_q}\right)^N [1 + (k + \varphi - \gamma - 1)\lambda]^n} \quad (3.20)$$

where $(\mu_0 = \min \{\mu_1, \mu_2, \dots, \mu_s\})$.

The result is sharp for the functions $F_w(z)$ given by (3.4).

Proof. Since Theorem 3.1 gives

$$\begin{aligned} & \sum_{k=p+r}^{\infty} \left\{ \frac{(1 + \xi)[k]_q \left(\frac{[k]_q}{[p]_q}\right)^N [1 + (k - 1)\lambda]^n}{2\xi([p]_q - \mu_w)} \right\}^2 a_{k,w}^2 \\ & \leq \sum_{k=p+r}^{\infty} \left\{ \frac{(1 + \xi)[k]_q \left(\frac{[k]_q}{[p]_q}\right)^N [1 + (k + \varphi - \gamma - 1)\lambda]^n}{2\xi([p]_q - \mu_w)} \right\}^2 \leq 1, \end{aligned}$$

then, for $(w = 1, 2, \dots, s)$ we have:

$$\sum_{k=p+r}^{\infty} \frac{1}{s} \left\{ \frac{(1 + \xi)[k]_q \left(\frac{[k]_q}{[p]_q}\right)^N [1 + (k + \varphi - \gamma - 1)\lambda]^n}{2\xi([p]_q - \mu_w)} \right\}^2 \left(\sum_{k=p+r}^{\infty} a_{k,w}^2 \right) \leq 1, \quad (3.21)$$

So, we may find the largest w_2 such that:

$$\sum_{k=p+r}^{\infty} \left\{ \frac{(1 + \xi)[k]_q \left(\frac{[k]_q}{[p]_q}\right)^N [1 + (k + \varphi - \gamma - 1)\lambda]^n}{2\xi([p]_q - \mu_w)} \right\}^2 \left(\sum_{k=p+r}^{\infty} a_{k,w}^2 \right) \leq 1, \quad (3.22)$$

Inequalities (3.21) and (3.22) leads to

$$\theta_2 \leq [p]_q - \frac{s[2\xi([p]_q - \mu_w)]^2}{2\xi(1 + \xi)[k]_q \left(\frac{[k]_q}{[p]_q}\right)^N [1 + (k + \varphi - \gamma - 1)\lambda]^n} \quad (3.23)$$

that is

$$\theta_2 \leq [p]_q - \frac{s[2\xi([p]_q - \mu_w)]^2}{2\xi(1 + \xi)[p+r]_q \left(\frac{[p+r]_q}{[p]_q}\right)^N [1 + (p+r + \varphi - \gamma - 1)\lambda]^n} \quad (3.24)$$

This completes the proof.

Putting $\mu_w = (w = 1, 2, \dots, s)$ in Theorem 3.2, we have:

Corollary 3.6. If $F_w(z) \in T_{p,q}(N, r, \mu_w, 1)$ ($w = 1, 2, \dots, s$) and $g(z)$ is defined by (3.19), then $g(z) \in T_{p,q}(N, r, \theta_3)$, where

$$\theta_3 \leq [p]_q - \frac{s[2\xi([p]_q - \mu_w)]^2}{2\xi(1 + \xi)[p + r]_q \left(\frac{[p + r]_q}{[p]_q}\right)^N [1 + (p + r + \varphi - \gamma - 1)\lambda]^n} \quad (3.25)$$

The result is sharp for the functions defined by (3.16).

Putting $\xi = 1$ in Theorem 3.2, we have:

Corollary 3.6. If $F_w(z) \in T_{p,q}(N, r, \mu_w, 1)$ ($w = 1, 2, \dots, s$) and $g(z)$ is defined by (3.19), then $(z) \in T_{p,q}(N, r, \theta_4)$, where

$$\theta_4 = [p]_q - \frac{s([p]_q - \mu_0)^2}{[p + r]_q \left(\frac{[p + r]_q}{[p]_q}\right)^N [1 + (p + r + \varphi - \gamma - 1)\lambda]^n} \quad (3.26)$$

where $(\mu_0 = \min \{\mu_1, \mu_2, \dots, \alpha_s\})$.

The result is sharp for the functions defined by (3.17)

Remark 3.7.

Putting $n = N = 0, q \rightarrow 1$ in the above results, we get the result of ([4]).

Putting $n = 0$ in the above results, we get the result of ([18])

4. Conclusion

Using the concept of quantum (or q-) calculus, a very vital area of study in the field of traditional mathematical analysis, we have introduced a family of analytic functions that are close-to-convex in the open unit disc U by means of p-q - Opoola differential operator. For the class defined, we discussed properties such coefficient estimate theorem and several modified-Hadamard products for functions belonging to the class. Our main results are obtained in Theorem 2.1, Theorem 3.1 and Theorem 3.4. Furthermore, by specializing in the parameters, several consequences of these new families are mentioned.

Authors’ Contributions

All the authors contributed significantly to writing this article. The authors read and approved the final manuscript.

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