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Original Article

Minimum Eccentric Dominating Energy of Graphs

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Abstract - *In this paper, for a graph $G = (V, E)$ of order k, the minimum eccentric dominating energy $\mathbb{E}_{ed}(G)$ is the sum of the eigenvalues obtained from the minimum eccentric dominating $k \times k$ matrix $\mathbb{A}_{ed}(G)$. $\mathbb{E}_{ed}(G)$ of standard graphs are computed. Properties, upper and lower bounds for $\mathbb{E}_{ed}(G)$ are established.*

Keywords - *Dominating set, Eccentricity, Eccentric dominating set, Graph theory, Minimum eccentric dominating energy.*

1. Introduction

In 1978, one of the present authors conceived the concept of energy of a graph, defined as the sum of absolute values of the eigenvalues of the adjacency matrix [3, 8]. Motivated by the remarkable success of the theory of graph energy [4, 8], many different variants of graph energy have been introduced and studied [5]. One of these is the "minimum dominating energy", put forward by Rajesh Kanna et al. [9].

For a graph $G = (V, E)$, let $\mathbb{A} = (a_{ij})$ be the minimum dominating matrix defined by

$$(a_{ij}) = \begin{cases} 1, & if\ v_i v_j \in E \\ 1, & if\ i = j\ and\ v_i \in D \\ 0, & otherwise \end{cases}$$

Where D is the dominating set with minimum cardinality [6]. If its eigenvalues are $\lambda_1, \lambda_2, \ldots, \lambda_n$, then the minimum dominating energy is defined as [9]

$$E_D(G) = \sum_{i=1}^{n} |\lambda_i|$$

The concept of eccentric domination was introduced by Janakiraman et al. [7]. For a graph, $G = (V, E)$, a set $S \subseteq V$ is said to be a dominating set if every vertex in $V - S$ is adjacent to some vertex in S [6]. The eccentricity $e(v)$ of a vertex v is the distance to a vertex farthest from v. Thus, $e(v) = max\{d(u, v) \mid u \in V\}$. For a vertex v, each vertex at a distance $e(v)$ from v is an eccentric vertex of v. The eccentric set of a vertex v is defined as $E(v) = \{u \in V(G) \mid d(u, v) = e(v)\}$. A dominating set $D \subseteq V(G)$ is an eccentric dominating set if, for every $v \in V - D$, there exists at least one eccentric vertex u of v in D. An eccentric dominating set with minimum cardinality is called a minimum eccentric dominating set. The eccentric domination number $\gamma_{ed}(G)$ of a graph, G equals the minimum cardinality of an eccentric dominating set. Inspired by Ref. [9], we now introduce the "minimum eccentric dominating energy" of a graph, denoted by $\mathbb{E}_{ed}(G)$.

In this paper, we find $\mathbb{E}_{ed}(G)$ of some standard classes of graphs often encountered in graph theory and its application [2]. These are the complete star, cocktail party, and crown graphs.

The cocktail party graph, denoted by K_{2k}, is the graph having the vertex set $V = \bigcup_{i=1}^{k}\{u_i, v_i\}$ and the edge set $E = \{(u_i, u_j), (v_i, v_j): i \neq j\} \cup \{(u_i, v_j), (v_i, u_j): i, j \in \mathbb{N}, i \neq j\}$. The crown graph H_k is the graph having 2 sets of vertices $\{u_1, u_2, \ldots, u_{k/2}\}$ and $\{v_1, v_2, \ldots, v_{k/2}\}$ with an edge from u_i to v_j Where $i \neq j$. The crown graph is a graph obtained from the complete bipartite graph by removing the horizontal edge between the paired nodes.

2. Minimum Eccentric Dominating Energy

In this section, the minimum eccentric dominating matrix and minimum eccentric dominating energy are defined and then computed for some standard graphs.

Definition 2.1. Let $G = (V, E)$ be a simple graph with vertex set $V(G) = \{v_1, v_2, \dots v_k\}, k \in \mathbb{N}$ and edge set E. Let D be a minimum eccentric dominating set of G. Then the minimum eccentric dominating matrix of G is the $k \times k$ matrix $\mathbb{A}_{ed}(G) = (e_{ij})$, whose

$$(e_{ij}) = \begin{cases} 1, & if\ v_j \in E(v_i)\ or\ v_i \in E(v_j) \\ 1, & if\ i = j\ and\ v_i \in D \\ 0, & otherwise \end{cases}$$

Definition 2.2. The characteristic polynomial of a minimum eccentric dominating matrix $\mathbb{A}_{ed}(G)$ is defined by $P_k(G, \psi) = det(\mathbb{A}_{ed}(G) - \psi I)$.

Definition 2.3. The eigenvalues of $\mathbb{A}_{ed}(G)$ are said to be the minimum eccentric dominating eigenvalues of G. Since $\mathbb{A}_{ed}(G)$ is symmetric, the eigenvalues of $\mathbb{A}_{ed}(G)$ are real. We label them in non-increasing order as $\psi_1 \geq \psi_2 \geq \cdots \geq \psi_k$.

Definition 2.4. The minimum eccentric dominating energy of G is

$$\mathbb{E}_{ed}(G) = \sum_{i=1}^{k} |\psi_i|$$

Remark 2.1. The trace of $\mathbb{A}_{ed}(G)$ it is the eccentric domination number of the respective graph

Example 2.1. In order to illustrate the above definitions, consider the 4-vertex graph H, depicted in Fig. 1, and its eccentricity properties shown in Table 1.

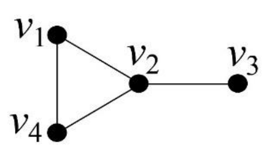

Fig. 1 Graph H is used to exemplify Definitions 2.1-2.4.

Table 1. Eccentricity properties of the vertices of the graph H

Vertex	Eccentricity e(v)	Eccentric vertex E(v)
v_1	2	v_3
v_2	1	v_1, v_3, v_4
v_3	2	$v_1 v_4$
v_4	2	v_3

The minimum eccentric dominating sets of H are $D_1 = \{v_1, v_3\}$, $D_2 = \{v_2, v_3\}$ and $D_3 = \{v_3, v_4\}$.
For $D_1 = \{v_1, v_3\}$

$$\mathbb{A}_{ed}(H) = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

The characteristic polynomial is $P_k(H, \psi) = \psi^4 - 2\psi^3 - 4\psi^2 + \psi + 1$. The minimum eccentric dominating eigenvalues are $\psi_1 \approx 3.1401$, $\psi_2 = 0.5712$, $\psi_3 = -0.4378$, $\psi_4 = -1.2735$. Thus the minimum eccentric dominating energy is $\mathbb{E}_{ed}(H) \approx 5.4226$.
For $D_2 = \{v_2, v_3\}$,

$$\mathbb{A}_{ed}(H) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

The characteristic polynomial is $P_k(H,\psi) = \psi^4 - 2\,\psi^3 - 4\,\psi^2$. The minimum eccentric dominating eigenvalues are $\psi_1 \approx 3.2361$, $\psi_2 = 0$, $\psi_3 = -1.2361$. The minimum eccentric dominating energy is $\mathbb{E}_{ed}(H) \approx 4.4722$.
For $D_3 = \{v_3, v_4\}$,

$$\mathbb{A}_{ed}(H) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

The characteristic polynomial is $P_k(H,\psi) = \psi^4 - 2\,\psi^3 - 4\,\psi^2 + \psi + 1$. The minimum eccentric dominating eigenvalues are $\psi_1 \approx 3.1401$, $\psi_2 \approx 0.5712$, $\psi_3 \approx -0.4378$, $\psi_4 \approx -1.2735$. The minimum eccentric dominating energy is $\mathbb{E}_{ed}(H) \approx 5.4226$.

Example 2.1 shows that the actual value of the minimum eccentric dominating energy depends on the eccentric dominating set used.

Theorem 2.1. The minimum eccentric dominating energy of the cocktail party graph k_{2k} is

$$\mathbb{E}_{ed}(K_{2k}) = \sum_{i=1}^{2k} |\psi_i| = \left[\left|\frac{1+\sqrt{5}}{2}\right| + \left|\frac{1-\sqrt{5}}{2}\right|\right] k. \qquad (\mathbf{1})$$

Proof. Let K_{2k} be a cocktail party graph with vertex set $V = \bigcup_{i=1}^{k}\{u_i, v_i\}$. Let D be the minimum eccentric dominating set and $|D| = k$. Then $D = \{u_1, u_2, \ldots u_k\}$ or $\{v_1, v_2, \ldots v_k\}$. Then the minimum eccentric dominating matrix is

$$\mathbb{A}_{ed}(K_{2k}) = \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & \cdots & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix}_{k\times k}$$

The characteristic polynomial is $P_k(K_{2k},\psi) = det(\mathbb{A}_{ed}(K_{2k}) - \psi I)$

$$= \begin{vmatrix} 1-\psi & 0 & 0 & 0 & \ldots & 0 & 0 & 1 & 0 \\ 0 & 1-\psi & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \\ 0 & 0 & 1-\psi & 0 & \cdots & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-\psi & \cdots & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & 0 & \cdots & -\psi & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & -\psi & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & -\psi & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & -\psi \end{vmatrix}$$

resulting in $P_k(K_{2k},\psi) = (\psi^2 - \psi - 1)k$.

The minimum eccentric dominating eigenvalues are

$$\psi = \frac{1+\sqrt{5}}{2}\ (k\ \ times)$$
$$\psi = \frac{1-\sqrt{5}}{2}\ (k\ \ times)$$

The minimum eccentric dominating energy of K_{2k} is then given by Eq. (1).

Theorem 2.2. For the star graph $S_k, k > 2$, the minimum eccentric dominating energy is

$$\mathbb{E}_{ed}(S_k) = |-1|(k-3) + \left|\frac{(k-1)+\sqrt{(k-1)^2+8}}{2}\right| + \left|\frac{(k-1)-\sqrt{(k-1)^2+8}}{2}\right|. \qquad (2)$$

Proof. Let S_k be the star graph with vertex set $V = \{v_1, v_2, \dots v_k\}$. The minimum eccentric dominating set is $D = \{v_1, v_k\}$, where v_k is the central vertex. Then

$$\mathbb{A}_{ed}(S_k) = \begin{pmatrix} 1 & 1 & 1 & & 1 & 1 & 1 \\ 1 & 0 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 1 & 0 & & 1 & 1 & 1 \\ & \vdots & & \ddots & & \vdots & \\ 1 & 1 & 1 & & 0 & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 0 & 1 \\ 1 & 1 & 1 & & 1 & 1 & 1 \end{pmatrix}_{k \times k}$$

The characteristic polynomial is $P_k(S_k, \psi) = det(\mathbb{A}_{ed}(S_k) - \psi I)$

$$= \begin{vmatrix} 1-\psi & 1 & 1 & & 1 & 1 & 1 \\ 1 & -\psi & 1 & \cdots & 1 & 1 & 1 \\ 1 & 1 & -\psi & & 1 & 1 & 1 \\ & \vdots & & \ddots & & \vdots & \\ 1 & 1 & 1 & & -\psi & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & -\psi & 1 \\ 1 & 1 & 1 & & 1 & 1 & 1-\psi \end{vmatrix}$$

From which we obtain $P_k(S_k, \psi) = \psi(\psi + 1)^{k-3}[\psi^2 - (k-1)\,\psi - 2]$

The minimum eccentric dominating eigenvalues are

$$\psi = 0$$

$$\psi = -1 \ \ (k - 3 \ \ times)$$

$$\psi = \frac{(k-1) + \sqrt{(k-1)^2 + 8}}{2}$$

$$\psi = \frac{(k-1) - \sqrt{(k-1)^2 + 8}}{2}$$

The minimum eccentric dominating energy of the star S_k is thus

$$\mathbb{E}_{\mathrm{ed}}(\mathrm{S_k}) = 0 + |-1|(\mathrm{k} - 3) + \left|\frac{(\mathrm{k}-1) + \sqrt{(\mathrm{k}-1)^2 + 8}}{2}\right| + \left|\frac{(\mathrm{k}-1) - \sqrt{(\mathrm{k}-1)^2 + 8}}{2}\right|$$

and Eq. (2) follows.

Theorem 2.3. For the complete graph $K_k, k \geq 2$, the minimum eccentric dominating energy is

$$\mathbb{E}_{ed}(K_k) = |(-1)|(k-2) + \left|\frac{(k-1) + \sqrt{k^2 - 2k + 5}}{2}\right| + \left|\frac{(k-1) - \sqrt{k^2 - 2k + 5}}{2}\right|. \quad (3)$$

Proof. Let K_k be the complete graph with vertex set $V = \{v_1, v_2, \dots v_k\}$. Its minimum eccentric dominating set is $D = \{v_1\}$. Then

$$\mathbb{A}_{ed}(K_k) = \begin{pmatrix} 1 & 1 & 1 & & 1 & 1 & 1 \\ 1 & 0 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 1 & 0 & & 1 & 1 & 1 \\ & \vdots & & \ddots & & \vdots & \\ 1 & 1 & 1 & & 0 & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 0 & 1 \\ 1 & 1 & 1 & & 1 & 1 & 1 \end{pmatrix}_{k \times k}$$

The characteristic polynomial is $P_k(K_k, \psi) = det(\mathbb{A}_{ed}(K_k) - \psi I)$

$$= \begin{vmatrix} 1-\psi & 1 & 1 & & 1 & 1 & 1 \\ 1 & -\psi & 1 & \cdots & 1 & 1 & 1 \\ 1 & 1 & -\psi & & 1 & 1 & 1 \\ & \vdots & & \ddots & & \vdots & \\ 1 & 1 & 1 & & -\psi & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & -\psi & 1 \\ 1 & 1 & 1 & & 1 & 1 & 1-\psi \end{vmatrix}$$

From which it follows $P_k(K_k, \psi) = (\psi + 1)^{k-2}[\psi^2 - (k-1)\psi - 1]$.

The minimum eccentric dominating eigenvalues are

$$\psi = -1 \ (k-2 \ times)$$

$$\psi = \frac{(k-1) + \sqrt{k^2 - 2k + 5}}{2}$$

$$\psi = \frac{(k-1) - \sqrt{k^2 - 2k + 5}}{2}$$

and thus the minimum eccentric dominating energy of K_k is given by Eq. (3).

Theorem 2.4. The minimum eccentric dominating energy of the crown graph H_k is

$$\mathbb{E}_{ed}(H_k) = \left[\left|\frac{1+\sqrt{5}}{2}\right| + \left|\frac{1-\sqrt{5}}{2}\right|\right]\frac{k}{2}. \quad (3)$$

Proof. Let H_k be the crown graph with vertex set $V = \cup_{i=1}^{k/2}\{u_i, v_i\}$. Let D be the minimum eccentric dominating set where $|D| = k/2$. Then $D = \{u_1, u_2, \ldots u_{k/2}\}$ or $\{v_1, v_2, \ldots v_{k/2}\}$. Then

$$\mathbb{A}_{ed}(H_k) = \begin{pmatrix} 1 & 0 & 0 & & 1 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 1 & & 0 & 0 & 1 \\ & \vdots & & \ddots & & \vdots & \\ 1 & 0 & 0 & & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & & 0 & 0 & 0 \end{pmatrix}_{k \times k}$$

The characteristic polynomial is $P_k(H_k, \psi) = det(\mathbb{A}_{ed}(H_k) - \psi I)$

$$= \begin{vmatrix} 1-\psi & 0 & 0 & & 1 & 0 & 0 \\ 0 & 1-\psi & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 1-\psi & & 0 & 0 & 1 \\ & \vdots & & \ddots & & \vdots & \\ 1 & 0 & 0 & & -\psi & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & -\psi & 0 \\ 0 & 0 & 1 & & 0 & 0 & -\psi \end{vmatrix}$$

From which, we calculate $P_k(H_k, \psi) = (\psi^2 - \psi - 1)^{k/2}$.

The minimum eccentric dominating eigenvalues are

$$\psi = \frac{1+\sqrt{5}}{2} \ \left(\frac{k}{2} \ times\right)$$

$$\psi = \frac{1-\sqrt{5}}{2} \ \left(\frac{k}{2} \ times\right)$$

and therefore, the minimum eccentric dominating energy of H_k is this given by Eq. (4).

3. Properties of Minimum Eccentric Dominating Eigenvalues

In this section, we discuss some properties of the eigenvalues of $\mathbb{A}_{ed}$ for cocktail party, complete, crown, and star graphs. Bounds for minimum eccentric dominating energy of some standard graphs are obtained.

Theorem 3.1. Let D be the minimum eccentric dominating set and $\psi_1, \psi_2, \dots \psi_k$ the eigenvalues of the minimum eccentric dominating matrix $\mathbb{A}_{ed}(G)$. If G is

1. any graph, then $\sum_{i=1}^{k} \psi_i = |D|$,
2. the cocktail party graph K_{2k}, then $\sum_{i=1}^{2k} \psi_i^2 = |D| + \sum_{i=1}^{2k} |E(v_i)|$,
3. the complete graph K_k and crown graphs H_k, then $\sum_{i=1}^{k} \psi_i^2 = |D| + \sum_{i=1}^{k} |E(v_i)|$,
4. the star graph S_k, then $\sum_{i=1}^{k} \psi_i^2 = |D| + \sum_{i=1}^{k} |E(v_i)| + (k-1)$.

Proof. 1. We know that the sum of eigenvalues of $\mathbb{A}_{ed}(G)$ is the trace of $\mathbb{A}_{ed}(G)$, implying

$$\sum_{i=1}^{k} \psi_i = \sum_{i=1}^{k} e_{ii} = |D|$$

2. The sum of squares of the eigenvalues of $\mathbb{A}_{ed}(K_{2k})$ is

$$\sum_{i=1}^{2k} \psi_i^2 = \sum_{i=1}^{2k}\sum_{j=1}^{2k} e_{ij}\, e_{ij} = \sum_{i=1}^{2k} (e_{ii})^2 + \sum_{i\neq j} e_{ij}e_{ij} = \sum_{i=1}^{2k} (e_{ii})^2 + 2\sum_{i<j} \left(e_{ij}\right)^2 = |D| + \sum_{i=1}^{2k} |E(v_i)|$$

Since $2\sum_{i<j}\left(e_{ij}\right)^2 = \sum_{i=1}^{2k} |E(v_i)|$.

3. The sum of squares of the eigenvalues of $\mathbb{A}_{ed}(K_k)$ and $\mathbb{A}_{ed}(H_k)$ is the trace of $[\mathbb{A}_{ed}(K_k)]^2$ and $[\mathbb{A}_{ed}(K_k)]^2$, respectively. Thus,

$$\sum_{i=1}^{k} \psi_i^2 = \sum_{i=1}^{k}\sum_{j=1}^{k} e_{ij}\, e_{ij} = \sum_{i=1}^{k} (e_{ii})^2 + \sum_{i\neq j} e_{ij}e_{ij} = \sum_{i=1}^{k} (e_{ii})^2 + 2\sum_{i<j} \left(e_{ij}\right)^2 = |D| + \sum_{i=1}^{k} |E(v_i)|$$

4. For the star graph S_k, the sum of squares of the eigenvalues of $\mathbb{A}_{ed}(S_k)$ is

$$\sum_{i=1}^{k} \psi_i^2 = \sum_{i=1}^{k}\sum_{j=1}^{k} e_{ij}\, e_{ij} = \sum_{i=1}^{k} (e_{ii})^2 + \sum_{i\neq j} e_{ij}e_{ij} = \sum_{i=1}^{k} (e_{ii})^2 + 2\sum_{i<j} \left(e_{ij}\right)^2 = |D| + \sum_{i=1}^{k} |E(v_i)| + (k-1)$$

Theorem 3.2. For the complete graph K_k and the crown graph H_k, if D is the minimum eccentric dominating set and $W = |\det \mathbb{A}_{ed}(G)|$, then

$$\sqrt{|D| + \sum_{i=1}^{k} |E(v_i)| + k(k-1)W^{2/k}} \leq \mathbb{E}_{ed}(G) \leq \sqrt{k\left(\sum_{i=1}^{k} |E(v_i)| + |D|\right)} \tag{5}$$

Proof. By the Cauchy-Schwarz inequality $\left(\sum_{i=1}^{k} a_i b_i\right)^2 \leq (\sum_{i=1}^{k} a_i^2)(\sum_{i=1}^{k} b_i^2)$, if $a_i = 1$ and $b_i = \psi_i$, then

$$\left(\sum_{i=1}^{k} |\psi_i|\right)^2 \leq \left(\sum_{i=1}^{k} 1\right)\left(\sum_{i=1}^{k} |\psi_i|^2\right)$$

$$\mathbb{E}_{ed}(G)^2 \leq k\left(|D| + \sum_{i=1}^{k} |E(v_i)|\right)$$

Implying the right-hand side inequality in (5).

Since the arithmetic mean is not smaller than the geometric mean, we have

$$\frac{1}{k(k-1)}\sum_{i\neq j} |\psi_i|\,|\psi_j| \geq \left[\prod_{i\neq j} |\psi_i|\,|\psi_j|\right]^{\frac{1}{k(k-1)}} = \left[\prod_{i=1}^{k} |\psi_i|^{2(k-1)}\right]^{\frac{1}{k(k-1)}} = \left[\prod_{i=1}^{k} |\psi_i|\right]^{\frac{2}{k}} = \left[\prod_{i=1}^{k} \psi_i\right]^{\frac{2}{k}}$$

$$\frac{1}{k(k-1)}\sum_{i\neq j}|\psi_i|\,|\psi_j| = |\,det\,\mathbb{A}_{ed}(G)|^{\frac{2}{k}} = W^{\frac{2}{k}}$$

and thus

$$\sum_{i\neq j}|\psi_i|\,|\psi_j| \geq k(k-1)W^{\frac{2}{k}}$$

We now have

$$\mathbb{E}_{ed}(G)^2 = \left(\sum_{i=1}^{k}|\psi_i|\right)^2 \geq \prod_{i=1}^{k}|\psi_i|^2 + \sum_{i\neq j}|\psi_i|\,|\psi_j| = \left(|D| + \sum_{i=1}^{k}|E(v_i)|\right) + k(k-1)W^{\frac{2}{k}}$$

which implies the left-hand side of the inequality in (5).

Theorem 3.3. For the cocktail party K_{2k}, if D is its minimum eccentric dominating set and $W = |\,det\,\mathbb{A}_{ed}(K_{2k})|$, then

$$\sqrt{|D| + \sum_{i=1}^{2k}|E(v_i)| + k(k-1)W^{2/k}} \leq \mathbb{E}_{ed}(K_{2k}) \leq \sqrt{k\left(\sum_{i=1}^{2k}|E(v_i)| + |D|\right)}$$

Proof. The proof is analogous as that of Theorem 3.2.

Theorem 3.4. For the star graph S_k, if D is its minimum eccentric dominating set and $W = |\,det\,\mathbb{A}_{ed}(S_k)|$, then

$$\sqrt{|D| + \sum_{i=1}^{k}|E(v_i)| + (k-1) + k(k-1)W^{2/k}} \leq \mathbb{E}_{ed}(S_k) \leq \sqrt{k\left(\sum_{i=1}^{k}|E(v_i)| + |D| + (k-1)\right)}$$

Proof. The proof is analogous as that of Theorem 3.2.

Theorem 3.5. If $\psi_1(G)$ is the largest minimum eccentric dominating eigenvalue of $\mathbb{A}_{ed}(G)$, then for the complete graph K_k and the crown graph H_k,

$$\psi_1(G) \geq \frac{1}{k}\left(|D| + \sum_{i=1}^{k}|E(v_i)|\right)$$

for the cocktail party graph K_{2k},

$$\psi_1(K_{2k}) \geq \frac{1}{2k}\left(|D| + \sum_{i=1}^{2k}|E(v_i)|\right)$$

and for the star graph S_k,

$$\psi_1(S_k) \geq \frac{1}{k}\left(|D| + \sum_{i=1}^{k}|E(v_i)| + (k-1)\right)$$

Proof. Let Y be a non-zero vector. Then by applying the Rayleigh-Ritz theorem [1],

$$\psi_1\big(\mathbb{A}_{ed}(G)\big) = \max_{Y\neq 0}\frac{Y^T\mathbb{A}_{ed}(G)\,Y}{Y^T\,Y}$$

$$\psi_1\big(\mathbb{A}_{ed}(G)\big) \geq \frac{U^T\mathbb{A}_{ed}(G)\,U}{U^T\,U} = \frac{|D| + \sum_{i=1}^{k}|E(v_i)|}{k}$$

where U is the unit matrix.
Analogously,

$$\psi_1\big(\mathbb{A}_{ed}(H_{2k})\big) \geq \frac{U^T\mathbb{A}_{ed}(H_{2k})\,U}{U^T\,U} = \frac{|D| + \sum_{i=1}^{2k}|E(v_i)|}{k}$$

and

$$\psi_1\big(\mathbb{A}_{ed}(S_k)\big) \geq \frac{U^T\mathbb{A}_{ed}(S_k)\,U}{U^T\,U} = \frac{|D| + \sum_{i=1}^{k}|E(v_i)| + (k-1)}{k}$$

4. Conclusion

In this paper, we define the minimum eccentric dominating energy of the graph and their properties are discussed. The minimum eccentric dominating energy for the family of graphs is determined, and their bounds are calculated.

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