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Short Communication

On a Mixed Arithmetic-Mean, Geometric-Mean, Harmonic-Mean Inequality

Kyumin Nam

Independent Researcher, Incheon, South Korea.

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Abstract *- In 1992, F. Holland conjectured a mixed arithmetic-mean, geometric-mean inequality, and it was proved by K. Kedlaya in 1994. In this short communication, we provide more extended inequality: a mixed arithmetic-mean, geometric-mean, and harmonic-mean inequality.*

Keywords *- Arithmetic mean, Geometric mean, Harmonic mean, Mixed mean, Mixed mean inequality.*

1. Introduction

In 1992, F. Holland [1] conjectured the following mixed arithmetic-mean, geometric-mean inequality:

$$\left(x_1 \cdot \frac{x_1+x_2}{2} \cdots \frac{x_1+x_2+\cdots x_n}{n}\right)^{1/n} \geq \frac{1}{n}\left(x_1 + \sqrt{x_1x_2} + \cdots + \sqrt[n]{x_1x_2\cdots x_n}\right) \quad (1)$$

Where $x_1, x_2, \ldots, x_n$ are positive real numbers with equality if and only if $x_1 = x_2 = \ldots = x_n$. And K. Kedlaya [2] proved it in 1994. We can extend this inequality, and the purpose of this paper is to address it.

2. Main Results

Before introducing the proof in [2], first, we'll state the following lemma from [2] without proof.

Lemma 1. The vectors $\mathbf{a}(i, j) = (a_1(i, j), a_2(i, j), \ldots, a_n(i, j))$ given by

$$a_k(i,j) = \frac{C(n-i,j-k)C(i-1,k-1)}{C(n-1,j-1)} = \frac{(n-i)!(n-j)!(i-1)!(j-1)!}{(n-1)!(k-1)!(n-i-j+k)!(i-k)!(j-k)!} \quad (2)$$

$(i, j = 1, 2, \ldots, n)$ satisfy

i. $a_k(i, j) \geq 0$ for all i, j, k,

ii. $a_k(i, j) = 0$ for $k > \min(i, j)$,

iii. $a_k(i, j) = a_k(j, i)$ for all i, j, k,

iv. $a_1(i, j) + a_2(i, j) + \ldots + a_n(i, j) = 1$ for all i, j,

v. $a_k(1, j) + a_k(2, j) + \ldots + a_k(n, j) = n / j$ for $k \leq j$, $a_k(1, j) + a_k(2, j) + \ldots + a_k(n, j) = 0$ for $k > j$.

And the proof of (1) in [2] followed as:

Proposition 2. Let $x_1, x_2, \ldots, x_n$ be positive real numbers. Then,

$$\left(\prod_{j=1}^{n} \frac{x_1+x_2+\cdots+x_j}{j}\right)^{1/n} \geq \frac{1}{n}\sum_{i=1}^{n} \sqrt[n]{x_1x_2\cdots x_i}. \quad (3)$$

There is equality if and only if $x_1 = x_2 = \ldots = x_n$.

Proof. Let us define the weighted arithmetic mean and geometric mean of tuple x as A(x, a) = $a_1x_1 + a_2x_2 + ... + a_nx_n$ and G(x, a) = $x_1^{a1}x_2^{a2}...x_n^{an}$ where a = (a_1, a_2, …, a_n) is an n-tuple of nonnegative real numbers such that $a_1 + a_2 + ... + a_n = 1$. By the AM-GM inequality [6, 7, 8, 9, 10, 11, 12, 13, 14, 15], A(x, a) ≥ G(x, a) with equality if and only if x_k is constant over all k for which $a_k > 0$. Let A(i, j) and G(i, j) be the means obtained by setting a = a(i, j) in A(x, a) and G(x, a). Using Lemma 1,

$$\frac{x_1+x_2+\cdots+x_j}{j} = \frac{1}{n}\sum_{k=1}^{n} x_k \sum_{i=1}^{n} a_k(i,j) = \frac{1}{n}\sum_{i=1}^{n} A(i,j) \geq \frac{1}{n}\sum_{i=1}^{n} G(i,j) \quad (4)$$

Taking the geometric mean of both sides over j, we get

$$\left(\prod_{j=1}^{n} \frac{x_1+x_2+\cdots+x_j}{j}\right)^{1/n} \geq \frac{1}{n}\prod_{j=1}^{n}\left(\sum_{i=1}^{n} G(i,j)\right)^{1/n} \quad (5)$$

By Hölder's inequality [3],

$$\frac{1}{n}\prod_{j=1}^{n}\left(\sum_{i=1}^{n} G(i,j)\right)^{1/n} \geq \frac{1}{n}\sum_{i=1}^{n}\prod_{j=1}^{n} G(i,j)^{1/n} \quad (6)$$

Equality holds only if every two g_1, g_2, …, g_n are proportional where g_i = (G(i, 1), G(i, 2), …, G(i, n)) (i = 1, 2, …, n). Since G(i, 1) = x_1 and G(i, n) = x_i for all i, this would imply that $g_1 = g_2 = … = g_n$ and that would imply $x_1 = x_2 = ... = x_n$. Also, by Lemma 1,

$$\prod_{j=1}^{n} G(i,j)^{1/n} = \prod_{k=1}^{n}\prod_{j=1}^{n} x_k^{a_k(i,j)/n} = \prod_{k=1}^{i} x_k^{1/i} = \sqrt[i]{x_1 x_2 \cdots x_i} \quad (7)$$

Combining (5), (6), and (7) completes the proof.

Also, in [4], the authors proved the mixed arithmetic-mean, harmonic-mean inequality for matrices. Here we provide proof for scalars.

Proposition 3. Let x_1, x_2, …, x_n be positive real numbers. Then,

$$\left[\frac{1}{n}\sum_{j=1}^{n}\left(\frac{x_1+x_2+\cdots+x_j}{j}\right)^{-1}\right]^{-1} \geq \frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{i}\sum_{k=1}^{i} x_k^{-1}\right)^{-1} \quad (8)$$

There is equality if and only if $x_1 = x_2 = ... = x_n$.

Proof. Let us define the weighted arithmetic mean and the geometric mean of tuple x as A(x, a) = $a_1x_1 + a_2x_2 + ... + a_nx_n$ and H(x, a) = $(a_1/x_1 + a_2/x_2 + ... + a_n/x_n)^{-1}$ where a = (a_1, a_2, …, a_n) is an n-tuple of nonnegative real numbers such that $a_1 + a_2 + ... + a_n = 1$. By the AM-HM inequality [6, 7, 8, 9, 10, 11, 12, 13, 14, 15], A(x, a) ≥ H(x, a) with equality if and only if x_k is constant over all k for which $a_k > 0$. Let A(i, j) and H(i, j) be the means obtained by setting a = a(i, j) in A(x, a) and G(x, a). Using Lemma 1,

$$\frac{x_1+x_2+\cdots+x_j}{j} = \frac{1}{n}\sum_{k=1}^{n} x_k \sum_{i=1}^{n} a_k(i,j) = \frac{1}{n}\sum_{i=1}^{n} A(i,j) \geq \frac{1}{n}\sum_{i=1}^{n} H(i,j) \quad (9)$$

Taking the harmonic mean of both sides over j, we get

$$\left[\frac{1}{n}\sum_{j=1}^{n}\left(\frac{x_1+x_2+\cdots+x_j}{j}\right)^{-1}\right]^{-1} \geq \left[\frac{1}{n}\sum_{j=1}^{n}\left(\frac{1}{n}\sum_{i=1}^{n} H(i,j)\right)^{-1}\right]^{-1} \quad (10)$$

From [3, 5],

$$\left[\frac{1}{n}\sum_{j=1}^{n}\left(\frac{1}{n}\sum_{i=1}^{n} H(i,j)\right)^{-1}\right]^{-1} \geq \frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{n}\sum_{j=1}^{n} H(i,j)^{-1}\right)^{-1} \quad (11)$$

Equality holds only if $h_1, h_2, \ldots, h_n$ are proportional where $h_i = (H(i, 1), H(i, 2), \ldots, H(i, n))$ $(i = 1, 2, \ldots, n)$. Since $H(i, 1) = x_1$ and $H(i, n) = x_i$ for all i, this would imply that $h_1 = h_2 = \ldots = h_n$ and that would imply $x_1 = x_2 = \ldots = x_n$. Also, by Lemma 1,

$$\frac{1}{n}\sum_{j=1}^{n} H(i,j)^{-1} = \frac{1}{n}\sum_{k=1}^{n}\sum_{j=1}^{n} a_k(i,j)x_k^{-1} = \frac{1}{n}\sum_{k=1}^{n} x_k{}^{-1}\sum_{j=1}^{n} a_k(i,j) = \frac{1}{i}\sum_{k=1}^{i} x_k{}^{-1} \quad (12)$$

Combining (10), (11), and (12) completes the proof.

Here, we can extend these inequalities to a mixed arithmetic-mean, geometric-mean, and harmonic-mean inequality. First, we define mixed arithmetic-mean, geometric-mean, and harmonic-mean of the first, second, and third kinds as follows:

$$\mathrm{AGH}_1(x_1, x_2, \cdots, x_n) = \left[\frac{1}{n}\sum_{i=1}^{n}\left(\prod_{j=1}^{i}\frac{x_1+x_2+\cdots+x_j}{j}\right)^{-1/i}\right]^{-1} \quad (13)$$

$$\mathrm{AGH}_2(x_1, x_2, \cdots, x_n) = \left[\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{n}\sum_{j=1}^{i}\sqrt[i]{x_1x_2\cdots x_j}\right)^{-1}\right]^{-1} \quad (14)$$

$$\mathrm{AGH}_3(x_1, x_2, \cdots, x_n) = \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{i}\sum_{j=1}^{i}\left(\sqrt[j]{x_1x_2\cdots x_j}\right)^{-1}\right]^{-1} \quad (15)$$

where $x_1, x_2, \ldots, x_n$ are positive real numbers. Then we obtain the following theorem:

Theorem 4. Let $x_1, x_2, \ldots, x_n$ be positive real numbers. Then,

$$\mathrm{AGH}_1(x_1, x_2, \cdots, x_n) \geq \mathrm{AGH}_2(x_1, x_2, \cdots, x_n) \geq \mathrm{AGH}_3(x_1, x_2, \cdots, x_n) \quad (16)$$

There is equality if and only if $x_1 = x_2 = \ldots = x_n$.

Proof. By Proposition 2,

$$\left[\frac{1}{n}\sum_{i=1}^{n}\left(\prod_{j=1}^{i}\frac{x_1+x_2+\cdots+x_j}{j}\right)^{-1/i}\right]^{-1} \geq \left[\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{n}\sum_{j=1}^{i}\sqrt[i]{x_1x_2\cdots x_j}\right)^{-1}\right]^{-1} \quad (17)$$

Equality holds if and only if $x_1 = x_2 = \ldots = x_n$. And by Proposition 3,

$$\left[\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{n}\sum_{j=1}^{i}\sqrt[i]{x_1x_2\cdots x_j}\right)^{-1}\right]^{-1} \geq \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{i}\sum_{j=1}^{i}\left(\sqrt[j]{x_1x_2\cdots x_j}\right)^{-1}\right]^{-1} \quad (18)$$

Equality holds if and only $x_1 = (x_1x_2)^{1/2} = \ldots = (x_1x_2\ldots x_n)^{1/n}$, i.e., $x_1 = x_2 = \ldots = x_n$. This completes the proof.

3. Conclusion

There are many versions of mixed mean and its inequalities. We just proved a mixed arithmetic-mean, harmonic-mean inequality and a mixed arithmetic-mean, geometric-mean, harmonic-mean inequality, which is just one of the mixed mean inequalities. Undoubtedly, more mixed mean inequality will be studied and discovered.

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