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Original Article

Matrix Associate with Holomorphic Functions Taking Values in Clifford Algebras

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Abstract *- In this paper, we introduce the matrix classes $\widetilde{D}\,(1,2,\dots,m;N)$ and it associates with functions taking values in Clifford algebras. This is an extension of the holomorphic function class from the complex plane to higher dimensional spaces. Some results and examples are presented with low-dimensional spaces.*

Keywords *- Holomorphic function, Clifford analysis, Reguler function, Cachy-Riemann system*.

1. Introduction

So far, as we know, the theory of a holomorphic function has not only reached its fullness and beauty in terms of structure but also enriched many applications in different fields.

In the theory of partial differential equations sense, the theory of a holomorphic function is essentially the theory of the solution of the following Cauchy-Riemann system.

$$\begin{cases} \dfrac{\partial u}{\partial x} - \dfrac{\partial v}{\partial y} = 0 \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} = 0. \end{cases}$$

The real part and imaginary part of the holomorphic function f(z) = u + iv are harmonic functions. But not with any two harmonic functions u and v, then u + iv is a holomorphic function: They must be pairs of harmonic functions associated together by a specified rule (conjugate rule). Here, the conjugate rule is the Cauchy-Riemann condition.

The ideas of complex analysis started in the middle of the 18th century, first of all in connection with the Swiss mathematician Leonhard Euler, and its main results in the 19th century have introduced by AugustinLouis Cauchy, Georg Friedrich Bernhard Riemann and Karl Theodor Wilhelm Weierstrass.

As more and more new problems emerge from the realities that need to be solved, more research has been done to expand the Cauchy-Riemann system (which is also an extension of the theory of a holomorphic function). Looking back at these expansions, one can see that; the authors find several ways to link the harmonic functions together.

As we know, to define a holomorphic function in a complex variable, there must be two harmonic functions that are adjoined together by the Cauchy- Riemann condition. With the addition of the number of equations, functions and variables, there are many new difficulties that appear; one suggested an alternative extension: to construct the theory of hyper-complex numbers and hyper-complex functions. Started by Moisil (see [11-14]) in 1931, this theory has been growing steadily and has many important applications using the results of Moisil, Theodorescu ([6-8]), Nef ([18]), Sobrero ([19]), Fueter ([20,21]), Iftimie ([1,2]), Delanghe ([3]), Goldschmidt ([22-24]), Gilbert ([25,26]), Colton ([27,28]), Sommen ([29]), Tutschke ([30,31]), etc.

It is necessary to build a class of matrices to represent and calculate holomorphic functions in space with high dimensions. In this paper, we introduce the matrix classes. $\widetilde{D}(1,2,\dots,m;N)$ and it associates with functions taking values in Clifford algebras.

2. Matrix $\widetilde{D}(1,2,\ldots,m;N)$ Classes

A holomorphic function $f(z)=u+iv$ an equivalent to a vector which has 2 components in $\mathbb{R}^2$. Its components satisfies linear first-order homogeneous partial equation (Cauchy-Riemann system)

$$\begin{cases}\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}=0\\ \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}=0\end{cases} \tag{2.1}$$

In ([21]), A.V. Bisatze had prescribed "Cauchy-Riemann type in 3-dimensional Euclidean space" by considering the following system

$$\begin{cases} \frac{\partial q_2}{\partial x_1}+\frac{\partial q_3}{\partial x_2}+\frac{\partial q_4}{\partial x_3}=0\\ \frac{\partial q_1}{\partial x_1}-\frac{\partial q_3}{\partial x_3}+\frac{\partial q_4}{\partial x_2}=0\\ \frac{\partial q_1}{\partial x_2}+\frac{\partial q_2}{\partial x_3}-\frac{\partial q_4}{\partial x_1}=0\\ \frac{\partial q_1}{\partial x_3}-\frac{\partial q_2}{\partial x_2}+\frac{\partial q_3}{\partial x_1}=0\end{cases} \tag{2.2}$$

(Moisil-Theodorescu system). Since the properties of the solution in the Cauchy-Riemann system is also true with the Moisil-Theodorescu system, then the vector $q=(q_1,q_2,q_3,q_4)$ Sao;otisfying Moisil-Theodorescu system can be called a "holomorphic vector" in $\mathbb{R}^3$.

In 1964, V-S.Vinogradov had represented the "Cauchy-Riemann type in 4-dimensional space" by investigating the following system.

$$\begin{cases}\frac{\partial u_1}{\partial x_1}-\frac{\partial u_2}{\partial x_2}-\frac{\partial u_3}{\partial x_3}-\frac{\partial u_4}{\partial x_4}=0\\ \frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}-\frac{\partial u_3}{\partial x_4}+\frac{\partial u_4}{\partial x_3}=0\\ \frac{\partial u_1}{\partial x_3}+\frac{\partial u_2}{\partial x_4}+\frac{\partial u_3}{\partial x_1}-\frac{\partial u_4}{\partial x_2}=0\\ \frac{\partial u_1}{\partial x_4}-\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}+\frac{\partial u_4}{\partial x_1}=0.\end{cases} \tag{2.3}$$

For each solution of (2.3) has 4 components (u_1,u_2,u_3,u_4), it has the same properties as the solution of the Cauchy-Riemann system, so it to be called a "holo-morphic vector" in $\mathbb{R}^4$.

There are many results that have expanded the Cauchy-Riemann system in different. In order to unify many different ways of extending that Cauchy-Riemann system into a common direction, consistently presenting the same method, and further can be expanded and generalized, we have a basic comment. as follows: each of these systems is associated with a square matrix of matrix $\widetilde{D}$ $(1,2,\ldots,m;N)$ which we will be defined as follows (see definition 1.1). For instance, the Moisil-Theodorescu system can be prescribed by following the matrix

$$\begin{pmatrix}0&1&2&3\\1&0&-3&2\\2&3&0&-1\\3&-2&1&0\end{pmatrix}. \tag{2.4}$$

The number 3 at the position which has 3-line and 2-column prescribes that, in the 3th-equations, the derivative of the 2th-component respect to x_3 which has a coefficient equal to +1, the number "-1" at the position which has 3-line and 4-column prescribes that, in the 3th-equations, derivative of the 4th-component respect to x_1 which has a coefficient equal to -1. (For equations (1.1), (1.2), (1.4) and the system can be investigated in [1], [2], [12],..., all of the coeffcients of the derivatives equal to 1 or -1).

By that denoting, the Cauchy-Riemann system can be presented by following the matrix

$$\begin{pmatrix}1&-2\\2&1\end{pmatrix} \tag{2.5}$$

and the system (2.3) can be represented by

$$\begin{pmatrix} 1 & -2 & -3 & -4 \\ 2 & 1 & -4 & 3 \\ 3 & 4 & 1 & -2 \\ 4 & -3 & 2 & 1 \end{pmatrix}. \tag{2.6}$$

The matrixes presented by (2.4), (2.5), (2.6), or the matrix which is considered in [1], [2], [12], ... belongs to $\widetilde{D}\,(1,2,\dots,m;N)$, which is defined in the following.

Definition 1.1. *Let A be a N-order square matrix ($N \geq 2$). The matrix A is called belong to $\widetilde{D}\,(1,2,\dots,m;N)$, if it has the following 2 properties:*

Property 1. *For any element of A which has only values: $0, \pm 1, \pm 2, \dots, \pm m$, ($2 \leq m \leq N$). If we do not care about the sign of these numbers, each row (and each column) contains all the integers 1, 2, ..., m, and each number only appears once (if $m = N$, there is no zero in A).*

Property 2. *Considering for any two columns (rows) of A, we have: if in a particular row (or column), there are two elements are i, k ($i, k \neq 0$), there exists only one another row (or column) which has two elements: $-k$ and i (or $k, = i$).*

It is easy to see that the matrixes (2.4), (2.5), (2.6) belong to $\widetilde{D}\,(1,2,3;4)$, $\widetilde{D}\,(1,2;2)$, $\widetilde{D}\,(1,2,3,4;4)$. In the following, we can prove that, for a given integer number $m \geq 2$, there exists a uniquely minimum $N_0 \geq m$ such that, $\widetilde{D}\,(1,2,\dots,m;N_0)$ is non-empty.

In the case $m = 2$, then $N_0 = 2$, and $m = 3,4$, then $N_0 = 4$, and in cases $m = 5,6,7,8$, we can close proof that $N_0 = 8$. There are 4 matrixes that depend on $\widetilde{D}\,(1,2,\dots,m;8)$, which applied for $m = 5,6,7,8$ resp.

$$\begin{pmatrix} 1 & -2 & -3 & -4 & -5 & 0 & 0 & 0 \\ 2 & 1 & 4 & -3 & 0 & -5 & 0 & 0 \\ 3 & -4 & 1 & 2 & 0 & 0 & -5 & 0 \\ 4 & 3 & -2 & 1 & 0 & 0 & 0 & -5 \\ 5 & 0 & 0 & 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 5 & -4 & -3 & 2 & 1 \\ 0 & 5 & 0 & 0 & -2 & 1 & -4 & 3 \\ 0 & 0 & 5 & 0 & -3 & 4 & 1 & -2 \end{pmatrix} \tag{2.7}$$

$$\begin{pmatrix} 1 & -2 & -3 & -4 & -5 & -6 & 0 & 0 \\ 2 & 1 & 0 & 5 & -4 & 0 & -6 & -3 \\ 3 & 0 & 1 & 0 & 6 & -5 & -4 & 2 \\ 4 & -5 & 0 & 1 & 2 & 0 & 3 & -6 \\ 5 & 4 & -6 & -2 & 1 & 3 & 0 & 0 \\ 6 & 0 & 5 & 0 & -3 & 1 & 2 & 4 \\ 0 & 6 & 4 & -3 & 0 & -2 & 1 & -5 \\ 0 & 3 & -2 & 6 & 0 & -4 & 5 & 1 \end{pmatrix} \tag{2.8}$$

$$\begin{pmatrix} 1 & -2 & -3 & -4 & -5 & -6 & -7 & 0 \\ 2 & 1 & -6 & 7 & 0 & -3 & -4 & -5 \\ 3 & -6 & -1 & 0 & -7 & 2 & -5 & 4 \\ 4 & -7 & 0 & 1 & 6 & 5 & 2 & -3 \\ 5 & 0 & 7 & 6 & -1 & -4 & 3 & 2 \\ 6 & 3 & 2 & -5 & -4 & 1 & 0 & -7 \\ 7 & 4 & -5 & -2 & 3 & 0 & 1 & 6 \\ 0 & 5 & 4 & 3 & 2 & 7 & -6 & 1 \end{pmatrix} \tag{2.9}$$

$$\begin{pmatrix} 1 & -2 & 3 & -4 & 5 & -6 & 7 & 8 \\ 2 & 1 & -4 & 3 & 6 & 5 & 8 & -7 \\ 3 & 4 & 1 & -2 & 7 & 8 & -5 & 6 \\ 4 & -3 & 2 & 1 & -8 & 7 & 6 & 5 \\ -5 & -6 & -7 & 8 & 1 & 2 & -3 & 4 \\ -6 & 5 & 8 & 7 & 2 & -1 & 4 & 3 \\ -7 & -8 & 5 & -6 & 3 & 4 & 1 & -2 \\ -8 & 7 & -6 & -5 & -4 & 3 & 2 & 1 \end{pmatrix} \tag{2.10}$$

3. Matrix Associated with Holomorphic Functions Taking Values in Clifford Algebras

In this section, we will show that the following equations

$$\left(\sum_{i=0}^{n} e_i \frac{\partial}{\partial x_i}\right)\left(\sum_A e_A f_A\right) = 0 \tag{3.1}$$

or

$$\left(\sum_{i=0}^{n} \alpha_i e_A \frac{\partial}{\partial x_i}\right)\left(\sum_A e_A f_A\right) = 0 \tag{3.2}$$

also associated with classes matrix $\widetilde{D}\,(1,2,\dots,m;N)$.

3.1. First Case

We rewrite (3.1) as follows.

$$\sum\nolimits_{i,A} e_i e_A \frac{\partial f}{\partial x_i} = 0$$

$$\Leftrightarrow \sum_s g_s e_s = 0.$$

For each e_s-fixed, then g_s is the sum of the term $\left(\pm \frac{\partial f}{\partial x_i}\right)$, we have

$$e_i e_A = e_s \quad \text{(a)}, \quad (\text{or } -e_s)$$

$$\text{(a)} \;\Rightarrow -e_i(e_i e_A) = -e_i e_s$$

$$\Leftrightarrow e_A^{(i)} = -e_i e_s \quad \text{(b)}, \; i = 0,1,2,\dots,n.$$

From (b) deduced, in each equation $g_s = 0$, we have $(n+1)$ derivatives $\frac{\partial f_A^{(i)}}{\partial x_i}$ are joined with coefficient ± 1.

One hand, for each f_A-fixed then $e_i e_A$ has $(n+1)$-distinguishing element vectors e_s. Thus $\frac{\partial f_A}{\partial x_i}$ will be appeared in $(n+1)$ equations. Therefore, the matrix associated with the system (3.1) has Property 1 of the $\widetilde{D}\,(1,2,\dots,m;N)$ classes.

On the other hand, suppose in g_s has two terms

$$e_i e_A = e_j e_B \quad \text{(a)} \quad (= e_s).$$

Therefore

$$-e_i e_i e_A = -e_i e_j e_B$$

$$\Leftrightarrow e_A = -e_i e_j e_B.$$

Multiply from the left-hand side above equation with $-e_j$, we have

$$-e_j e_A = e_j e_i e_j e_B$$

$$\Leftrightarrow -e_j e_A = e_i e_B \quad \text{(b)} \quad (= e_s').$$

From (a) and (b) deduces: when we consider two fixed-columns f_A and f_B respectively, if one certain row contains i and j, then in other row will contain $(-j)$ and i, that means which has Property 2 of the $\widetilde{D}\,(1,2,\dots,m;N)$ classes.

To sum up, the system (3.1) associated with the $\widetilde{D}\,(1,2,\dots,m;N)$ classes.

Examples 3.1. *In the Quaternion algebra, $dimC = 4$, we have*

$$\left(e_0 \frac{\partial}{\partial x_0} + e_1 \frac{\partial}{\partial x_1} + e_2 \frac{\partial}{\partial x_2}\right)(e_0 u_0 + e_1 u_1 + e_2 u_2 + e_1 e_2 u_3) = 0$$

$$\Leftrightarrow e_0\left(\frac{\partial u_0}{\partial x_0} - \frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2}\right) + e_1\left(\frac{\partial u_0}{\partial x_1} + \frac{\partial u_1}{\partial x_0} + \frac{\partial u_3}{\partial x_2}\right)$$

$$+e_2\left(\frac{\partial u_0}{\partial x_2}+\frac{\partial u_2}{\partial x_0}-\frac{\partial u_3}{\partial x_1}\right)+e_1e_2\left(-\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}+\frac{\partial u_3}{\partial x_0}\right)=0$$

$$\Leftrightarrow\begin{cases}\frac{\partial u_0}{\partial x_0}-\frac{\partial u_1}{\partial x_1}-\frac{\partial u_2}{\partial x_2}=0\\ \frac{\partial u_0}{\partial x_1}+\frac{\partial u_1}{\partial x_0}+\frac{\partial u_3}{\partial x_2}=0\\ \frac{\partial u_0}{\partial x_2}+\frac{\partial u_2}{\partial x_0}-\frac{\partial u_3}{\partial x_1}=0\\ -\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}+\frac{\partial u_3}{\partial x_0}=0.\end{cases}$$

This system is associated with the following matrix

$$\begin{pmatrix}1 & -2 & -3 & 0\\ 2 & 1 & 0 & 3\\ 3 & 0 & 1 & -2\\ 0 & -3 & 2 & 1\end{pmatrix}.$$

Examples 3.2. *In case* $dimC=8$. *Acting operator*

$$D=e_0\frac{\partial}{\partial x_0}+e_1\frac{\partial}{\partial x_1}+e_2\frac{\partial}{\partial x_2}+e_3\frac{\partial}{\partial x_3}$$

on function

$$u=e_0u_0+e_1u_1+e_2u_2+e_1e_2u_4+e_1e_3u_5+e_2e_3u_6+e_1e_2e_3u_7$$

then the equation $D_u=0$ *has the following form*

$$e_0\left(\frac{\partial u_0}{\partial x_0}-\frac{\partial u_1}{\partial x_1}-\frac{\partial u_2}{\partial x_2}-\frac{\partial u_3}{\partial x_3}\right)+e_1\left(\frac{\partial u_0}{\partial x_1}+\frac{\partial u_1}{\partial x_0}+\frac{\partial u_4}{\partial x_2}+\frac{\partial u_5}{\partial x_3}\right)+$$
$$+e_2\left(\frac{\partial u_0}{\partial x_2}+\frac{\partial u_2}{\partial x_0}-\frac{\partial u_4}{\partial x_1}+\frac{\partial u_6}{\partial x_3}\right)+e_3\left(\frac{\partial u_0}{\partial x_3}+\frac{\partial u_3}{\partial x_0}-\frac{\partial u_5}{\partial x_1}-\frac{\partial u_6}{\partial x_2}\right)+$$
$$+e_1e_2\left(-\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}+\frac{\partial u_4}{\partial x_0}-\frac{\partial u_7}{\partial x_3}\right)+e_1e_3\left(-\frac{\partial u_1}{\partial x_3}+\frac{\partial u_3}{\partial x_1}+\frac{\partial u_5}{\partial x_0}+\frac{\partial u_7}{\partial x_2}\right)+$$
$$+e_2e_3\left(-\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}+\frac{\partial u_6}{\partial x_0}-\frac{\partial u_7}{\partial x_1}\right)+e_1e_2e_3\left(\frac{\partial u_4}{\partial x_3}-\frac{\partial u_5}{\partial x_2}+\frac{\partial u_6}{\partial x_1}+\frac{\partial u_7}{\partial x_0}\right)=0.$$

Therefore, we obtain a matrix belonging to $\widetilde{D}\,(1,2,3,4;8)$ *classes as follows*

$$\begin{pmatrix}1 & -2 & -3 & -4 & 0 & 0 & 0 & 0\\ 2 & 1 & 0 & 0 & 3 & 4 & 0 & 0\\ 3 & 0 & 1 & 0 & -2 & 0 & 4 & 0\\ 4 & 0 & 0 & 1 & 0 & -2 & -3 & 0\\ 0 & -3 & 2 & 0 & 1 & 0 & 0 & -4\\ 0 & -4 & 0 & 2 & 0 & 1 & 0 & 3\\ 0 & 0 & -4 & 3 & 0 & 0 & 1 & -2\\ 0 & 0 & 0 & 0 & 4 & -3 & 2 & 1\end{pmatrix}.$$

Examples 3.3. *In case* $dimC=16$, *we are acting operator*

$$D=e_0\frac{\partial}{\partial x_0}+e_1\frac{\partial}{\partial x_1}+e_2\frac{\partial}{\partial x_2}+e_3\frac{\partial}{\partial x_3}+e_4\frac{\partial}{\partial x_4}$$

on the following function

$$u=e_0u_0+e_1u_1+e_2u_2+e_3u_3+e_4u_4+e_1e_2u_5+e_1e_3u_6+e_1e_4u_7+e_2e_3u_8+e_2e_4u_9+e_3e_4u_{10}+e_1e_2e_3u_{11}$$
$$+e_1e_2e_4u_{12}+e_1e_3e_4u_{13}+e_2e_3e_4u_{14}+e_1e_2e_3e_4u_{15}$$

then we obtain a matrix belonging to $\widetilde{D}\ (1,2,3,4,5;16)$ *as follows*

$$\begin{array}{cccccccccccccccc}
1 & -2 & -3 & -4 & -5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 1 & 0 & 0 & 0 & 3 & 4 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 1 & 0 & 0 & -2 & 0 & 0 & 4 & 5 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 1 & 0 & 0 & -2 & 0 & -3 & 0 & 5 & 0 & 0 & 0 & 0 & 0 \\
5 & 0 & 0 & 0 & 1 & 0 & 0 & -2 & 0 & -3 & -4 & 0 & 0 & 0 & 0 & 0 \\
0 & -3 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -4 & -5 & 0 & 0 & 0 \\
0 & -4 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 3 & 0 & -5 & 0 & 0 \\
0 & -5 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 3 & 4 & -5 & 0 \\
0 & 0 & -4 & 3 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 & 0 & 0 & 4 & 0 \\
0 & 0 & -5 & 0 & 3 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 & 0 & -3 & 0 \\
0 & 0 & 0 & -5 & 4 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 4 & -3 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & 5 \\
0 & 0 & 0 & 0 & 0 & 5 & 0 & -3 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & -4 \\
0 & 0 & 0 & 0 & 0 & 0 & 5 & -4 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -4 & 3 & 0 & 0 & 0 & 1 & -2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5 & 4 & -3 & 2 & 1
\end{array}$$

3.2. Second Case

We rewrite (3.2) as follows

$$\sum_{i=1}^{m}\sum_{A}\alpha_i e_{A_i} e_A \frac{\partial f_A}{\partial x_i} = 0 \Leftrightarrow \sum_{s} g_s e_s = 0.$$

For each e_s-fixed, then g_s is the sum of the term $\left(\pm\frac{\partial f_A}{\partial x_i}\right)$, we have

$$\alpha_i e_{A_i} e_A = e_s \quad \text{(a)} \quad (\text{or } -e_s\ )$$

$$\text{(a)} \ \Rightarrow\ (\alpha_i \overline{e_{A_i}})\alpha_i e_{A_i} e_A = (\alpha_i \overline{e_{A_i}}) e_s$$

$$\Leftrightarrow e_A = \left(\alpha_i \overline{e_{A_i}}\right) e_s \quad \text{(b).}$$

In (b), we consider for each $i = 1,2,\dots,m$, we obtain m vectors $e_A^{(i)}$, therefore we have m derivatives $\frac{\partial f_A^{(i)}}{\partial x_i}$, thus, each row will has full of integers: $1,2,\dots,m$.

On the other hand, for each A-fixed then $\alpha_i e_{A_i} e_A$, $(i = 1,2,\dots,m)$ has m −distinguishing vectors, which means each column of the matrix will have full of integers: $1,2,\dots,m$.

Thus, the matrix associated with the system (3.2) has Property 1 of the $\widetilde{D}\ (1,2,\dots,m;N)$ classes.

On the other hand, suppose in g_s has two terms with the same values (equal to e_s)

$$\alpha_i e_{A_i} e_A = \alpha_j e_{A_i} e_B \quad \text{(c)}$$

We have to prove that, from (c), we obtain

$$\alpha_j e_{A_i} e_A = -\,\alpha_i e_{A_i} e_B. \quad \text{(d)}$$

Multiply from the left-hand side of equation (c) with $\overline{e_{A_i}}$, we have

$$\alpha_i\left(\overline{e_{A_i}} e_{A_i}\right) e_A = \alpha_j \overline{e_{A_i}} e_{A_i} e_B$$

After that, multiply the above equation with e_{A_j}, then

$$\alpha_i e_{A_j} e_A = \ \alpha_j\left(e_{A_j}\overline{e_{A_i}}\right) e_{A_j} e_B$$

$$\Leftrightarrow \alpha_i e_{A_j} e_A = \ \alpha_j\left(-e_{A_i}\overline{e_{A_j}}\right) e_{A_j} e_B$$

$$\Leftrightarrow \alpha_i e_{A_j} e_A = \ -\,\alpha_j e_{A_i} e_B$$

$$\Leftrightarrow (\alpha_i\alpha_j)\alpha_i e_{A_j} e_A = (\alpha_i\alpha_j)(-\alpha_j e_{A_i} e_B)$$

$$\Leftrightarrow \alpha_j e_{A_j} e_A = -\alpha_i e_{A_i} e_B \quad \text{(d).}$$

Thus, the matrix associated with the system (2.59) has Property 2 of the $\widetilde{D}\,(1,2,\dots,m;N)$ classes.

Examples 3.4. In Quaternion algebra, with 2^2 -dimensional, we have $2 = 4.0 + 2$. Thus, we have "sub-space", which has basis elements $\{e_0, e_1, e_2, e_1 e_2\}$ (i.e. whole Quaternion algebra) is inversible. There we can introduce a generalized operator.

$$T = \sum_{i=0}^{3} \alpha_i\, e_i \frac{\partial}{\partial x_i}$$

where $e_3 = e_1 e_2, \alpha_i = \pm 1$. And it is associated with a matrix belong to $\widetilde{D}\,(1,2,3,4;4)$.

a) Let

$$T = e_0 \frac{\partial}{\partial x_0} + e_1 \frac{\partial}{\partial x_1} + e_2 \frac{\partial}{\partial x_2} + e_3 \frac{\partial}{\partial x_3} \;,$$

$$u = e_0 u_0 + e_1 u_1 + e_2 u_2 + e_3 u_3.$$

Then $Tu = 0$ leads to the following equation

$$e_0\left(\frac{\partial u_0}{\partial x_0} - \frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2} - \frac{\partial u_3}{\partial x_3}\right) + e_1\left(\frac{\partial u_0}{\partial x_1} + \frac{\partial u_1}{\partial x_0} - \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) +$$

$$+ e_2\left(\frac{\partial u_0}{\partial x_2} + \frac{\partial u_1}{\partial x_3} + \frac{\partial u_2}{\partial x_0} - \frac{\partial u_3}{\partial x_1}\right) + e_3\left(\frac{\partial u_0}{\partial x_3} - \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} + \frac{\partial u_3}{\partial x_0}\right) = 0.$$

This system is associated with the following matrix

$$\begin{pmatrix} 1 & -2 & -3 & -4 \\ 2 & 1 & -4 & 3 \\ 3 & 4 & 1 & -2 \\ 4 & -3 & 2 & 1 \end{pmatrix}.$$

b) Let

$$T = e_0 \frac{\partial}{\partial x_0} + e_1 \frac{\partial}{\partial x_1} - e_2 \frac{\partial}{\partial x_2} + e_3 \frac{\partial}{\partial x_3}.$$

Then $Tu = 0$ leads to the following equation

$$e_0\left(\frac{\partial u_0}{\partial x_0} - \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} - \frac{\partial u_3}{\partial x_3}\right) + e_1\left(\frac{\partial u_0}{\partial x_1} + \frac{\partial u_1}{\partial x_0} - \frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}\right) +$$

$$+ e_2\left(-\frac{\partial u_0}{\partial x_2} + \frac{\partial u_1}{\partial x_3} + \frac{\partial u_2}{\partial x_0} - \frac{\partial u_3}{\partial x_1}\right) + e_3\left(\frac{\partial u_0}{\partial x_3} + \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} + \frac{\partial u_3}{\partial x_0}\right) = 0.$$

This system is associated with the following matrix

$$\begin{pmatrix} 1 & -2 & 3 & -4 \\ 2 & 1 & -4 & -3 \\ -3 & 4 & 1 & -2 \\ 4 & 3 & 2 & 1 \end{pmatrix}.$$

4. Conclusion

The results in the paper are mainly presented for the low-dimensional R spaces. We will try to generalize to the general case with arbitrarily dimensional spaces $\mathbb{R}^n$.

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