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*Original Article*

# Existence and Uniqueness Solutions of System Caputo-Type Fractional-Order Boundary Value Problems using Monotone Iterative Method

Audubmbar Kumar Mule[1], Sidheshwar S. Bellale[2]

[1]*Department of Mathematics, Dayanand Science College, Latur, India.*
[2]*Mathematics Research center, Dayanand Science College, Latur, India.*

[1]*Corresponding Author : audumbar21@gmail.com*

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***Abstract*** *- In this paper, we investigate the existence and uniqueness solutions of nonlinear boundary value problems for a system of Caputo-type nonlinear fractional differential equations of the form:*

$$\begin{cases} {}^{c}D_{a^+}^{q;\psi} u_i(t) = F_i\big(t, u_1(t), u_2(t)\big) \quad t \in J = [a,b], \\ \phi\big(v_i(a), v_i(b)\big) = 0. \end{cases}$$

*To develop a monotone iterative technique by introducing upper and lower solutions to Caputo-type fractional differential equations with nonlinear boundary conditions. The monotone method yields monotone sequences which converge uniformly and monotonically to extremal solutions.*

***Keywords*** *- $\psi$-Caputo fractional derivative, Upper and lower solutions, Monotone iterative method.*

***Mathematics Subject Classification(2020)*** *- 26A33, 26A48, 34A08.*

## 1. Introduction

Fractional differential equations or fractional differential systems have numerous applications in diverse and widespread fields of science and technology [4, 10, 22]. The study of fractional calculus and its applications see more details [12, 13, 20]. The approach to obtain the existence and uniqueness of solutions for the nonlinear fractional differential systems, in general, has been through the fixed point theorem method [3, 6, 9, 16, 23, 24, 25, 26]. In this paper investigates the existence and uniqueness using the method of lower and upper solutions combined with the monotone iterative technique [5, 8, 24, 28, 29].

The monotone method is useful for nonlinear equations and systems because it reduces the problem to sequences of linear equations. Specifically, if the nonlinear system is unwieldy and too difficult to solve explicitly, then the monotone method may be beneficial. If one can find upper and lower solutions to the original system that are less unwieldy and satisfy the particular requirements, then the monotone method implements a technique for constricting sequences from these upper and lower solutions. These sequences are solutions to linear equations and converge uniformly and monotonically to maximal and minimal solutions [11, 12, 14, 16, 17, 18, 19, 25].

Motivated by the work see [7], we determine the existence criteria of extremal solution for following system $\psi$-Caputo type fractional differential equations in a Caputo sense with nonlinear boundary conditions

$$\begin{cases} {}^{c}D_{a^+}^{q;\psi} u_i(t) = F_i\big(t, u_1(t), u_2(t)\big) \quad t \in J = [a,b], \\ \phi\big(v_i(a), v_i(b)\big) = 0. \end{cases}$$

The rest of the paper is arranged in the following way.

In section 2, definitions and basic results are discussed that play a vital role in the main results. These results are useful in main results proving that the sequences developed in the generalized monotone method converge to coupled minimal and maximal solutions of the non-linear system of the fractional differential equation. Finally, under the uniqueness assumption, we prove that there exists a unique solution to the non-linear system of $\psi$-Caputo fractional differential equation.

## 2. Definitions and Basic Results

In this section, we recall some known definitions and known results which are useful to develop our main result.

**Definition 2.1** [4, 1] The $\psi$-Riemann-Liouville fractional integral of order $q$ is defined by

$$I_{a+}^{q;\psi}u(t) = \frac{1}{\Gamma(q)}\int_0^t \psi'\left(\psi(t)-\psi(s)\right)^{q-1}u(s)ds, t > a.$$

**Definition 2.2** [1] Let $\psi, u \in C^n(J,\mathbb{R})$. The $\psi$-Riemann-Liouville derivative of the order of a function $u$ with $(n-1 < q \leq n)$ can be written as

$$\begin{aligned} D_{a+}^{q;\psi}u(t) &= \left(\frac{D_t}{\psi'(t)}\right)^n I_{a+}^{n-q;\psi}u(t) \\ &= \frac{1}{\Gamma(n-q)}\left(\frac{D_t}{\psi'(t)}\right)^n \int_0^t \psi'\left(\psi(t)-\psi(s)\right)^{n-q-1}u(s)ds, \end{aligned}$$

where $n = [q]+1 (n \in \mathbb{N})$, and $D_t = \frac{d}{dt}$.

**Definition 2.2** [1] Let $\psi, u \in C^n(J,\mathbb{R})$. The $\psi$-Caputo derivative of the order of a function $u$ with $(n-1 < q \leq n)$ can be written as

$$D_{a+}^{q;\psi}u(t) = I_{a+}^{n-q;\psi}u_\psi^{[n]}(t)$$

where $u_\psi^{[n]}(t) = \left(\frac{D_t}{\psi'(t)}\right)^n u(t)$, $n = [q]+1$ for $q \notin \mathbb{N}$ and $n = q$ for $q \in \mathbb{N}$.

One has

$$^cD_{a^+}^{q;\psi}u(t) = \begin{cases} \int_0^t \psi'\left(\psi(t)-\psi(s)\right)^{n-q-1}u_\psi^{[n]}(s)ds, & if q \notin \mathbb{N}, \\ u_\psi^{[n]}(t), & if q \in \mathbb{N} \end{cases}$$

**Definition 2.3** [2] One and two-parameter Mittag-Leffler function is defined as

$$E_q(t) = \sum_{k=0}^{\infty}\frac{(t)^k}{\Gamma(qk+1)} \quad t \in \mathbb{R}, q > 0$$

$$E_{q,\beta}(t) = \sum_{k=0}^{\infty}\frac{(t)^k}{\Gamma(qk+\beta)} \quad q,\beta > 0, t \in \mathbb{R}$$

**Lemma 2.1** [1] Let $p, q > 0$, and $u \in C(J,\mathbb{R})$, for every $t \in J$
i. $^cD_{a^+}^{q;\psi} I_{a^+}^{q;\psi}u(t) = u(t)$,
ii. $I_{a^+}^{q;\psi}\, {}^cD_{a^+}^{q;\psi}u(t) = u(t) - u(a)$, $0 < q \leq 1$.

iii. $I_{a^+}^{q;\psi}\big(\psi(t)-\psi(a)\big)^{p-1}=\frac{\Gamma(p)}{\Gamma(p-q)}\big(\psi(t)-\psi(a)\big)^{p+q-1}$,
iv. ${}^cD_{a^+}^{q;\psi}\big(\psi(t)-\psi(a)\big)^{p-1}=\frac{\Gamma(p)}{\Gamma(p-q)}\big(\psi(t)-\psi(a)\big)^{p-q-1}$,
v. ${}^cD_{a^+}^{q;\psi}\big(\psi(t)-\psi(a)\big)^{k}=0, \forall k<n\in\mathbb{N}$

**Lemma 2.2** [27] Let $q\in(0,1)$ and $x\in\mathbb{R}$, one has i. $E_{q,1}$ and $E_{q,q}$ are non-negative. ii. $E_{q,1}(x)\le 1, E_{q,q}(x)\le\frac{1}{\Gamma(q)}$, for any $x<0$.

**Lemma 2.3** [7] Let $q\in(0,1)$, $\lambda\in\mathbb{R}$ and $g\in C(J,\mathbb{R})$, then the linear problem

$$\begin{cases} {}^cD_{a^+}^{q;\psi}u(t)+\lambda u(t)=g(t), & t\in J. \\ u(a)=u_a, \end{cases}$$

has a unique solution as

$$\begin{aligned} u(t) &= u_aE_{q,1}\big(-\lambda\big(\psi(t)-\psi(a)\big)^q\big) \\ &+\int_0^t\psi'\,\big(\psi(t)-\psi(s)\big)^{q-1}E_{q,q}\big(-\lambda\big(\psi(t)-\psi(a)\big)^q\big)g(s)ds, \end{aligned}$$

where $E_{p,q}(.)$ is the two-parametric Mittag-Leffler function

**Lemma 2.4** [Comprising result] [7]Let $q\in(0,1)$ and $\lambda\in\mathbb{R}$ if $\gamma\in C(J,\mathbb{R})$,

$$\begin{cases} {}^cD_{a^+}^{q;\psi}\gamma(t)\ge-\lambda\gamma(t), & t\in(a,b]. \\ \gamma(a)\ge 0, \end{cases}$$

then $\gamma(t)\ge 0$ for all $t\in J$.

## 3. Main Results

In this section, we develop a monotone method for the system $\psi$-Caputo fractional differential equations (3.7) using coupled lower and upper solutions, respectively.

**Definition 3.1** The functions $f_i\in C(J,\mathbb{R}$ such that ${}^cD_{a^+}^{q;\psi}f_i(t)$ exist and is continuous on $J$ and is known to be a solution (1.1).Further,$f_i$ gives the statistics of the equation ${}^cD_{a^+}^{q;\psi}u_i(t)=F_i\big(t,u_1(t),u_2(t)\big)$, for each $t\in J$ and the nonlinear boundary conditions

$$\phi\big(f_i(a),f_i(b)\big)=0$$

**Definition 3.2** If the functions $v_i(x,t),\quad w_i(x,t)\in C^{2,q}[Q_T,\mathbb{R}]$ are called the lower and upper solutions of if

$$\begin{cases} {}^cD_{a^+}^{q;\psi}v_i(t)\le F_i\big(t,v_1(t),v_2(t)\big) & t\in[a,b], \\ \phi\big(v_i(a),v_i(b)\big)\le 0 \end{cases}$$

$$\begin{cases} {}^cD_{a^+}^{q;\psi}w_i(t)\ge F_i\big(t,w_1(t),w_2(t)\big) & t\in[a,b], \\ \phi\big(w_i(a),w_i(b)\big)\ge 0 \end{cases}$$

**Theorem 3.1** Let $F:J\times\mathbb{R}\to\mathbb{R}$ be continuous. Assume that
(i) There exist $v_i(t)$ and $w_i(t)$ as lower and upper solutions of problem (1.1) in $C(J,\mathbb{R})$ respectively, with $v_i(t)\le w_i(t), t\in J$.
(ii) There exists a constant $k_i>0$ with

$$F_i(t,u_2) - F_i(t,u_1) \geq -k_i(u_2 - u_1) \quad for \quad v_i(t) \leq u_1 \leq u_2 \leq w_i(t), t \in J$$

(iii) There exists non-negative constants $M, N$ with $v_i(a) \leq x_1 \leq x_2 \leq w_i(a)$, $v_i(b) \leq y_1 \leq y_2 \leq w_i(b)$, such that

$$\phi_i(x_2, y_2) - \phi_i(x_1, y_1 \leq M(x_2 - x_1) - N(y_2 - y_1)$$

Then there exist monotone sequences $\{v_i^n(t)\}$ and $\{w_i^n(t)\}$ such that $v_i^n(t) \to v_i(t)$ and $w_i^n(t) \to w_i(t)$ as $n \to \infty$ uniformly on $J$, to the extremal solutions of (1.1) in the sector $[v_i, w_i]$ where

$$= \{u_i \in C(J, \mathbb{R}) : v_i(t) \leq u_i(t) \leq w_i(t), t \in J\}$$

**Proof.**

We construct the sequences $\{v_i^{n+1}(t)\}$ and $\{w_i^{n+1}(t)\}$ and $k_i > 0$, we consider the following fractional differential equations

$$\begin{cases} {}^cD_{a^+}^{q;\psi} v_i^{n+1}(t) = F_i(t, v_i^n(t)) - k\left(v_i^{n+1}(t) - v_i^n(t)\right) & t \in J, \\ v_i^{n+1}(a) = v_i^n(a) - \frac{1}{c}\phi\left(v_i^n(a), v_i^n(b)\right) \end{cases} \tag{3.1}$$

$$\begin{cases} {}^cD_{a^+}^{q;\psi} w_i^{n+1}(t) = F_i(t, w_i^n(t)) - k\left(w_i^{n+1}(t) - w_i^n(t)\right) & t \in J, \\ w_i^{n+1}(a) = w_i^n(a)) - \frac{1}{c}\phi\left(w_i^n(a), w_i^n(b)\right) \end{cases} \tag{3.2}$$

By Lemma 3 and equation (3.1),(3.2)preserve at most one solution in $C(J, \mathbb{R}$ we have

$$v_i^{n+1}(t) = \left(v_i^n(a)) - \frac{1}{c}\phi\left(v_i^n(a), v_i^n(b)\right)\right) E_{q,1}\left(-k_i\left((\psi)_i(t) - \psi_i(a)\right)^q\right)$$
$$+ \int_a^t \psi_i'(s)\left(\psi_i(t) - \psi_i(a)\right)^{q-1} E_{q,q}\left(-k_i\left(\psi_i(t) - \psi_i(s)\right)^q\right)\left(F_i\left(s, v_i^n(s)\right) + k_i\left(v_i^{n+1}(s)\right)\right) ds \quad t \in J,$$

$$w_i^{n+1}(t) = \left(w_i^n(a)) - \frac{1}{c}\phi\left(w_i^n(a), w_i^n(b)\right)\right) E_{q,1}\left(-k_i\left((\psi)_i(t) - \psi_i(a)\right)^q\right)$$
$$+ \int_a^t \psi_i'(s)\left(\psi_i(t) - \psi_i(a)\right)^{q-1} E_{q,q}\left(-k_i\left(\psi_i(t) - \psi_i(s)\right)^q\right)\left(F_i\left(s, w_i^n(s)\right) + k_i\left(w_i^{n+1}(s)\right)\right) ds \quad t \in J.$$

**Step 1**: The sequences $\{v_i^{n+1}(t)\}, \{w_i^{n+1}(t)\}$ $(n \geq 1)$ are lower and upper solutions of (), respectively. We prove that $v_i^0(t) \leq v_i^1(t)$. Let $\rho_i(t) = v_i^1(t) - v_i^0(t)$. Then equation (3.1) and Definition 3.2, we have

$$\begin{aligned} {}^cD_{a^+}^{q;\psi}\rho_i(t) &\overset{c}{=} D_{a^+}^{q;\psi} v_i^1(t) \overset{c}{-} D_{a^+}^{q;\psi} v_i^0(t) \\ &\geq F_i\left(t, v_i^0(t)\right) - k_i\left(v_i^1(t) - v_i^0(t)\right) - F_i\left(t, v_i^0(t)\right) \\ &= -k_i\rho_i(t). \end{aligned}$$

Since $\rho_i(a) = -\frac{1}{c}\phi\left(v_i^0(a), v_i^0(b)\right) \geq 0$, $\rho_i(t) \geq 0$, for $t \in J$ by Lemma 4. Thus $v_i^0(t) \leq v_i^1(t)$. Assume that $v_i^{k-1}(t) \leq v_i^k(t)$. Now we show that $v_i^k(t) \leq v_i^{k+1}(t)$. Let $\rho_i(t) = v_i^k(t) - v_i^{k+1}(t)$

$$\begin{aligned} {}^cD_{a^+}^{q;\psi}\rho_i(t) &\overset{c}{=} D_{a^+}^{q;\psi} v_i^k(t) \overset{c}{-} D_{a^+}^{q;\psi} v_i^{k+1}(t) \\ &\geq F_i\left(t, v_i^k(t)\right) - k_i\left(v_i^{k+1}(t) - v_i^k(t)\right) - F_i\left(t, v_i^k(t)\right) \\ &= -k_i\rho_i(t). \end{aligned}$$

Since $\rho_i(a) = -\frac{1}{c}\phi\left(v_i^k(a), v_i^k(b)\right) \geq 0$, $\rho_i(t) \geq 0$, for $t \in J$ by Lemma 4. Thus $v_i^k(t) \leq v_i^{k+1}(t)$. Hence by mathematical induction, we have

$$v_i^0(t) \leq v_i^1(t) \leq \ldots \leq v_i^k(t) \leq v_i^{k+1}(t) \leq \ldots \leq v_i^n(t) \tag{3.3}$$

Next, we prove that $w_i^1(t) - w_i^0(t)$, $t \in J$. Let $\rho_i(t) = w_i^0(t) - w_i^1(t)$. Then equation (3.1) and Definition 3.2, we have

$$\begin{aligned} {}^cD_{a^+}^{q;\psi}\rho_i(t) &\overset{c}{=} D_{a^+}^{q;\psi}w_i^0(t) \overset{c}{-} D_{a^+}^{q;\psi}w_i^1(t) \\ &\geq F_i\left(t, w_i^1(t)\right) - k_i\left(w_i^0(t) - w_i^1(t)\right) - F_i\left(t, w_i^1(t)\right) \\ &= -k_i\rho_i(t). \end{aligned}$$

Since $\rho_i(a) = -\frac{1}{c}\phi\left(w_i^0(a), w_i^0(b)\right) \geq 0$, $\rho_i(t) \geq 0$, for $t \in J$ by Lemma 4. Thus $w_i^1(t) \leq w_i^0(t)$. Assume that $w_i^k(t) \leq w_i^{k-1}(t)$. Now we show that $w_i^{k+1}(t) \leq w_i^k(t)$. Let $\rho_i(t) = w_i^{k+1}(t) - w_i^k(t)$

$$\begin{aligned} {}^cD_{a^+}^{q;\psi}\rho_i(t) &\overset{c}{=} D_{a^+}^{q;\psi}w_i^{k+1}(t) \overset{c}{-} D_{a^+}^{q;\psi}w_i^k(t) \\ &\leq F_i\left(t, w_i^{k+1}(t)\right) - k_i\left(w_i^{k+1}(t) - w_i^k(t)\right) - F_i\left(t, w_i^{k+1}(t)\right) \\ &= -k_i\rho_i(t). \end{aligned}$$

Since $\rho_i(a) = -\frac{1}{c}\phi\left(w_i^{k+1}(a), w_i^{k+1}(b)\right) \geq 0$, $\rho_i(t) \geq 0$, for $t \in J$ by Lemma 4. Thus $w_i^{k+1}(t) \leq w_i^k(t)$. Hence by mathematical induction, we have

$$w_i^n(t) \leq w_i^{n-1}(t) \leq \ldots \leq w_i^k(t) \leq w_i^{k-1}(t) \leq \ldots \leq w_i^1(t) \leq w_i^0(t) \tag{3.4}$$

Now to Prove that $v_i^1(t) \leq w_i^1(t)$. Let $\rho_i(t) = w_i^1(t) - v_i^1(t)$. Using equations (3.1) and (3.2) together with assumptions (ii) and (iii), we have

$$\begin{aligned} {}^cD_{a^+}^{q;\psi}\rho_i(t) &= F_i\left(t, w_i^0(t)\right) - F_i\left(t, v_i^0(t)\right) - k_i\left(w_i^1(t) - w_i^0(t)\right) + k_i(v_i^1(t) - v_i^0) \\ &\geq -k_i\left(w_i^0(t) - v_i^0(t)\right) - k_i\left(w_i^1(t) - w_i^0(t)\right) + k_i\left(v_i^1(t) - v_i^0(t)\right) \\ &= -k_i\rho_i(t). \end{aligned}$$

Since

$$\begin{aligned} \rho_i(a) &= \left(w_i^0(a) - v_i^0(t)\right) - \frac{1}{c}\left(\phi\left(w_i^0(a), w_i^0(b)\right) - \phi\left(v_i^0(a), v_i^0(b)\right)\right) \\ &\geq \frac{d}{c}((w_i^0(b) - v_i^0(b) \\ &\geq 0, \end{aligned}$$

we have $v_i^1(t) \leq w_i^1(t)$, $t \in J$ by Lemma 4. Hence $v_i^0(t) \leq v_i^1(t) \leq w_i^1(t) \leq w_i^0(t)$.

By mathematical inductions and equations (3.3) and (3.4), we get

$$v_i^0(t) \leq v_i^1(t) \leq \ldots \leq v_i^n(t) \leq w_i^n(t) \leq \ldots \leq w_i^1(t) \leq w_i^0(t) \tag{3.5}$$

We prove that $v_i^0(t), w_i^0(t)$ are extremum solutions of (1.1). Since $v_i^0$ and $w_i^0$ are lower and upper solutions of (1.1), assumptions (ii)and (iii), we get

$$
\begin{aligned}
{}^{c}D_{a^+}^{q;\psi} v_i^0(t) &= F_i\left(t, v_i^0(t)\right) - k_i\left(v_i^1(t) - v_i^0(t)\right)\\
&\leq F_i\left(t, v_i^1(t)\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\phi\left(v_i^1(a), v_i^1(b)\right) &\leq \phi(v_i^0(a), v_i^0(b) + c\left(v_i^1(a) - v_i^0(a)\right) - d\left(v_i^1(b) - v_i^0(b)\right)\\
&= -d\left(v_i^1(b) - v_i^0(b)\right)\\
&\leq 0.
\end{aligned}
$$

$$
\begin{aligned}
{}^{c}D_{a^+}^{q;\psi} w_i^0(t) &= F_i\left(t, w_i^0(t)\right) - k_i\left(w_i^1(t) - w_i^0(t)\right)\\
&\geq F_i\left(t, w_i^1(t)\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\phi\left(w_i^1(a), w_i^1(b)\right) &\geq \phi(w_i^0(a), w_i^0(b) + c\left(w_i^1(a) - w_i^0(a)\right) - d\left(w_i^1(b) - w_i^0(b)\right)\\
&= -d\left(w_i^1(b) - w_i^0(b)\right)\\
&\geq 0.
\end{aligned}
$$

Therefore, $v_i^1(t)$ ,$w_i^1(t)$ is the lower and upper solution of (1.1), respectively. By induction, Hence $v_i^n(t), w_i^n(t)$ are lower and upper solutions of (1.1), respectively.

**Step 2**:$v_i^n \to v_i$ and $w_i^n \to w_i$

First, we prove that $\{v_i^n\}$ is uniformly bounded. By considering supposition Hypothesis 2, we have

$$
F_i\left(t, v_i^0(t)\right) + k_i v_i^0(t) \leq F_i\left(t, v_i^n(t)\right) + k_i v_i^n(t) \leq F_i\left(t, w_i^0(t)\right) + k_i w_i^0(t), \quad t \in J
$$

That is

$$
\begin{aligned}
0 &\leq F_i\left(t, v_i^n(t)\right) - F_i\left(t, v_i^0(t)\right) + k_i\left(v_i^n(t) - v_i^0(t)\right)\\
&\leq F_i\left(t, w_i^0(t)\right) - F_i\left(t, v_i^0(t)\right) + k_i\left(w_i^0(t) - v_i^0(t)\right)
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
|F_i\left(t, v_i^n(t)\right) - F_i\left(t, v_i^0(t)\right) + k_i\left(v_i^n(t) - v_i^0(t)\right)| &\leq |F_i\left(t, w_i^0(t)\right) - F_i\left(t, v_i^0(t)\right)\\
&\quad + k_i\left(w_i^0(t) - v_i^0(t)\right)|.
\end{aligned}
$$

Thus

$$
\begin{aligned}
|F_i\left(t, v_i^n(t)\right) + k_i(v_i^n(t)| &\leq |F_i\left(t, v_i^n(t)\right) - F_i\left(t, v_i^0(t)\right) + k_i\left(v_i^n(t) - v_i^0(t)\right)|\\
&\quad + |F_i\left(t, v_i^0(t)\right) + k_i(v_i^0(t)|\\
&\leq |F_i\left(t, w_i^0(t)\right) - F_i\left(t, v_i^0(t)\right) + k_i\left(w_i^0(t) - v_i^0(t)\right)|\\
&\quad + |F_i\left(t, v_i^0(t)\right) + k_i(v_i^0(t)|\\
&\quad + \leq 2|F_i\left(t, v_i^0(t)\right) + k_i(v_i^0(t)| + |F_i\left(t, v_i^0(t)\right) + k_i(v_i^0(t)|.
\end{aligned}
$$

Since $v_i^0, F_i$ are continuous on $J$, we can see a constant $C$ independent of $n$ with

$$|F_i\big(t, v_i^n(t)\big) + k_i(v_i^n(t)| \leq C \tag{3.6}$$

Furthermore, from Hypothesis 3, we have

$$v_i^0(a) - \frac{1}{c}\phi\left(v_i^0(a), v_i^0(b)\right) \leq v_i^n(a) - \frac{1}{c}\phi\left(w_i^0(a), w_i^0(b)\right) \leq w_i^0(a) - \frac{1}{c}\phi\left(v_i^0(a), v_i^0(b)\right)$$

That is

$$\begin{aligned} 0 \quad &\leq v_i^n(a) - v_i^0(a) - \frac{1}{c}\phi\big(v_i^n(a), v_i^n(b)\big) - \phi\left(v_i^0(a), v_i^0(b)\right) \\ &\leq w_i^0(a) - v_i^0(a) - \frac{1}{c}\phi\left(w_i^n(0), w_i^0(b)\right) - \phi\left(v_i^0(a), v_i^0(b)\right). \end{aligned}$$

Hence, we have

$$\begin{aligned} &|v_i^n(a) - v_i^0(a) - \frac{1}{c}\phi\big(v_i^n(a), v_i^n(b)\big) - \phi\left(v_i^0(a), v_i^0(b)\right)| \\ &\leq |v_i^n(a) - v_i^0(a) - \frac{1}{c}\phi\big(v_i^n(a), v_i^n(b)\big) - \phi\left(v_i^0(a), v_i^0(b)\right)| \\ &\leq |v_i^n(a) - v_i^0(a) - \frac{1}{c}\phi\big(v_i^n(a), v_i^n(b)\big) - \phi\left(v_i^0(a), v_i^0(b)\right)|. \end{aligned}$$

Thus

$$\begin{aligned} |v_i^n(a) - \frac{1}{c}\phi\big(v_i^n(a), v_i^n(b)\big)| \quad &\leq |v_i^n(a) - v_i^0(a) - \frac{1}{c}\phi\big(v_i^n(a), v_i^n(b)\big) - \phi\left(v_i^0(a), v_i^0(b)\right)| \\ &+ |v_i^0(a) - \frac{1}{c}\phi\left(v_i^0(a), v_i^0(b)\right)| \\ &\leq 2|v_i^0(a) - \frac{1}{c}\phi\left(v_i^0(a), v_i^0(b)\right)| + |w_i^0(a) - \frac{1}{c}\phi\left(w_i^0(a), w_i^0(b)\right)|. - \end{aligned}$$

Since $v_i^0, w_i^0$ and $\phi$ are continuous functions; we can see a constant $D$ independent of $n$ with

$$|v_i^n(a) - \frac{1}{c}\phi\big(v_i^n(a), v_i^n(b)\big)| \leq D \tag{3.7}$$

Moreover, by (3.1) and (3.2), we have

$$\begin{aligned} |v_i^{n+1}(t)| \quad &= |v_i^n(a) - \frac{1}{c}\phi\big(v_i^n(a), v_i^n(b)\big)| E_{q,1}\big(-k_i\big(\psi(t) - \psi(a)\big)^q\big) \\ &+ \int_a^t \psi'(s)\big(\psi(t) - \psi(s)\big)^{v-1} E_{q,q}\left(-k_i\big(\psi(t) - \psi(s)\big)\right)^q |F(s, v_i^n(s) + k_i v_i^n(t)| ds, \end{aligned}$$

Using Lemma 2 along with (3.6) and (3.7), we have

$$\begin{aligned} |v_i^{n+1}(t)| \quad &= D + \frac{C}{\Gamma(q)}\int_a^t \psi'(s)\big(\psi(t) - \psi(s)\big)^{v-1} ds, \\ &\leq D + \frac{C\big(\psi(t) - \psi(s)\big)^q}{\Gamma(q+1)}. \end{aligned}$$

Hence, $v_i^n$ is uniformly bounded in $C(J, \mathbb{R})$. Similarly $w_i^n$ is uniformly bounded $C(J, \mathbb{R})$. Next, to prove that the sequence $v_i^n$ and $w_i^n$ are equi-continuous on $J$. Choosing $t_1, t_2 \in J$, with $t_1 \leq t_2$. By (3.6),(3.7) and Lemma 2, we have

$$
\begin{aligned}
|v_i^{n+1}(t_2) - v_i^{n+1}(t_1)| \quad & \leq |v_i^n(a) - \frac{1}{c}\phi\big(v_i^n(a), v_i^n(b)\big)||E_{q,1}\big(-k_i\big(\psi(t_2)-\psi(a)\big)^q\big) \\
& -E_{q,1}\big(-k_i\big(\psi(t_1)-\psi(a)\big)^q\big)| \\
& \leq \int_a^{t_1} \frac{\psi'(s)\left[\big(\psi(t_1)-\psi(s)\big)^{v-1} - \big(\psi(t_2)-\psi(s)\big)^{v-1}\right]}{\gamma(q)} |F(s, v_i^n(s) + k_i v_i^n(s)|ds \\
& + \int_{t_1}^{t_2} \frac{\psi'(s)\left[\big(\psi(t_2)-\psi(s)\big)^{v-1}\right]}{\Gamma(q)} |F(s, v_i^n(s) + k_i v_i^n(s)|ds \\
& \leq D|E_{q,1}\big(-k_i\big(\psi(t_2)-\psi(a)\big)^q\big) - E_{q,1}\big(-k_i\big(\psi(t_1)-\psi(a)\big)^q\big)| \\
& + \frac{2C\big(\psi(t_2)-\psi(t_1)\big)^q}{\Gamma(q+1)}.
\end{aligned}
$$

By the continuity of $E_{q,1}\big(-k_i\big(\psi(t_1)-\psi(a)\big)^q\big)$ on $J$, the right-hand-side of the preceding inequality approaches zero, when $t_1 \to t_2$. This implies that $\{v_i^{n+1}(t)\}$ is equi-continuous on $J$. Similarly $\{w_i^{n+1}(t)\}$ is equi-continuous on $J$. Hence, by using the Ascoli-Arzelas theorem, the subsequences converge to $v_i^*(t)$ and $w_i^*(t)$. Hence the monotonic sequences combined with $v_i^n(t)$ and $w_i^n(t)$ yields $\lim_{n\to\infty} v_i^n(t) = v_i^*(t)$ and $\lim_{n\to\infty} w_i^n(t) = w_i^*(t)$, uniformly on $t \in J$ and limit functions $v_i^*, w_i^*$ satisfy (1.1)

**Step 3:** $v_i^*$ and $w_i^*$ are maximal solutions of (1.1)in $[v_i^0, w_i^0]$ Let $u_i \in [v_i^0, w_i^0]$ be any solution of (1.1). Suppose that

$$v_i^n(t) \leq u_i(t) \leq w_i^n(t), \quad t \in J \tag{3.8}$$

for some $n \in \mathbb{N}$. To prove that $u_i(t) \leq v_i^n(t)$ Let $\rho_i(t) = u_i(t) - v_i^n(t)$. Then from, we have

$$
\begin{aligned}
{}^cD_{a^+}^{q;\psi}\rho_i(t) \quad & = F_i\big(t, u_i(t)\big) - F_i\Big(t, v_i^0(t)\Big) - k_i\Big(v_i^{n+1}(t) - v_i^n(t)\Big) \\
& \geq -k_i\big({}_i(t) - v_i^n(t)\big) + k_i\Big(v_i^{n+1}(t) - v_i^n(t)\Big) \\
& = -k_i\rho_i(t).
\end{aligned}
$$

Furthermore

$$
\begin{aligned}
v_i^{n+1}(a) \quad & = (v_i^n(a) - \frac{1}{c}(\phi\big(v_i^n(a), v_i^n(b)\big) \\
& = (v_i^n(a) - \frac{1}{c}(\phi\big(u_i(a), u_i(b)\big) - \frac{1}{c}(\phi\big(v_i^n(a), v_i^n(b)\big) \\
& \leq u_i(a) - \frac{d}{c}((u_i(b) - v_i^n(b)\big) \\
& \leq u_i(a)
\end{aligned}
$$

that is $\rho_i \geq 0$. By Lemma 4, we have $\rho_i \geq 0$, $t \in J$ which implies that

$$v_i^n(t) \leq u_i(t), \quad t \in J$$

Next, we prove that $w_i^{n+1}(t) \leq u_i(t)$ Let $\rho_i(t) = w_i^{n+1}(t) \leq u_i(t)$. Then from, we have

$$
\begin{aligned}
{}^cD_{a^+}^{q;\psi}\rho_i(t) \quad & = F_i\Big(t, w_i^{n+1}(t)\Big) - F_i\big(t, u_i(t)\big) - k_i\big(u_i(t) - u_i(t)\big) \\
& \geq -k_i\big(w_i 6n(t) - u_i(t)\big) + k_i\big(u_i(t) - u_i(t)\big) \\
& = -k_i\rho_i(t).
\end{aligned}
$$

Furthermore

$$
\begin{aligned}
u_i(a) &= (u_i(a) - \frac{1}{c}(\phi(u_i(a), u_i(b)) \\
&= (u_i(a) - \frac{1}{c}(\phi(w_i^n(a), w_i^n(b)) - \frac{1}{c}(\phi(u_i(a), u_i(b)) \\
&\leq w_i^n(a) - \frac{d}{c}((w_i^n(b) - u_i(b)) \\
&\leq w_i^n(a)
\end{aligned}
$$

that is $\rho_i \geq 0$. By Lemma 4, we have $\rho_i \geq 0$, $t \in J$ which implies that

$$u_i(t) \leq w_i^n(t), \quad t \in J$$

Hence,

$$v_i^n(t) \leq u_i(t) \leq w_i^n(t), \quad t \in J$$

By (4.8) is satisfied on $J$ for all $n \in \mathbb{N}$. For $n \to \infty$ on (3.8), we have

$$v_i^* \leq u_i \leq w_i^*.$$

Hence $v_i^*, w_i^*$ are the extremal solutions of (1.1)in $[v_i^0, w_i^0]$

**Theorem 3.2** Let all the assumptions of Theorem 3.1 hold. Further, there exist non-negative constants M and N such that the function $f_i$ satisfies the condition

$$f_i(x, u_1, u_2) - f_i(x, v_1, v_2) \leq M(u_1 - v_1) + N((u_2 - v_2),$$

for $v_i^0(t) \leq u_i \leq w_i^0(t)$. Then the problem $u_i(t)$ of (1.1) has a unique solution.
**Proof.** We know $v_i^0(t) \leq w_i^0(t)$ on $J$. It is sufficient to prove that $v_i(t)^0 \geq w_i^0(t)$ on $J$. Consider $\rho_i(t) = w_i^0(t) - v_i^0(t)$.Then we have

$$
\begin{aligned}
{}^cD_{a^+}^{q;\psi}\rho_i(t) &= F_i\left(t, w_i^0(t)\right) - F_i\left(t, v_i^0(t)\right) - k_i\left(w_i^0(t) - w_i^0(t)\right) + k_i(v_i^0(t) - v_i^0) \\
&\geq -k_i\left(w_i^0(t) - v_i^0(t)\right) - k_i\left(w_i^0(t) - w_i^0(t)\right) + k_i\left(v_i^0(t) - v_i^0(t)\right) \\
&= -k_i\rho_i(t).
\end{aligned}
$$

Since

$$
\begin{aligned}
\rho(a) &= \left(w_i^0(a) - v_i^0(t)\right) - \frac{1}{c}\left(\phi\left(w_i^0(a), w_i^0(b)\right) - \phi\left(v_i^0(a), v_i^0(b)\right)\right) \\
&\geq \frac{d}{c}((w_i^0(b) - v_i^0(b) \\
&\geq 0,
\end{aligned}
$$

we have $w_i^0(t) \geq v_i^0(t)$, $t \in J$ . By Lemma 4, we know $p_i \geq 0$, implying that $w_i^0(t) \geq v_i^0(t)$ on $J$. Hence $v_i(t) = u_i(t) = w_i(t)$.

## 4. Conclusion

In this work, initially, we have investigated by using a monotone iterative method together with upper and lower solutions for boundary value problems involving a generalized system of Caputo derivative of fractional order. The monotone method yields monotone sequences which converge uniformly and monotonically to extremal (maximal and minimal) solutions of (1.1). We have proven that the unique solution of $u_i(t)$ of the system.

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