International Journal of Mathematics Trends and Technology
ISSN: 2231-5373/ https://doi.org/10.14445/22315373/IJMTT-V69I6P506

Volume 69 Issue 6, 39-47, June 2023

Original Article

Exact Squaring the Circle with Straightedge and Compass by Secondary Geometry

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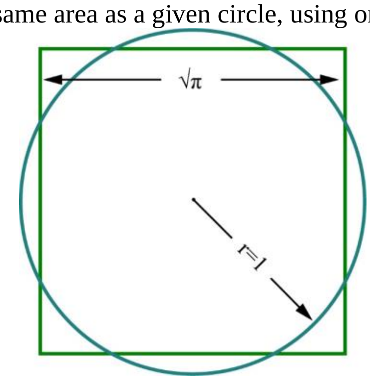
Received: 16 April 2023 Revised: 24 May 2023 Accepted: 07 June 2023 Published: 17 June 2023

Abstract *- There are three classical problems remaining from ancient Greek mathematics, which are extremely influential in the development of geometry. They are "Trisecting An Angle", "Squaring The Circle", and "Doubling The Cube" problems. I solved the first one – Trisecting An Angle and published its paper in the International Journal Of Mathematics Trends And Technology (Volume 69, May 2023). It is difficult to give an accurate date when the problem of Squaring The Circle first appeared. The present article studies what has become the most famous for these problems, namely the problem of squaring the circle or the quadrature of the circle as it is sometimes called. One of the fascinations of this problem is that it has been of interest throughout the whole of the history of mathematics. From the oldest mathematical documents known up to the mathematics of today, the problem and related problems concerning π have interested professional & non-professional mathematicians for millenniums. The problem of Squaring The Circle is stated: Using only a straightedge and a compass, is it possible to construct a square with an area equal to the area of a given circle? I adopt the technique "ANALYSIS" to solve accurately the "Squaring The Circle" problem with only a straightedge & compass by secondary Geometry. Upstream from this method of exact "squaring the circle", we can deduce, conversely, to get a new Mathematical challenge "Circling the Square", with a straightedge & a compass. In addition, this research result can be used for further research in the "CUBING THE SPHERE" challenge, with only "a straightedge & a compass" using only secondary Geometry.*

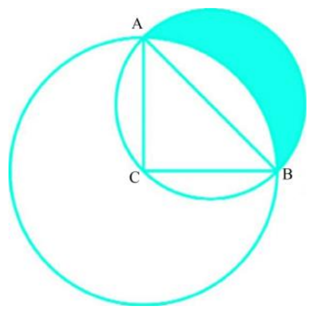
Keywords *- Squaring the circle, Quadrature of the circle, Make a circle squared, Find a square area same as the circle, Circling the square, Make a square rounded.*

1. Introduction

Doubling a cube, trisecting an angle, and squaring the circle are the problems in geometry first proposed in Greek mathematics, which were extremely influential in the development of Geometry. Although these are closely linked, I chose to solve the Squaring The Circle problem as my second study after solving the "Trisecting An Angle" exactly & successfully (Published in IJMTT Volume 69 – May 2023). The history of the "squaring the circle" problem dates back millennia to around 450 B.C. (nearly 2,500 years), according to Quanta, a science and mathematics magazine. Mathematician Anaxagoras of Clazomenae was imprisoned for radical ideas about the sun, and while in prison, he worked on the now-iconic problem involving a compass and straightedge [1]. The present article studies what has become the most famous for these problems, namely the problem of squaring the circle or the quadrature of the circle as it is sometimes called. One of the fascinations of this problem is that it has been of interest throughout the whole of the history of mathematics. From the oldest mathematical documents known up to today's mathematics, the problem and related problems concerning π have interested both professional and non-professional mathematicians. Squaring the circle is the challenge of constructing a square with the area of a given circle by using only a finite number of steps with a compass and straightedge. In geometry, "straightedge and compass" construction is also known as Euclidean construction or classical construction. According to mathematicians, "squaring the circle" means to construct a square with the same area as a given circle, using only a compass and a straightedge.



First of all, it is not saying that a square of equal area with a circle does not exist. If the circle has area A, then a square with a side “square root” of A has the same area. Secondly, it is not saying that it is impossible since it is possible under the restriction of using only a straightedge and a compass [2]. About three thousand years ago, there were three well-known ancient Greek problems. Among them, the squaring the circle problem was meticulously studied by Hippocrates. Hippocrates was the first to use a plane construction to find a square with an area equal to a figure with circular sides. He squared certain lunes and also the sum of a lune and a circle. The problem of squaring a circle is stated: Using only a straightedge and a compass, is it possible to construct a square with an area equal to the area of a given circle? In 1882, mathematician Ferdinand von Lindemann proved that pi is an irrational number, which means that it is impossible to construct a square in the problem of squaring a circle mentioned by Hippocrates. In his attempts to square the circle, Hippocrates was able to find the areas of certain moons or crescent-shaped figures contained between two intersecting circles. In the figure below, Hippocrates asserted that the shaded green part (called the crescent moon) has an area equal to the area of triangle ABC. But no one understands how he calculated it.



A major step forward in proving that the circle could not be squared using a straightedge and a compass occurred in 1761 when Lambert proved that π was irrational. This was not enough to prove the impossibility of squaring the circle with a straightedge and a compass since certain algebraic numbers can be constructed with these tools. In 1775, the Paris Académie des Sciences passed a resolution that stated that no further attempted solutions submitted to them would be examined! The exact question posed by Anaxagoras was answered in 1882 when the German mathematician Ferdinand von Lindemann proved that squaring the circle is impossible with classical tools! A few years later, the Royal Society in London also banned consideration of any further 'proofs' of squaring the circle as large numbers of amateur mathematicians tried to achieve fame by presenting the Society with a solution! This decision of the Royal Society was described by De Morgan about 100 years later as the official blow to circle-squarers! [3]. Despite the proof of the impossibility of "squaring the circle," the problem has continued to capture the imagination of mathematicians and the general public alike, and it remains an important topic in the history and philosophy of mathematics. In 2022, I did solve the “Trisecting An Angle” problem with straightedge & compass [4] and published it in the IJMTT journal as a counter-proof to the *Wantzel, L. (1837)* [5].

In seeking solutions to problems, geometers developed a special technique, which they called “analysis.” They assumed the problem to have been solved and then, by investigating the properties of this solution, worked back to find an equivalent problem that could be solved based on the givens. To obtain the formally correct solution to the original problem, then, geometers reversed the procedure: first, the data were used to solve the equivalent problem derived in the analysis, and, from the solution obtained, the original problem was then solved. In contrast to analysis, this reversed procedure is called “synthesis” [4]. I adopted the technique “ANALYSIS” to solve accurately the “Squaring The Circle” problem with only a straightedge & compass by secondary Geometry.

“The great Tao is simple, very simple!”
(Lão Tử - Đạo Đức Kinh: Đại Đạo thì giản dị, rất giản dị !)

2. Proofs for News General Propositions

2.1. Definition 1: “Conical-Arc” shape

Given a circle (O, r) and an angle $\widehat{BAC}$ with its vertex outside the circle such that the bisector of the angle passes through the centre O of the circle, then the special shape formed by the 2 sides of the angle and arc $\widehat{DE}$ can be called a Conical-Arc (in Figure 1 below, the red shape ADE is a Conical-Arc). If $\widehat{BAC}$ is a right angle, then the shape ADE is called a Right-Conical-Arc.

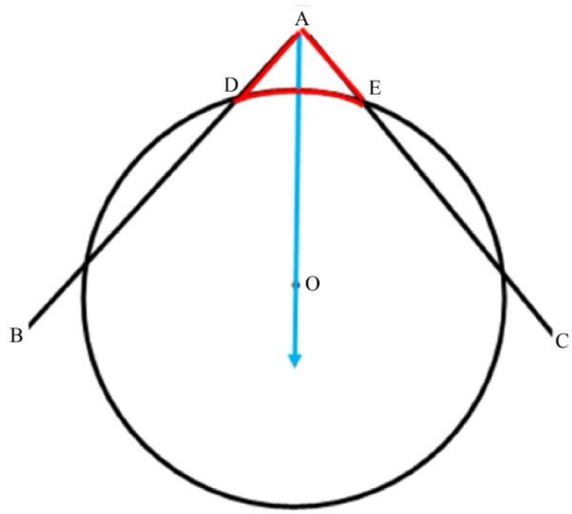


Fig. 1 The Conical-Arc ADE

2.2. Theorem 1

If there is a square ABCD of area πr^2 (assumed, yellow colour) of which the centre coincides with the centre of a given circle (O, r), then ABCD occupies a stretch in between the inscribed square A"B"C"D" (red colour) of the circle and the circumscribed square A'B'C'D' (blue colour) of the circle (with 4 sides of A'B'C'D' being tangents to the circle in Figure 2 below). OR,

Given $\begin{cases} \alpha.) \text{ a circle (O, r) – in black colour of Figure 2 below;} \\ \beta.) \text{ a circumscribing square A'B'C'D' (blue) with area } 4r^2; \\ \gamma.) \text{ an inscribed square A"B"C"D" (red) with area } 2r^2; \end{cases}$

then $\begin{cases} \alpha.) \text{ a square ABCD with area } \pi r^2 \text{ (yellow) occupies a stretched area in between A'B'C'D' \& A"B"C"D";} \\ \beta.) \text{ the given circle (O, r) contains the inscribed circle (O, } \frac{r\sqrt{\pi}}{2}\text{) – yellow dashes - of ABCD;} \\ \gamma.) \text{ 4 sides of the square ABCD intersect the circle (O, r) to make 4 equal "small segments" of the circle (Figure 2 below).} \end{cases}$

Proof:

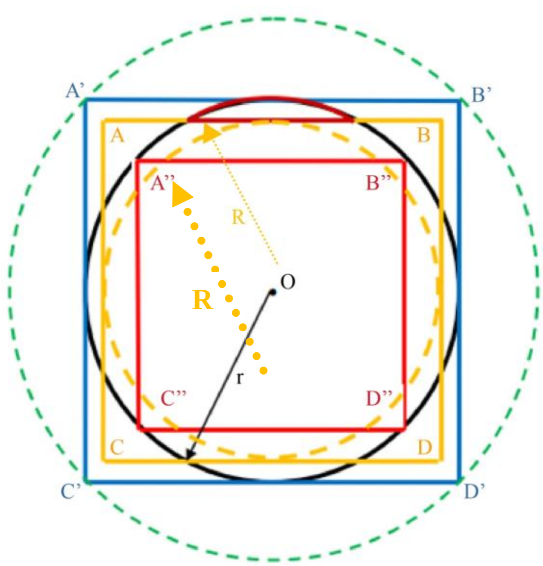


Fig. 2 A small segment (red) located above side AB of square ABCD.

Assume there exists an unknown square ABCD (yellow) of area πr^2, having the same centre O with the circle and having the sides parallel to sides of the circumscribed square A'B'C'D' (blue) of the circle (O, r) in Figure 2 above. That means A'B'C'D' is a circumscribing square (blue colour) of the given circle (O, r), and A''B''C''D'' (red) is an inscribing square of the circle (O,r). Let these squares have the same centre O as the circle (O,r) and let them have parallel sides.

The square A'B'C'D' (blue colour) has the following properties:

A'B' = C'D' = A'C' = B'C' = 2r

and

$$\text{A'B'C'D' area} = 4r^2 \qquad (1)$$

Similarly, A''B''C''D'' (red colour) has properties:

A''B'' = C''D'' = A''C' = B''C'' = $r\sqrt{2}$

and

$$\text{Area of A''B''C''D''} = 2r^2 \qquad (2)$$

Then from (1), (2) and $2 < \pi < 4$, we get:

$$2r^2 < \text{Area } \pi r^2 \text{ of the square ABCD} < 4r^2 \qquad (3)$$

α.) From (3), the square A'B'C'D' of area $4r^2$ contains square ABCD of area πr^2 *(assumed, yellow colour)*, which in turn contains the inscribed A"B"C"D" (with area 2*r, red colour in Figure 2 above*) of the given circle (O, r).

Therefore, square ABCD occupies the stretched area *(yellow colour in the above Figure 2)* in between squares A'B'C'D' & A''B''C''D'', as required.

β.) Let R be the radius of the inscribed circle (yellow dashes in above Figure 2) of the square ABCD, then $R = \frac{r\sqrt{\pi}}{2}$, which is half of the side $r\sqrt{\pi}$ of the square ABCD (yellow colour). And then $R = \frac{r\sqrt{\pi}}{2} < r$ to the result that the given circle (O, r) contains the inscribed circle (O, $R=\frac{r\sqrt{\pi}}{2}$) of ABCD, as $\frac{\sqrt{\pi}}{2} < 1$. Therefore the given circle (O, r) contains the circle (O, $\frac{r\sqrt{\pi}}{2}$), as required.

γ.) By (3), the area πr^2 of ABCD is less than the area $4r^2$ of A'B'C'D'. The circle (O, r) is the inscribed circle of square A'B'C'D'. Therefore, the 4 sides of square ABCD intersect the circle (O, r) to make 4 equal segments (*which are formed by arc chords and 4 small arcs of the circle (O, r), as described in the above Figure 2, - the red shape near the top of Figure 2 – attached to side AB - illustrates one of the mentioned circle segments*), as required.

2.3. Theorem 2: "Analysis Method."

Given a circle (O,r). If there exists a square ABCD with the area πr^2 (assumed), of which the centre is located at centre O of the circle (O,r), then 4 sides of the square ABCD are overlapped 4 non-consecutive sides of a regular octagon abcdefgh, which is inscribed in the circle (O, r).

***Proof*: "Analysis Method"**

Assume there exists a square ABCD of area πr^2, of which the centre is located at the centre of the given circle (O, r); then by section γ of the above Theorem 1, the 4 circle segments, formed by the circle (O, r) and 4 sides of ABCD, are equal.

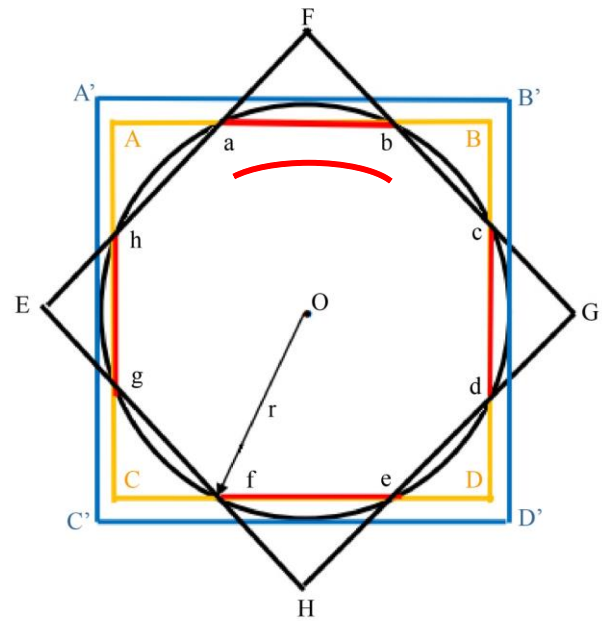


Fig. 3 abcdefgh is the regular octagon (red & black colours) inscribed in Circle (O, r).

Consider the area of Conical-Arc Aah in Figure 3 above (as defined in Definition 1 above) of corner A of the square ABCD. From the expression {area πr^2 of ABCD = area πr^2 of the circle (O, r)}, we get the expressions:

{the area of the Conical-Arc Aha = the area of the circle segment ab (red colour)} (4)

Similarly to (4),

{the area of the Conical-Arc Bbc = the area of the circle segment cd} (5)

{the area of the Conical-Arc Dde = the area of the circle segment ef} (6)

{the area of the Conical-Arc Cfg = the area of the circle segment gh} (7)

Note that all the above expressions (4), (5), (6) & (7) are illustrated in the above Figure 3.

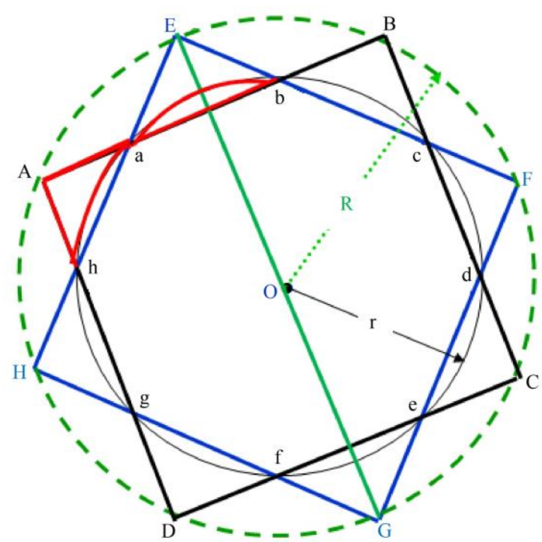


Fig. 4 2 inscribed squares ABCD (red and black colours) & EFGH (blue colour) of the circle (O, R)

Let centre O of the circumscribed circle (O, R) – green colour - of square ABCD be the same centre O as the given circle (O, r) – black colour. Then lengthen the arc chord ah of (O, r) in Figure 4 above that meets the circumscribed circle (O, R) – marked green dashes in Figure 4 above - at E and H. And then, connect the diameter of (O, R), which gets through E & O. From E, draw a symmetric chord to EH that meets the green dashes Circle (O, R) at F. The special octagon abcdefgh inscribed in the circle (O,r) with 4 equal & parallel side pairs, and Section γ of Theorem 1 shows that EF is the symmetric chord of EH through the symmetric EG-axis (green colour). From Section γ of Theorem 1 above, the distances between O and the 2 chords ha & bc are the same, and this equality shows chord EF (Figure 4) in the green dashes Circle (O, R) overlaps chord bc of the circle (O, r). Similarly, chord FG in the green dashes Circle (O, R) also overlaps chord de (Figure 4 above) of the given black circle (O, r). By Section γ of Theorem 1, FG // EH, then chords EF & GH of the green dashes Circle (O, R) are equal and parallel. This implies

$$EF = FG = GH = HE \qquad (8)$$

and

$$\text{EFGH (blue) is the inscribed square of the circle (O, R)} \qquad (9)$$

Then (8) and (9) show that the areas of the two squares ABCD & EFGH (blue) are the same and equal to πr^2. The sides of the equal squares ABCD (black) & EFGH (blue) above show 8 chords ab, bc, cd, de, ef, fg, gh & ha of the given black circle (O, r) are equal. Therefore, these 8 equal chords show the shape abcdefgh is a regular octagon that inscribes in the given circle (O, r), as required.

2.4 Definition 2: 2D-SQUARING RULER

Given a circle (O, r) with an area πr^2, then a regular octagon, inscribed in the circle (O, r), is defined as a 2D-Squaring Ruler (colour red in Figure 5 below).

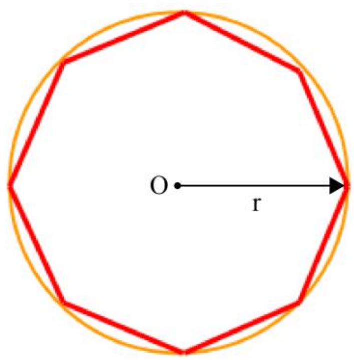


Fig. 5 a 2D-Squaring ruler (red) of the circle (o,r) (organg)

2.5. Theorem 3:

Given a circle (O, r) with the area πr^2, then the square ABCD of an area πr^2 is formed by lengthening the 4 non-consecutive sides of the regular octagon abcdefgh (red colour in Figure 6 below), which is the inscribed regular octagon of the circle (this circle O is the circumscribed circle of the octagon abcdefgh, which is defined 2D-Squaring Ruler in Definition 2 above).

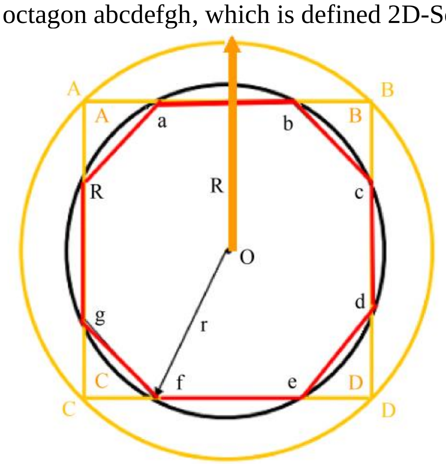


Fig. 6 The given circle(O,r) and the regular octagon abcdefgh

Proof:

Given a circle (O, r), then by Definition 2 above, we can construct a 2D-Squaring Ruler abcdefgh (red colour) inscribed in the circle (back colour in Figure 6 above). And then, by Theorem 2 & Theorem 3 above, the square ABCD (coloured orange in Figure 6 above) is square with the area πr^2, which is equal to the area πr^2 of the given circle. Thus the circle (O, r) is squared exactly and accurately into the square ABCD (yellow colour), as required.

3. Exact "Squaring The Circle" method with Straightedge and Compass

Given a circle (O, r) with the area πr^2, a straightedge and a compass.

Step 1: Use the given straightedge & compass to construct a 2D-Squaring Ruler, which is exactly a regular octagon (red colour in Figure 7 below), inscribed in the given circle (O, r).

Step 2: Lengthen any 2 pairs of the parallel sides of the 2D-Squaring Ruler (red colour in Figure 7 below), which meet at 4 points A, B, C and D. Then ABCD (yellow colour in Figure 7 below) is square with the area πr^2, as required.

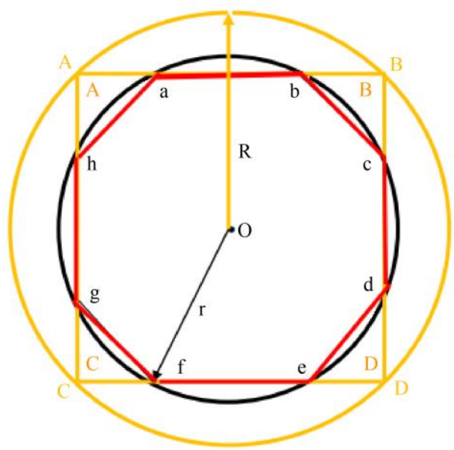


Fig. 7 Square ABCD is a result of the exact "Squaring the circle"

Results of this research show that the square ABCD has the exact area πr^2*;* therefore, if the given circle to the square is a unit circle r = 1, then in terms of Geometry, π can exactly be constructive / expressed by a square with an area π. This square comes from the *2D-Squaring Ruler* of the unit circle (O, 1), by *Step 1 & Step 2*, above.

Moreover,

- In above Figure 6, AB = $\sqrt{\pi r^2}$ = $r\sqrt{\pi}$. Then if the given circle to the square is a unit circle r = 1, then in terms of Algebraic Geometry, $\sqrt{\pi}$ can be exactly constructive or expressed by a geometric LENGTH. This length is a side of the square resulting from the solution of "squaring the unit circle (O, 1) problem" by *Step 1 & Step 2* above.
- We can find out a geometric length of the radius R of the circumscribed circle (O, R) of square ABCD (coloured yellow in Figure 7 above), in terms of π, as follows:
  From the right angle triangle OAB in Figure 7 above, we get $R^2 + R^2 = \pi r^2$; then

$$R = \frac{r\sqrt{2\pi}}{2}$$

4. Discussion and Conclusion

Can mathematicians use a compass and a straightedge to construct a square of equal area to a given circle? Surprisingly, mathematicians are still working on this question. In January 2022, a paper posted online by Andras Máthé and Oleg Pikhurko of the University of Warwick and Jonathan Noel of the University of Victoria was the latest to join in this ancient tradition challenge. These authors show how a circle can be squared by cutting it into pieces that can be visualized and possibly drawn. It's a result that builds on a rich history. Mathematicians named this method *"the equidecomposition",* but it is also theoretical proof that the problem can be solved (without a straightedge & compass) by cutting the circle into pieces and rearranging it into a square, and none knows the number of pieces. Nevertheless, no computer existed in the ancient Greek era.

Results of my independent research show that the square ABCD, constructed by compass & straightedge, has the exact area πr^2, therefore if the given circle to the square is a unit circle r = 1, then in terms of Geometry, π can exactly be constructive / expressed by a square with an area π. This square comes from the *2D-Squaring Ruler* of the unit circle (O,1), by Section 1. and Section 2., above.

Moreover,

- In above Figure 6, $AB = \sqrt{\pi r^2} = r\sqrt{\pi}$. Then if the given circle to the square is a unit circle r = 1, then in terms of Algebraic Geometry, $\sqrt{\pi}$ can be exactly constructive or expressed by a geometric LENGTH. This length is a side of the square that comes from the solution of "squaring the unit circle (O, 1) problem" by Section 1. & Section 2., above.
- We can find out a geometric length – in term of π - for the radius R of the circumscribed circle (O, R) of ABCD (coloured yellow in Figure 6 and Figure 7 above), as follow:

$$R = \frac{r\sqrt{2\pi}}{2}$$

- This research result is also a new method to calculate the arithmetic value of π as follows: with an accurate length of 2 metres given by the *International Bureau of Weights and Measures (BIPM)* or the *International System of Units,* we use a compass to construct a circle, of which area is accurately 4π. Then use a straightedge & a compass to construct an accurate square, of which area is also accurately 4π, as proved above. Laser measurement can be used to measure as much accurately as possible to have the arithmetic value of 4π, say, it is equal to A. And then,

$$4\pi = A \quad => \quad \pi = \frac{A}{4}$$

The above arithmetic value of π could be the nearest value of π we have ever seen.

My construction method is quite different from approximation and based on the use of a straightedge and a compass within secondary Geometry so that any secondary student can solve this problem for any given circle. Moreover, the method shows that. $\sqrt{\pi}$ can be expressed accurately in 1-D space and π can be expressed accurately in 2-D space in terms of Geometry. This Geometrical expression of the irrational number π could be an interesting field for mathematicians in the 21st Century. In other words, algebraic geometry can express exactly any irrational number kπ, $k \subset R$.

Upstream from this method of exact "squaring the circle", we can deduce, conversely, to get a new mathematical challenge "Circling the Square" with a straightedge & a compass. In detail, we can describe it as follows:

Given a square with side a, $a \subset R$, then use a straightedge & a compass to construct an accurate circle, which has the exact area a^2. Then, how to solve this new geometry problem is still an open research interest in order to get a circle area without the traditional constant π.

In addition, this research result can be used for further research in the "CUBING THE SPHERE" challenge, with only "a straightedge & a compass" using only secondary Geometry.

5. Acknowledgements

I greatly acknowledge the constructive suggestions by the friends and reviewers who took part in the evaluation of the developed theorems.

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