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Original Article

Multiplicity Results of Second-Order Singular Nonlinear Differential Equation with a Parameter

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Abstract *- In a periodic differential equation with a singular nonlinearity, we prove the existence and multiplicity of positive periodic solutions of the equation through a basic application of Krasnoselskiĭ's-Guo fixed point theorem and the positivity of the associated Green's function.*

Keywords *- Positive periodic solution, Krasnoselskiĭ's-Guo fixed point, Green's function.*

1. Introduction

In recent years, the periodic problem for the second-order singular nonlinear equation

$$y''(t)+a(t)y(t)=\gamma\frac{h(t)}{y^{\rho}(t)}+\gamma e(t) \tag{1.1}$$

has deserved the attention of many experts in differential equations, where γ is a positive paramaeter, ρ is a positive constant, $e(t)\in L^p(\mathrm{R}/\omega\mathrm{Z},\mathrm{R}^+)$, here $\mathrm{R}^+:=(0,+\infty)$ and ω is a positive constant, $a(t),h(t)\in L^p(\mathrm{R}/\omega\mathrm{Z},\mathrm{R})$ are ω-periodic functions. From a mathematical perspective, differential equations with singularities can be divided into three classes based on the types of singularities involved in the equations of attractive, repulsive and indefinite (attractive or repulsive) types. Each of the above classes has its own properties, which means that their research methods are different. At present, there have been many results in the study of this type of equation.

There are a number of methods that are widely used, starting with the pioneering paper of Lazer and Solimini. In particular, the method of Schauder fixed point theorem [11, 15], Krasnoselskiĭ's-Guo fixed point theorem [8, 15], the theory of upper and lower solutions [10], Leray-Schauder alternative principle [2], Leray-Schauder degree theory [3, 6] and coincidence degree theory [12] are important tools.

Interest in the study of such equations began with Lazer and Solimini [10]. They investigated the existence of positive periodic solutions to equation (1.1) by the theory of upper and lower solutions for $\gamma=1$, $1\le\rho\le+\infty$, $e(t)\in L^p(\mathrm{R}/\omega\mathrm{Z},\mathrm{R})$ and $a(t)\equiv 0$, $b(t)\equiv\pm 1$ (if $b(t)=1$ it is a repulsive singular equation, and $b(t)=-1$ it is an attractive singular equation). After that, the use of methods Krasnoselskiĭ's-Guo fixed point theorem and Schauder fixed point theorem for Torres [14] and Wang [15] respectively proved that equation (1.1) has a periodic positive solution with a singularity of repulsive type, where $\gamma=1$, $a(t),e(t)\in L^p(\mathrm{R}/\omega\mathrm{Z},\mathrm{R}^+)$, $b(t)\in C(\mathrm{R},\mathrm{R}^+)$ is ω-periodic function. Later, by Schauder fixed point theorem, Liu, Cheng and Wang [11] 2020 considered the existence of a positive periodic solution for equation (1.1) with a singularity of attractive type, where $\gamma=1$, $e(t)\in L^p(\mathrm{R}/\omega\mathrm{Z},\mathrm{R})$, $a(t),h(t)\in C(\mathrm{R},\mathrm{R}^-)$, here $\mathrm{R}^-:=(-\infty,0)$. Recently, Han and Cheng [8] 2022 discussed the existence of positive periodic solutions for equation (1.1) with indefinite weights, where $\gamma=1$, $a(t)\in L^p(\mathrm{R}/\omega\mathrm{Z},\mathrm{R})$, $e(t)\in C(\mathrm{R},\mathrm{R})$. Their proofs were based on Krasnoselskiĭ's-Guo fixed point theorem.

Although there have been many research results on equation (1.1) [5, 9], there is still room for improvement. In this paper, applying Krasnoselskiĭ's-Guo fixed point theorem and the positivity of the associated Green's function, we consider the existence and multiplicity of positive periodic solutions to equation (1.1). It is worth mentioning that since $b(t) \in L^p(\mathbb{R}/\omega\mathbb{Z}, \mathbb{R})$ this indicates that $\frac{h(t)}{y^\rho(t)}$ it is an indefinite type $y = 0$. Furthermore, depending on the value of γ taken, we can obtain the existence of one positive periodic solution and two periodic solutions to equation (1.1), respectively.

2. Preparations and Notations

Our proof relies on the following lemmas, which we describe in detail next.

Lemma 2.1. (Krasnoselskiĭ's-Guo fixed point theorem [4, P. 94]) Let Y be a Banach space and K is a cone in Y. Assume that S_1 and S_2 are open subsets of Y with $0 \in S_1$, $\overline{S}_1 \subset S_2$. Let

$$T : K \cap \left(\overline{S}_2 \setminus S_1\right) \to K$$

be a completely continuous operator such that one of the following conditions holds:

(i) $\|Ty\| \ge \|y\|$ for $y \in K \cap \partial S_1$ and $\|Ty\| \le \|y\|$ for $y \in K \cap \partial S_2$.

(ii) $\|Ty\| \le \|y\|$ for $y \in K \cap \partial S_1$ and $\|Ty\| \ge \|y\|$ for $y \in K \cap \partial S_2$.

Then T has a fixed point in the set $K \cap \left(\overline{S}_2 \setminus S_1\right)$.

We will use the concept of Green's functions to write the periodic problem as an equivalent fixed point problem. A general mechanism for constructing Green's functions is described in [1]. The following one lemma is common to us and can be seen in some related literature (see, e.g., [13, Corollary 2.3]).

Lemma 2.2. (see [13, Corollary 2.3]) Define

$$\wp(\alpha) = \begin{cases} \dfrac{2\pi}{\alpha\omega^{1+2/\alpha}} \left(\dfrac{\Gamma\left(\frac{1}{\alpha}\right)}{\Gamma\left(\frac{1}{2} + \frac{1}{\alpha}\right)} \right)^2, & 1 \le \alpha < \infty, \\ \dfrac{4}{\omega}, \alpha = \infty, \end{cases}$$

Where Γ is the Gamma function, i.e., $\Gamma(t) = \int_0^{+\infty} y^{t-1} e^{-y} dy$. Assume that $a(t) \ge 0$ for almost every $t \in [0, \omega]$ and $a(t) \in L^p(\mathbb{R}/\omega\mathbb{Z})$. Furthermore, let

$$\|a\|_p := \left(\int_0^\omega |a(t)|^p dt \right)^{\frac{1}{p}} < \wp(2p^*), \tag{2.1}$$

where $p^* = \frac{p}{p-1}$ if $1 \le p < \infty$ we have the Green's function $G(t,s) > 0$ for $(t,s) \in [0,\omega] \times [0,\omega]$.

Remark 2.1. (see [7]) In the special case $a(t) \equiv \varsigma^2$ with $\varsigma > 0$. Green's function has the following form

$$G(t,s) \begin{cases} \dfrac{\cos\varsigma(t - s - \frac{\omega}{2})}{2\varsigma \sin\frac{\varsigma\omega}{2}}, 0 \le s \le t \le \omega, \\ \dfrac{\cos\varsigma(t - s + \frac{\omega}{2})}{2\varsigma \sin\frac{\varsigma\omega}{2}}, 0 \le t < s \le \omega. \end{cases}$$

If $\varsigma < \frac{\pi}{\omega}$ we have Green's function $G(t,s) > 0$ for $(t,s) \in [0,\omega] \times [0,\omega]$.

Besides, we will use the notations

$$\mathrm{G}_* := \min_{0\le s,t\le\omega} \mathrm{G}(t,s),\ \mathrm{G}^* := \max_{0\le s,t\le\omega} \mathrm{G}(t,s),\ \sigma := \frac{\mathrm{G}_*}{\mathrm{G}^*}. \tag{2.2}$$

On the basis of (2.2), it is clear that $0 < \mathrm{G}_* < \mathrm{G}^*$ and $0 < \sigma \le 1$.

Where $\mathrm{C}_\omega := \{y \in C(\mathrm{R},\mathrm{R}), y(t+\omega) \equiv y(t), \text{for all } t \in \mathrm{R}\}$ with norm $\|y\| := \max_{t\in\mathbb{R}} |y(t)|$. It is easy to verify that K it is a cone in C_ω. Finally, we provide several definitions

$$\mathrm{h}^+(t) := \max\{\mathrm{h}(t),0\},\ \mathrm{h}^-(t) := -\min\{\mathrm{h}(t),0\},\ \bar{\mathrm{h}} := \frac{1}{\omega}\int_0^\omega \mathrm{h}(t)dt,\ \mathrm{e}^* := \max_{t\in[0,\omega]} \mathrm{e}(t),\ \mathrm{e}_* := \min_{t\in[0,\omega]} \mathrm{e}(t).$$

3. Main Result

Based on several lemmas from the previous section, we can obtain the following theorem.

Theorem 3.1. Assume that equation (2.1) holds. Then the following one of conclusions holds.

(i) There exists $\gamma_0 > 0$ such that equation (1.1) has a positive periodic solution for $\gamma > \gamma_0$;

(ii) For all sufficiently small $\gamma > 0$, equation (1.1) has two positive periodic solutions.

Proof. A positive periodic solution of equation (1.1) is just a fixed point of the map T defined by

$$(\mathrm{T}_\gamma y)(t) := \gamma\int_0^\omega \mathrm{G}(t,s)\left(\frac{\mathrm{h}(s)}{y^\rho(s)} + \mathrm{e}(s)\right)ds.$$

According to Lemma 2.2, for all $(t,s) \in [0,\omega]\times[0,\omega]$ we have $\mathrm{G}(t,s) > 0$.

(i)Our proof relies on Lemma 2.1. First, define

$$\mathrm{S}_{R_1} := \{y \in \mathrm{C}_\omega : \|y\| < R_1\} \text{ and } \mathrm{S}_{R_2} := \{y \in \mathrm{C}_\omega : \|y\| < R_2\},$$

where R_1 and R_2 are two positive constants and

$$R_2 > R_1 > \frac{1}{\sigma}\left(\frac{\|\mathrm{h}^-\|}{\mathrm{e}_*}\right)^{\frac{1}{\rho}} := \delta\cdot$$

Step 1. We claim that $\mathrm{T}_\gamma\left(\mathrm{K}\cap\left(\overline{\mathrm{S}}_{R_2}\setminus \mathrm{S}_{R_1}\right)\right) \subset \mathrm{K}$. Actually, for $y \in \mathrm{K}\cap\left(\overline{\mathrm{S}}_{R_2}\setminus \mathrm{S}_{R_1}\right)$ we have

$$\sigma R_1 < y(t) < R_2, \text{ for all } t \in \mathrm{R}.$$

Because $R_1 > \delta$ we obtain

$$\begin{aligned}\frac{\mathrm{h}(t)}{y^\rho(t)} + \mathrm{e}(t) &= \frac{\mathrm{h}^+(t)}{y^\rho(t)} - \frac{\mathrm{h}^-(t)}{y^\rho(t)} + \mathrm{e}(t)\\ &> -\frac{\mathrm{h}^-(t)}{y^\rho(t)} + \mathrm{e}(t)\\ &> -\frac{\|\mathrm{h}^-(t)\|}{y^\rho(t)} + \mathrm{e}(t)\\ &> 0,\end{aligned} \tag{3.1}$$

for all $t \in \mathbb{R}$. From (2.2) and (3.1), it can be seen that

$$\begin{aligned}\min_{t\in[0,\omega]}(T_\gamma y)(t) &:= \min_{t\in[0,\omega]}\gamma\int_0^\omega G(t,s)\left(\frac{h(s)}{y^\rho(s)}+e(s)\right)ds\\ &\geq \gamma G_*\int_0^\omega G(t,s)\left(\frac{h(s)}{y^\rho(s)}+e(s)\right)ds\\ &= \gamma\sigma G^*\int_0^\omega G(t,s)\left(\frac{h(s)}{y^\rho(s)}+e(s)\right)ds\\ &\geq \gamma\sigma\max_{t\in[0,\omega]}\int_0^\omega G(t,s)\left(\frac{h(s)}{y^\rho(s)}+e(s)\right)ds\\ &= \sigma\|T_\gamma y\|,\end{aligned}$$

Which implies $T_\gamma\left(K\cap\left(\overline{S}_{R_2}\setminus S_{R_1}\right)\right)\subset T$. Besides, by applying the Arzela-Ascoli theorem, we can easily prove that $T_\gamma : K\cap\left(\overline{S}_{R_2}\setminus S_{R_1}\right)\to T$ it is a completely continuous operator.

Step 2. We prove that

$$\|T_\gamma y\|\geq\|y\|,\ \text{for}\ y\in K\cap\partial S_{R_1}. \tag{3.2}$$

In fact, for $y\in K\cap\partial S_{R_1}$, it is obvious that $\|y\|=R_1$ and

$$\sigma R_1 < y(t) < R_1,\ \text{for all}\ t\in R.$$

It can be obtained from (3.1) that

$$\begin{aligned}(T_\gamma y)(t) &= \gamma\int_0^\omega G(t,s)\left(\frac{h(s)}{y^\rho(s)}+e(s)\right)ds\\ &= \gamma\int_0^\omega G(t,s)\left(\frac{h^+(s)}{y^\rho(s)}-\frac{h^-(s)}{y^\rho(s)}+e(s)\right)ds\\ &\geq \gamma\int_0^\omega G(t,s)\frac{h^+(s)}{y^\rho(s)}ds\\ &\geq \gamma\frac{G_*\omega\overline{h^+}}{R_1^\rho}.\end{aligned}$$

Now for a fixed number $R_1>\delta$, there exists $\gamma_0>\dfrac{R_1^{\rho+1}}{G_*\omega\overline{h^+}}>0$ such that for $\gamma>\gamma_0$ (3.2) holds.

Step 3. We prove that

$$\|T_\gamma y\|\geq\|y\|,\ \text{for}\ y\in K\cap\partial S_{R_2}. \tag{3.3}$$

Actually, for any $y\in K\cap\partial S_{R_2}$, we can obviously get $\|y\|=R_2$ and

$$\sigma R_2 < y(t) < R_2,\ \text{for all}\ t\in R.$$

From (3.1), it can be seen that

$$
\begin{aligned}
(T_\gamma y)(t) &= \gamma\int_0^{\omega} G(t,s)\left(\frac{h(s)}{y^{\rho}(s)} + e(s)\right)ds \\
&= \gamma\int_0^{\omega} G(t,s)\left(\frac{h^{+}(s)}{y^{\rho}(s)} - \frac{h^{-}(s)}{y^{\rho}(s)} + e(s)\right)ds \\
&\le \gamma\left(\frac{G^{*}\omega\overline{h^{+}}}{\left(\sigma R_2\right)^{\rho}} - \frac{G_{*}\omega\overline{h^{-}}}{R_2^{\rho}} + G^{*}\omega e^{*}\right).
\end{aligned}
$$

It is obvious that we can choose R_2 large enough such that

$$
(T_\gamma y)(t) \le \gamma\left(\frac{G^{*}\omega\overline{h^{+}}}{\left(\sigma R_2\right)^{\rho}} - \frac{G_{*}\omega\overline{h^{-}}}{R_2^{\rho}} + G^{*}\omega e^{*}\right) < R_2.
$$

Therefore, (3.3) is satisfied. According to Lemma 2.1, we know that T_γ it has a fixed point. Thus, equation (1.1) has a positive periodic solution.

(ii) First, define

$$
S_{r_1} := \left\{ y \in C_{\omega} : \|y\| < r_1 \right\} \text{ and } S_{r_2} := \left\{ y \in C_{\omega} : \|y\| < r_2 \right\},
$$

where r_1 and r_2 are two positive constants and $r_2 > \left(\frac{G_{*}\omega\overline{h^{+}}}{\eta}\right)^{\frac{1}{\rho+1}} > r_1 > \delta$, here $\eta > 0$ is a constant and $\gamma\eta > 1$.

According to Step 1 of (i), we can know that $T_\gamma\left(K \cap \left(\overline{S}_{r_2} \setminus S_{r_1}\right)\right) \subset T$ and $T_\gamma : K \cap \left(\overline{S}_{r_2} \setminus S_{r_1}\right) \to T$ is a completely continuous operator.

Afterwards, let us consider

$$
\|T_\gamma y\| \ge \|y\|, \text{ for } y \in K \cap \partial S_{r_1}. \tag{3.4}
$$

In fact, for $y \in K \cap \partial S_{r_1}$, we can obviously get $\|y\| = r_1$ and

$$
\sigma r_1 < y(t) < r_1, \text{ for all } t \in R.
$$

It can be obtained from (3.1) that

$$
\begin{aligned}
(T_\gamma y)(t) &= \gamma\int_0^{\omega} G(t,s)\left(\frac{h(s)}{y^{\rho}(s)} + e(s)\right)ds \\
&= \gamma\int_0^{\omega} G(t,s)\left(\frac{h^{+}(s)}{y^{\rho}(s)} - \frac{h^{-}(s)}{y^{\rho}(s)} + e(s)\right)ds \\
&\ge \gamma\int_0^{\omega} G(t,s)\frac{h^{+}(s)}{y^{\rho}(s)}ds \\
&\ge \gamma\frac{G_{*}\omega\overline{h^{+}}}{r_1^{\rho}} \\
&\ge \gamma\eta r_1 \ge r_1
\end{aligned}
$$

since $\gamma\eta > 1$. Thus (3.4) holds.

Then we consider

$$
\|T_\gamma y\| \le \|y\|, \text{ for } y \in K \cap \partial S_{r_2}. \tag{3.5}
$$

Actually for $y \in K \cap \partial S_{r_2}$, we can obviously get $\|y\| = r_2$ and

$$\sigma r_2 < y(t) < r_2, \text{ for all } t \in R.$$

From (3.1), it can be seen that.

$$\begin{aligned}(T_\gamma y)(t) &= \gamma \int_0^\omega G(t,s)\left(\frac{h(s)}{y^\rho(s)} + e(s)\right)ds \\ &= \gamma \int_0^\omega G(t,s)\left(\frac{h^+(s)}{y^\rho(s)} - \frac{h^-(s)}{y^\rho(s)} + e(s)\right)ds \\ &\le \gamma\left(\frac{G^*\omega\overline{h^+}}{(\sigma R_2)^\rho} - \frac{G_*\omega\overline{h^-}}{R_2^\rho} + G^*\omega e^*\right) \\ &\le \gamma\left(\frac{G^*\omega\overline{h^+}}{(\sigma r_2)^\rho} + G^*\omega e^*\right).\end{aligned}$$

There exists $\gamma_1 > 0$ such that

$$\gamma_1 < \frac{\sigma^\rho r_2^{\rho+1}}{G^*\omega\overline{h^+} + \sigma^\rho r_2^\rho G^*\omega e^*}.$$

For $\gamma < \gamma_1$, (3.5) holds.

It follows from Lemma 2.1 that T_γ has a fixed point $y_1 \in K \cap \left(\overline{S}_{r_2} \setminus S_{r_1}\right)$, which is a positive periodic solution of equation (1.1) for $\gamma < \gamma_1$ and satisfies $r_1 < \|y_1\| < r_2$.

On the other hand, define

$$S_{r_3} := \left\{y \in C_\omega : \|y\| < r_3\right\} \text{ and } S_{r_4} := \left\{y \in C_\omega : \|y\| < r_4\right\},$$

where r_3 and r_4 are two positive constants and $r_4 > \left(\frac{G_*\omega\overline{h^+}}{\eta'}\right)^{\frac{1}{\rho+1}} > r_3 > r_2 > \delta$, here $\eta' > 0$ is a constant, $\gamma\eta' > 1$ and $\gamma' < \gamma$.

Similarly, according to Step 1 of (i), we can know that $T_\gamma\left(K \cap \left(\overline{S}_{r_4} \setminus S_{r_3}\right)\right) \subset T$ and $T_\gamma : K \cap \left(\overline{S}_{r_4} \setminus S_{r_3}\right) \to T$ is a completely continuous operator.

Afterwards, let us prove that

$$\|T_\gamma y\| \ge \|y\|, \text{ for } y \in K \cap \partial S_{r_3}. \tag{3.6}$$

In fact, for any $y \in K \cap \partial S_{r_3}$, it is clear that $\|y\| = r_3$ and

$$\sigma r_3 < y(t) < r_3, \text{ for all } t \in R.$$

From (3.1), it can be seen that.

$$\begin{aligned}(T_\gamma y)(t) &= \gamma\int_0^\omega G(t,s)\left(\frac{h(s)}{y^\rho(s)}+e(s)\right)ds\\ &= \gamma\int_0^\omega G(t,s)\left(\frac{h^+(s)}{y^\rho(s)}-\frac{h^-(s)}{y^\rho(s)}+e(s)\right)ds\\ &\geq \gamma\int_0^\omega G(t,s)\frac{h^+(s)}{y^\rho(s)}ds\\ &\geq \gamma\frac{G_*\omega\overline{h^+}}{r_3^\rho}\\ &\geq \gamma\eta' r_3 \geq r_3\end{aligned}$$

since $\gamma\eta' > 1$, thus (3.6) holds.

Then we prove that

$$\|T_\gamma y\| \leq \|y\|, \text{ for } y \in K \cap \partial S_{r_4}. \tag{3.5}$$

Actually, for any $y \in K \cap \partial S_{r_4}$, it is clear that $\|y\| = r_4$ and

$$\sigma r_4 < y(t) < r_4, \text{ for all } t \in R.$$

It can be obtained from (3.1) that

$$\begin{aligned}(T_\gamma y)(t) &= \gamma\int_0^\omega G(t,s)\left(\frac{h(s)}{y^\rho(s)}+e(s)\right)ds\\ &= \gamma\int_0^\omega G(t,s)\left(\frac{h^+(s)}{y^\rho(s)}-\frac{h^-(s)}{y^\rho(s)}+e(s)\right)ds\\ &\leq \gamma\left(\frac{G^*\omega\overline{h^+}}{(\sigma r_4)^\rho}-\frac{G_*\omega\overline{h^-}}{r_4^\rho}+G^*\omega e^*\right)\\ &\leq \gamma\left(\frac{G^*\omega\overline{h^+}}{(\sigma r_4)^\rho}+G^*\omega e^*\right).\end{aligned}$$

There exists $\gamma_2 > 0$ satisfying

$$\gamma_2 < \min\left\{\frac{\sigma^\rho r_4^{\rho+1}}{G^*\omega\overline{h^+}+\sigma^\rho r_4^\rho G^*\omega e^*}, \gamma_1\right\}.$$

Therefore, $\gamma < \gamma_2$, we know (3.7) holds.

It follows from Lemma 2.1 that T_γ has a fixed point $y_2 \in K \cap \left(\overline{S}_{r_4} \setminus S_{r_3}\right)$, which is a positive periodic solution of equation (1.1) for $\gamma < \gamma_2$ and satisfies $r_3 < \|y_2\| < r_4$. Noting that

$$r_1 < \|y_1\| < r_2 < r_3 < \|y_2\| < r_4,$$

we can deduce that y_1 and y_2 are two desired distinct positive periodic solutions of equation (1.1) for $\gamma < \gamma_2$.

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