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Original Article

Distributive Character of Multiplication N-groups

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Abstract *- In this paper, Uniserial N-groups and Bezout N-groups are defined. Defining localized near ring, localized N-groups, localized N-subgroups and localized ideals of N-groups the related results are discussed. It is observed that the various characteristics of DN-groups, multiplication N-groups, Uniserial N-groups and Bezout N-groups are also investigated. It is also seen that in arithmetical near rings, uniserial N-groups and DN-groups lead to multiplication N-groups.*

Keywords *- Near rings, Localized N-groups, Multiplication N-groups, Distributive N-groups.*

AMS Subject Classification Codes*: 16Y30.*

1. Introduction

In this study, the concepts of multiplication N-groups and cyclic N-groups in the near rings are defined by Elaheh Khodadaapour and Tahereh Roodbarilor. They discussed the relationship between multiplication N-groups and cyclic N-groups. The left N-group E is regarded as unitary and N as a commutative near ring with zero symmetric. This paper's foundational ideas are all referenced in [6, 9]. Here, we defined the fundamental definition and outcomes required for this paper. The symbols $\leq_N, \trianglelefteq_N$ and $\triangleleft$ are used to mean N-subgroup, normal N-group and ideal respectively. $\mathrm{Max}(N)$ represents the collection of all maximal ideals of N. Most of the definitions have been extracted from [9].

Definition 1.1 If the following standards are satisfied, a nonempty set N combined with the binary operations "$+$" and "$.$" is referred to as right near ring.
i. $(N,+)$ is a group(not necessarily abelian).
ii. $(N,.)$ is a semi group.
iii. $(p+b)c = pc + bc), \forall p, b, c \in N$.

Definition 1.2 An additive group $(E,+)$ is referred to be a left N-group, if $\exists$ a map $N \times E \to E, (n,u) \to nu$ in which the following standards are satisfied-
i. $(m+n)u = mu + nu$.
ii. $(mn)u = m(nu)$.
It is to be noted that N is itself an N-group over itself. If for $1 \in N$ such that $1.u = u\ \forall\ u \in E$, then E is called an unitary N-group.

In the event that A is a subgroup of $(E,+)$ and $NA \subseteq A$ for any $A \subseteq E$, then E is referred to as an N-subgroup.
If F is a normal subgroup of $(E,+)$ with $na \in F,\ \forall\ n \in N, a \in F$, then F is referred to be a normal N-subgroup of E.
If D is a normal subgroup of $(E,+)$ such that $n(a+e) - ne \in D,\ \forall\ n \in N, a \in D, e \in E$, then D is referred to as an ideal of E.
Let $A \triangleleft E$. Then the set $\frac{E}{A} = \{a + A : a \in E\}$ forms an N-group under the operations $(k+A)+(s+A) = (k+s)+A$ and $m(s+A) = ms + A, \forall s, k \in E, m \in N$, called quotient N-group.
E is called cyclic if $E = nl$ for some $n \in N,\ l \in E$.
When $x \in E$, Nx is referred to as the principal N-subgroup of E.
E is known as the principal N-group (PNG) if each $A \leq_N E$ is principal.

E is referred to as an ideal N-group if each $A \leq_N E$ is an ideal.
If $K \lhd E$, then the N-subgroup of $\frac{E}{K}$ is referred as a subfactor of E.
If for any (left or right) $J \lhd N$ with $I \subseteq J$ implies $J = I$ or $J = N$, then $I \lhd N$ is referred to be (left or right)maximal.
If for any $J \leq_N E$ with $I \subseteq J$ implies $J = I$ or $J = E$, then $I \leq_N E$ is referred to be maximal.
If k or $1 - k$ is invertible in it for any $k \in N$ or N has a unique maximal N-subgroup, then N is called local.
N is called strongly regular if for any $n \in N\ \exists\ m \in N$ such that $n = n^2 m$ or mn^2.
Jacobson radical, $J(N) = \{\cap I: I \in Max(N)\}$.
If $\exists\ I, J \lhd N$ such that $IJ \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$, then $P \lhd N$ is called prime.
If $(P + T) \cap C = P \cap C + T \cap C\ \forall\ P, T, C \leq_N E$, then E is called left DN-group.
A fully DN-group is a DN-group if each factor group is DN-group.
E is referred to generated finitely if $\exists$ a finite set $\{e_1, e_2, e_3, \dots e_n\}$ such that $E = n_1e_1 + n_2e_2 + n_3e_3 + \cdots + n_ne_n,\ e_i \in E, n_i \in N,\ i = 1,2, \dots n$.

Definition 1.3 E is Uniserial if any two of its N-subgroups are comparable to each other.

Definition 1.4 E is called Bezout N-group if any of its N-subgroup which is generated finitely, is cyclic.

Lemma 1.1 [9][2.72 corollary] Maximal ideal of a near ring with unity is a prime ideal.

Lemma 1.2 N-subgroups of an ideal DN-group E are also ideal DN-groups.

Proof. Let $T \leq_N E$. If $T_1, T_2, T_3 \leq_N T$, then $A_1, A_2, A_3 \leq_N E$ also. So the result.

2. Localized N-groups

Definition 2.1 $H \subseteq N$ is called multiplicative closed if $p \in H$ implies $p^{-1} \in H$ or $1 \in H$ and $p, y \in H$ imply $py \in H$.

It is to be noted that for any $p \in H, pp^{-1} = p^{-1}p = 1 \in H$ and $p = (p^{-1})^{-1}$.

Definition 2.2 Let S be a multiplicative closed subset of a commutative near ring N with identity. Define "$+$" and "$.$" in $(S^{-1}N, +, .)$ by- $s_1^{-1}n_1 + s_2^{-1}n_2 = (s_1s_2)^{-1}(s_2n_1 + s_1n_2)$ and $(s_1^{-1}n_1).(s_2^{-1}n_2) = (s_1s_2)^{-1}(n_1n_2),\ \forall\ s_1, s_2 \in S, n_1, n_2 \in N$. Then for any $s_1^{-1}n_1, s_2^{-1}n_2, s_3^{-1}n_3 \in S^{-1}N$ we have, $(s_1s_2)^{-1} \in S$ and $s_2n_1 + s_1n_2 \in N$. So, $(s_1s_2)^{-1}(s_2n_1 + s_1n_2) \in S^{-1}N \Rightarrow s_1^{-1}n_1 + s_2^{-1}n_2 \in S^{-1}N$. Now, $(s_1^{-1}n_1 + s_2^{-1}n_2) + s_3^{-1}n_3 = (s_1s_2)^{-1}(s_2n_1 + s_1n_2) + s_3^{-1}n_3 = (s_1s_2s_3)^{-1}\{s_3(s_2n_1 + s_1n_2) + (s_1s_2)n_3\}$ [since $(N, .)$ is associative] $= (s_1s_2s_3)^{-1}\{(s_2n_1 + s_1n_2)s_3 + (s_1s_2)n_3\}$ [since $S \subseteq N$ and $(N, .)$ is commutative]
$= (s_1s_2s_3)^{-1}\{(s_2n_1)s_3 + (s_1n_2)s_3 + (s_1s_2)n_3\}$ [by right distributive law] $= (s_1s_2s_3)^{-1}\{s_3(s_2n_1) + s_3(s_1n_2) + (s_1s_2)n_3\}$ [since $S \subseteq N$ and $(N, .)$ is commutative] $= (s_1s_2s_3)^{-1}\{s_3(s_2n_1) + s_3(s_1n_2) + (s_1s_2)n_3\}$ [since $(N, .)$ is commutative and has right distributive property]
$= (s_1s_2s_3)^{-1}\{(s_2s_3)n_1) + (s_1s_3)n_2) + (s_1s_2)n_3\}$ [since $(N, .)$ is associative and commutative]. Similarly, we can show that $s_1^{-1}n_1 + (s_2^{-1}n_2 + s_3^{-1}n_3) = (s_1s_2s_3)^{-1}\{(s_2s_3)n_1) + (s_1s_3)n_2) + (s_1s_2)n_3\}$. $\therefore (s_1^{-1}n_1 + s_2^{-1}n_2) + s_3^{-1}n_3 = s_1^{-1}n_1 + (s_2^{-1}n_2 + s_3^{-1}n_3)$.
For any $s^{-1}x \in S^{-1}N$ we have, $0 + s^{-1}x = s^{-1}0 + s^{-1}x = s^{-1}(0 + x)$[since $s^{-1} \in N, N$ has the right distributive property and $(N, .)$ is commutative] $= s^{-1}x$. Similarly, $s^{-1}x + 0 = s^{-1}x$. Thus the identity 0 of $(N, +)$ is the identity of $(S^{-1}N, +)$. For any $s^{-1}x \in S^{-1}N$, $-s^{-1}x$ is the inverse of $s^{-1}x$ as $s^{-1}x + (-s^{-1}x) = s^{-1}(x - x) = s^{-1}0 = 0 = (-s^{-1}x) + s^{-1}x$. Also, $(S^{-1}N, .)$ is closed by definition. Since N is commutative, $s_1^{-1}n_1(s_2^{-1}n_2s_3^{-1}n_3) = (s_1^{-1}n_1s_2^{-1}n_2)s_3^{-1}n_3$. Now, for any $s, y \in S \subseteq N$ we have, $s^{-1}, y^{-1}, s^{-1}y^{-1}, (sy)^{-1} \in S \subseteq N$. $\therefore s^{-1}y^{-1} = (s^{-1}.1)(y^{-1}.1)$[since N has the unity] $= (sy)^{-1}(1.1)$[by hypothesis] $= (sy)^{-1}$ [since N has the unity]. $\therefore (s_1^{-1}n_1 + s_2^{-1}n_2).s_3^{-1}n_3) = (s_1s_2)^{-1}(s_2n_1 + s_1n_2)s_3^{-1}n_3) = (s_1s_2s_3)^{-1}(s_2n_1 + s_1n_2)n_3)$ [since $(N, .)$ is associative]
$= (s_1s_2s_3)^{-1}(s_2n_1n_3 + s_1n_2n_3) = (s_1s_2s_3)^{-1}.1.(s_2n_1n_3 + s_1n_2n_3)$[since N has the unity] $= (s_1s_2s_3)^{-1}.s_3^{-1}s_3.(s_2n_1n_3 + s_1n_2n_3)$ [by definition of S]
$= (s_1s_3s_2s_3)^{-1}(s_2s_3n_1n_3 + s_1s_3n_2n_3)$[since N has the right distributive property and $(N, .)$ is commutative] $=$
$(s_1s_3)^{-1}(n_1n_3) + (s_2s_3)^{-1}(n_2n_3) = s_1^{-1}n_1.s_3^{-1}n_3 + s_2^{-1}n_2.s_3^{-1}n_3$ The above conditions shows that $S^{-1}N$ is a near ring, called localized near ring.

Note that if $h \in H$, then $h^{-1}0 = 0$, where 0 is the identity of $(N, +)$.

Definition 2.3 Let S be a multiplicative closed subset of a commutative near ring N with identity. Then as above $(S^{-1}N,+)$ is a group. If we define a map $S^{-1}N \times S^{-1}E \to S^{-1}E$ by $(s_1^{-1}n, s_2^{-1}e) \to (s_1s_2)^{-1}(ne)$ i. e. $s_1^{-1}n.s_2^{-1}e = (s_1s_2)^{-1}(ne)$. Then we get, $(s_1^{-1}n_1 + s_2^{-1}n_2)(s^{-1}e) = (s_1s_2)^{-1}(s_2n_1 + s_1n_2)s^{-1}e = (s_1s_2s)^{-1}(s_2n_1 + s_1n_2)e = (s_1s_2s)^{-1}(s_2n_1s + s_1n_2s) = (s_1s_2)^{-1}s^{-1}(s_2n_1e + s_1n_2e)$ [as $(xy)^{-1} = x^{-1}y^{-1}$] $= (s_1s_2)^{-1}$ $(s^{-1}s_2n_1e + s^{-1}s_1n_2e)$[as $s^{-1} \in N$ and N is commutative] $= s_1^{-1}(s^{-1}n_1e) + s_2^{-1}(s^{-1}n_2e)$ [by hypothesis] $= (s_1^{-1}s^{-1})(n_1e) + (s_2^{-1}s^{-1})(n_2e)$ [since N is commutative] $= (s_1s)^{-1}$ $(n_1e) + (s_2s)^{-1})(n_2e)$ [since $(xy)^{-1} = x^{-1}y^{-1}$] $= (s_1^{-1}n_1.s^{-1}e) + (s_2^{-1}n_2.s^{-1}e)$[by hypothesis]. This shows that $S^{-1}E$ is an $S^{-1}N$-group called localized N-group of E or simply $S^{-1}E$ is an N-group.

Definition 2.4 Let S be a multiplicative closed subset of a commutative N with unity. If $A \leq_N E$, then $S^{-1}A$ is called $S^{-1}N$-subgroup of $S^{-1}E$ if $n(s^{-1}a) = s^{-1}(na) \in S^{-1}A$, for some $s \in S, a \in A, n \in N$. This $S^{-1}N$-subgroup is called localized N-subgroup of E or simply $S^{-1}A$ is an N-subgroup of E.

Definition 2.5 Let S be a multiplicative closed subset of a commutative N with unity. For $I \subseteq N$, $S^{-1}I$ is an ideal of $S^{-1}N$ if $S^{-1}I$ is an additive normal subgroup of $S^{-1}N$ and $s_1^{-1}x.s_2^{-1}n$, $s_1^{-1}n_1(s_2^{-1}n_2 + s^{-1}x) - s_1^{-1}n_1s_2^{-1}n_2 \in S^{-1}I$, for some $s, s_1, s_2 \in S, n, n_1, n_2 \in N, x \in I$. This ideal $S^{-1}I$ is called localized ideal of N or simply $S^{-1}I$ is an ideal of N.

Definition 2.6 Let $P \lhd N$ be prime without unity. Then $S = N\backslash P$ is multiplicative closed subset of N because if $d, b \in S$, then $db \in N$ and $d, b \not\in P$. Also, P is prime, $db \not\in P$ and so $db \in S$. Also $1 \not\in P, 1 \in N \Rightarrow 1 \in S$. Then $S^{-1}N$ is called localization of N at P and denoted by N_P. Therefore $N_P = (N\backslash P)^{-1}N = S^{-1}N$.
Localization of E at a prime ideal P, $E_P = S^{-1}E = (N\backslash P)^{-1}E$.

Lemma 2.1 $SS = S$ if S is a multiplicative closed subset of N.

Proof. If $x \in SS$, then $x = s_1s_2 \in S$, for some $s_1, s_2 \in S$. So, $SS \subseteq S$.
Again if $s \in S$, then $s = 1.s$ [since N has identity]. Since $1 \in S$ and S is Multiplicative closed, therefore $s = 1.s \in SS$. Thus $S \subseteq SS$ and hence $SS = S$.

Lemma 2.2 Let S be a multiplicative closed subset of N and $I \lhd N$. Then $S^{-1}I \lhd S^{-1}N$.

Proof. Since $I \lhd N$, for any $x, y \in I$ and $n, n_1, n_2 \in N$, $xn \in I, n + x - n \in I$ and $n_1(n_2 + x) - n_1n_2 \in I$. Now, for any $s_1, s_2 \in S$ we have, $s_1^{-1}x - s_2^{-1}y = (s_1s_2)^{-1}(s_2x - s_1y) \in S^{-1}I$ and $s_1^{-1}n + s_2^{-1}x - s_1^{-1}n = (s_1s_2)^{-1}(s_2n + s_1x) + s_1^{-1}(-n) = (s_1s_2s_1)^{-1}[s_1(s_2n + s_1x) - s_1s_2n]$. Now, $s_1 \in S \Rightarrow s_1 \in N \Rightarrow s_1x \in I$. Also $s_2n \in N$ and so $s_1(s_2n + s_1x) - s_1s_2n \in I$. Again, $s_1s_2s_1 \in S$ and therefore $(s_1s_2s_1)^{-1}[s_1(s_2n + s_1x) - s_1s_2n] \in S^{-1}I$. Now, $s_2^{-1}x.s_1^{-1}n = (s_2s_1)^{-1}(xn) \in S^{-1}I$ and $s_1^{-1}n_1(s_2^{-1}n_2 + s^{-1}x) - s_1^{-1}n_1s_2^{-1}n_2 = s_1^{-1}n_1[(s_2s)^{-1}(sn_2 + s_2x)] - (s_1s_2)^{-1}(n_1n_2) = (s_1s_2s)^{-1}$ $n_1(sn_2 + s_2x) - (s_1s_2)^{-1}(n_1n_2) = (s_1s_2s)^{-1}[n_1(sn_2 + s_2x) - s(n_1n_2)] \in S^{-1}I$ [since $s_2x \in I$]. This shows that $S^{-1}I$ is an ideal of $S^{-1}N$.

Since by lemma 1.1, maximal ideal in N with unity is prime ideal. So, $P \in Max(N)$ implies $S = N\backslash P$ is closed subset as shown earlier and localized near ring, localized N-groups, localized N-subgroups, localized ideals are defined. Now, utilizing this idea, we will demonstrate some findings.

Lemma 2.3 Let $X \leq_N E$. Then $X_P \leq_N E_P$, $\forall$ $P \in Max(N)$.

Proof. We have, $X_P = S^{-1}X, E_P = S^{-1}E$. Since $X \leq_N E$, X is subgroup of E and so $NX \subseteq X$. Now, $a, b \in X_P$ implies $a = s_1^{-1}x_1, b = s_2^{-1}x_2$, for some $s_1, s_2 \in S, x_1, x_2 \in X$. So, $a - b = (s_1s_2)^{-1}(s_2x_1 - s_1x_2)$. Since $s_2x_1, s_1x_2 \in X$ and X is subgroup of E, $s_2x_1 - s_1x_2 \in X$. Also, $s_1.s_2 \in S$ [since S is multiplicative closed]. Therefore, $a - b \in S^{-1}X$. Also, let $y \in N_P.X_P$. Then $y = nx$, for some $n \in N_P, x \in X_P$. $\therefore n = s_1^{-1}n_1$ and $x = s_2^{-1}x_1$, for some $n_1 \in N, x_1 \in X \Rightarrow nx = (s_1s_2)^{-1}(n_1x_1) \in S^{-1}X$. Thus, $y = nx \in X_P$. Hence the result.

Lemma 2.4 If $E_P = 0$, $\forall$ $P \in Max(N)$, then $\exists$ $s \in S = N\backslash P$ such that $se = 0$ $\forall$ $e \in E$.

Proof. $E_P = 0 \Rightarrow S^{-1}E = 0$. So, for any $e \in E$ there exists $s \in S$ such $s^{-1} \in S$ and $(s^{-1})^{-1}e = 0$[Since S is closed]. Also $s = (s^{-1})^{-1}$. Thus the result.

Theorem 2.1 For an N-group E, $E = 0$ if and only if $E_P = 0,\ \forall\ P \in Max(N)$.
Proof. If $E = 0,\ \exists\ s \in (N\backslash P)^{-1}$ such that $se = 0$. So, $E_P = 0$. Let $e \in E$. Then $Ann(e) \lhd N$. Let $Ann(e) \lhd N$ be proper. Then $\exists\ P \in Max(N)$ such that $Ann(e) \subseteq P$ [since every proper ideal in N is contained in a maximal ideal]. Since $E_P = 0,\ \exists$ $s \in S = N\backslash P$ such that $se = 0$[by lemma 2.4]. $\therefore s \in Ann(e) \subseteq P \Rightarrow s \in P$-which contradicts $s \in N\backslash P$. Thus $Ann(e) = N \Rightarrow$ $ne = 0,\ \forall\ n \in N$. Since $1 \in N, 1e = 0 \Rightarrow e = 0 \Rightarrow E = 0$.

Corollary 2.1 $A_P = B_P \Rightarrow A = B,\ \forall\ A, B \leq_N E,\ P \in Max(N)$.
Proof. Let $a \in A$. Then $s_1^{-1}a \in S^{-1}A$, for some $s_1 \in S = N\backslash P \Rightarrow s_1^{-1}a \in S^{-1}B$[since $A_P = B_P$] $\Rightarrow s_1^{-1}a = s_2^{-1}b$, for some $s_2 \in S, b \in B \Rightarrow s_1^{-1}a - s_2^{-1}b = 0 \Rightarrow (s_1s_2)^{-1}(s_2a - s_1b) = 0 \Rightarrow (s_2a - s_1b)_P = 0 \Rightarrow (s_2a - s_1b) = 0$[by using **theorem 2.1**] $\Rightarrow$ $s_2a = s_1b \Rightarrow s_2^{-1}s_2a = s_2^{-1}S_1b \Rightarrow a = 1.a \in NB \subseteq B$. Thus $A \subseteq B$. Similarly, $B \subseteq A$. Hence $A = B$.

Lemma 2.5
If $I \lhd N$, then $I_P \lhd N_P,\ \forall\ P \in Max(N)$

Proof. As in lemma 2.2 we can prove the result.

Theorem 2.2 An ideal N-group E is a DN-group if and only if E_P is a also DN-group, $\forall\ P \in Max(N)$.
Proof. Since E is a DN-group, $(D \cap T) + (K \cap T) = (D + K) \cap T,\ \forall\ D, K, T \leq_N E$. Now, to show $(D_P + K_P) \cap T_P = (D_P + T_P) \cap (K_P + T_P),\ \forall\ D_P, K_P, T_P \leq_N E\ E_P$. It is enough to show that, $D_P + K_P = (D + K)_P$ and $D_P \cap K_P = (D \cap K)_P$. Let $x \in D_P + K_P \Rightarrow x = a + b$, where $a \in D_P, b \in K_P$. So, $a = s_1^{-1}a_1, b = s_2^{-1}b_1$ for some $a_1 \in D, b_1 \in K$. $\therefore x = s_1^{-1}a_1 + s_2^{-1}b_1 = (s_1s_2)^{-1}(s_2a_1 + s_1b_2)$. Since $D, K \leq_N E$, $s_2a_1 \in D, s_1b_2 \in K$. Also, since S is multiplicative closed, $s_1, s_2 \in S \Rightarrow s_1.s_2 \in S$. $\therefore$ $x \in S^{-1}(D + K)$. $\therefore D_P + K_P \subseteq (D + K)_P$.
Let $y \in (D + K)_P = S^{-1}(D + K)$. Then, $y = s^{-1}(x + b)$, for some $s \in S, x \in D, b \in K$. Since $S^{-1}x$ is an ideal, $s^{-1}(x + b) - s^{-1}b \in S^{-1}x \Rightarrow y = s^{-1}(x + b) \in S^{-1}x + S^{-1}b \subseteq S^{-1}D + S^{-1}K = D_P + K_P$. $\therefore (D + K)_P \subseteq D_P + K_P$. Thus $D_P + K_P = (D + K)_P$.
Again, let $x \in D_P \cap K_P = S^{-1}D \cap S^{-1}K$. Then, $x = s_1^{-1}a = s_2^{-1}b$,for some $s_1, s_2 \in S, a \in D, b \in K$. Since $s_1^{-1}, s_2^{-1} \in S = N\backslash P$ and $D, K \leq_N E$, $s_1^{-1}a \in D, s_2^{-1}b \in K$. $\therefore x \in D \cap K$. Since $D \cap K \leq_N E$ and $S = N\backslash P \subseteq N$, $S(D \cap K) \subseteq D \cap K$ and so $D \cap K \subseteq S^{-1}(D \cap K)$. $\therefore x \in S^{-1}(D \cap K) = (D \cap K)_P$. $\therefore D_P \cap K_p \subseteq (D \cap K)_P$. So we get, $D_P \cap K_P = (D \cap K)_P$. Thus $(D_P + K_P) \cap T_P = (D + K)_P \cap T_P = [(D + K) \cap T]_P = [(D \cap T) + (K \cap T)]_P = (D \cap T)_P + (K \cap T)_P = (D_P \cap T_P) + (K_P \cap T_P)$. But by lemma 2.3, $D_P, K_P, T_P \leq_N E_P$. Hence E_P is a DN-group.
Let E_P be DN-group, then for any $D, K, T \leq_N E$, $(D_P + K_P) \cap T_P = (D_P \cap T_P) + (K_P \cap T_P) \Rightarrow ((D + K) \cap T)_P = ((D \cap T) + (K \cap T))_P \Rightarrow (D + K) \cap T = (D \cap T) + (K \cap T)$[by **corollary 2.1**]. This shows that E is a DN-group.

Proposition 2.1 An ideal N-group E generated finitely if and only if E_P generated finitely, $\forall\ P \in Max(N)$.
Proof. Let $e_p \in E_P = S^{-1}E = (N\backslash P)^{-1}$. Then $e_P = s^{-1}e$, for some $s \in S, e \in E$. Since E generated finitely, $e = n_1e_1 + n_2e_2 + \cdots + n_ne_n$, where $n_i \in N, e_i \in E, i = 1,2,..n$. Since $S^{-1}n_1e_1$ is an ideal, $s^{-1}(n_1e_1 + n_2e_2) - s^{-1}n_2e_2 \in S^{-1}n_1e_1 \Rightarrow$ $s^{-1}(n_1e_1 + n_2e_2) = s_1^{-1}n_1e_1 + s^{-1}n_2e_2$, for some $s_1 \in S = n_1(s_1^{-1}e_1) + n_2(s_2^{-1}e_2)$[since N is commutative], where $s = s_2$. In the same way, it can be extended to a finite number n of steps, i. e $e_P = s^{-1}e = s^{-1}(n_1e_1 + n_2e_2 + n_3e_3 + \cdots + n_ne_n) = n_1(s_1^{-1}e_1) + n_2(s_2^{-1}e_2) + n_3(s_3^{-1}e_3) + \cdots n_n(s_n^{-1}e_n)$, where $s_i \in S, e_i \in E$ and $n_i \in N$, for $i = 1,2,3, \ldots n$. This shows that E_P generated finitely.

Lemma 2.6 If E_P is cyclic N-group, then $\frac{N_P}{I_P} \cong E_P$, for some $P \in Max(N)$.
Proof. Let E_P be cyclic N-group generated by e_p. Now, let us define a function $\phi: N_P \to E_P$ by $\phi(n_p) = (ne)_p$, where $n \in N, e \in E$. i. e.$\phi(s^{-1}n) = s^{-1}(ne)$,where $s \in S = N\backslash P$. Clearly, ϕ is well defined and onto. For any $m_p, n_p \in N_P$ we have, $\phi(m_P + n_p) = \phi(s_1^{-1}m + s_2^{-1}n) = \phi((s_1s_2)^{-1})(s_2m + s_1n)) = (s_1s_2)^{-1}((s_2m + s_1n)e) = (s_1s_2)^{-1}(s_2me + s_1ne) =$ $s_1^{-1}(me) + s_2^{-1}(ne) = (me)_p + (ne)_p = \phi(m_p) + \phi(n_p)$. Also, for any $n_p \in N_p, x_p \in E_p$, we have $\phi(n_px_p) = \phi((nx)_p) = ((nx)e)_p = n_p(xe)_p = n_p\phi(x_p)$. $\therefore \frac{N_P}{\ker\phi} \cong E_P$. Since $\ker\phi$ is an ideal, taking $\ker\phi = I_p$ we get, $\frac{N_P}{I_P} \cong E_P$, where I_p is an ideal of N_P.

3. Multiplication N-groups

Definition 3.1 N is referred to be arithmetical if N considered as N-group is a DN-group or $N_P = (N\backslash P)^{-1}N$ is Uniserial, $\forall$ $P \in Max(N)$.

Example 3.1 If $E = N = \{0, s, b, m\}$ is the Klein's 4-groups given by the following table-

.	0	s	b	m
0	0	0	0	0
s	0	0	s	s
b	0	s	b	b
m	0	s	m	m

+	0	s	b	m
0	0	s	b	m
s	s	0	m	b
b	b	m	0	s
m	m	b	s	0

Then $(E, +, .)$ is a near ring and N-group over itself.
$P = \{0\}, L = \{0, s\}$, $E \leq_N E$ as $NP = P, NL = L$ and $NN = N$ such that $P \subset L \subset N$.
We have, $P + L = L + P = L, P + E = E + P = E, L + E = E + L = E, P + P = P$.
and $(P + L) \cap E = L = (P \cap E) + (L \cap E), (P + E) \cap L = L = (P \cap L) + (E \cap L), (L + E) \cap P = P = (L \cap P) + (E \cap P), (L + P) \cap E = L = (L \cap E) + (P \cap E), ((E + P) \cap L = L = (E \cap L) + (P \cap L), (E + L) \cap P = P = (E \cap P) + (L \cap P)$.
Thus E is a DN-group and hence E is arithmetical.

Definition 3.2 If an N-subgroup A of E has the form IE for some $I \lhd N$, it is referred to be multiplication.

Definition 3.3 E is referred to as a multiplication N-group if A is multiplication $\forall A \leq_N E$.

Example 3.2 Example of a multiplication N-group.
Let $N = (E, +, .) = \{0, s, b, k\}$ be the Klein's 4-groups under the operations given below-

.	0	s	b	k
0	0	0	0	0
s	0	0	s	s
b	0	s	k	b
k	0	s	b	k

+	0	s	b	k
0	0	s	b	k
s	s	0	k	b
b	b	k	0	s
k	k	b	s	0

Then $(N, +, .)$ is a near ring as well as N-group over itself. $D = \{0\}, K = \{0, s\}, E \leq_N E$. Also, $D, K, N \lhd N$ such that $D = DE, K = KE$ and $E = NE$. Thus E is a multiplication N-group.

Theorem 3.1 If $K \lhd N$ such that $K \subseteq J(N)$ and E is multiplication N-group, then $KE = 0$ implies $E = 0$.
Proof. Let $x \in E$. Since E is a multiplication N-group, therefore by definition of multiplication N-group, $Nx = JE$, for some $J \lhd N$[since Nx is a principal N-subgroup]. Now, $KE = 0 \Rightarrow JKE = 0 \Rightarrow KJE = 0$[since N is commutative] $\Rightarrow KNx = 0$. Since $x \in Nx, ax = 0 \forall a \in K \Rightarrow a^{-1}ax = 0$[since $a \in K \subseteq J(N)$] $\Rightarrow x = 0$[since E is unitary] $\Rightarrow E = 0$.

Definition 3.4 $(A_P : E_P) = \{n_P \in N_P : n_P E_P \subseteq A_P\}$, for any $A_P \leq_N E_P$.

Definition 3.5 An $I_P \leq_N N_P$ is called an ideal of N_P if $x_P - y_P \in I_P, n_P + x_P - n_P \in I_P, n_P(n'_P + y_P) - n_P n'_P \in I_P, \forall x_P, y_P \in I_P, n_P, n'_P \in N_P$.

Definition 3.6 E_P is referred to as a multiplication N-group if for every $A_P \leq_N E_P$, $A_P = I_P E_P$, for some $I_P \lhd N_P$.

Theorem 3.2 Every cyclic localized N-group is a localized multiplication N-group.
Proof. Let E_P be cyclic generated by e_P, for some $e \in E$. Let $A_P \leq_N E_P$. Now, $(A_P: E_P) = \{n_P \in N_P: n_P E_P \subseteq A_P\}$. So, $(A_P: E_P)E_P \subseteq A_P$. Let $a_P \in A_P \subseteq E_P \Rightarrow a_P = n_P e_P$, for some $n \in N$. Now, for any $m_P \in E_P$ we have, $n_P m_P = n_P n'_P e_P$, for some $n' \in N$ $= n'_P(n_P e_P)$ [since N is commutative] $= n'_P a_P \in N_P A_P \subseteq A_P$ [since $A_P \leq_N E_P$]. $\therefore n_P E_P \subseteq A_P \Rightarrow n_P \in (A_P: E_P) \Rightarrow a_P \in (A_P: E_P)E_P \Rightarrow A_P \subseteq (A_P: E_P)E_P$. $\therefore A_P = (A_P: E_P)E_P$. Let $x_P, y_P \in (A_P: E_P)$ and $n_P, n'_P \in N_P$. Since $A_P \leq_N E_P$, for any $e \in E$, $(x_P - y_P)e_P = x_P e_P - y_P e_P \in A_P$. $\therefore (x_P - y_P)E_P \subseteq A_P \Rightarrow x_P - y_P \in (A_P: E_P)$. Since N is commutative $n_P + x_P - n_P = x_P \in (A_P: E_P)$ and $n_P(n'_P + y_P) - n_P n'_P = n_P y_P$. But for any $e \in E, (n_P y_P)e_p = (y_P n_P)e_P = y_P(n_P e_P) \in y_P E_P \subseteq A_P$. $\therefore n_P y_P \in (A_P: E_P)$. Thus $(A_P: E_P)$ is an ideal of N_P and hence E_P is a multiplication N-group.

Theorem 3.3 Every localized multiplication N-group over local N is cyclic.
Proof. Let E_P be multiplication N-group over local N. $\therefore E_P = I_P E_P$, for some $I_P \lhd N_P \Rightarrow E_P = I_P E_P \subseteq N_P E_P \subseteq E_P$. $\therefore E_P = N_P E_P$. So, for any $e \in E, N_P e_P \subseteq N_P E_P = E_P \Rightarrow N_P e_P \subseteq E_P$. Let $e_P \in E_P$ and $a \in N$. Since N is local, a or $1 - a$ is invertible in it. If a is invertible, then $a_P e_P \in N_P E_P \Rightarrow a_P e_P = n_p e_P$, for some $n \in N$. $\Rightarrow (s_1^{-1}a)(s_2^{-1}e) = (s_3^{-1}n)(s_4^{-1}e)$, for some $s_1, s_2, s_3, s_4 \in S, n \in N$ $\Rightarrow (s_1 s_2)^{-1}(ae) = (s_3 s_4)^{-1}(ne)$ $\Rightarrow a^{-1}(s_1 s_2)^{-1}(ae) = a^{-1}(s_3 s_4)^{-1}(ne)$ $\Rightarrow (s_1 s_2)^{-1}(a^{-1}(ae)) = (s_3 s_4)^{-1}(a^{-1}(ne))$ [since N is commutative] $\Rightarrow (s_1 s_2)^{-1}(e) = (s_3 s_4)^{-1}((a^{-1}n)e)$ $\Rightarrow (s_1 s_2)^{-1}(e) = (s_3^{-1}(a^{-1}n))(s_4^{-1}e)$ $\Rightarrow e_p \in N_P e_P \Rightarrow E_P \subseteq N_P e_P$. Thus $E_P = N_P e_P$ and hence E_P is cyclic.

Theorem 3.4 Localized multiplication N-group is also multiplication N-group.
Proof. Let $M' \leq_N S^{-1}E = E_P$. Then $\exists$ $M \leq_N E$ such that $M' = S^{-1}M$. Since E is a multiplication N-group, $M = IE$, for some $I \lhd N$. Then $M' = S^{-1}(IE) = (SS)^{-1}(IE)$[using lemma 2.1]. So, $M' = (S^{-1}I)(S^{-1}E)$. Also, by lemma 2.2, $S^{-1}I$ is an ideal of $S^{-1}N$. Thus the result.

Corollary 3.1 Since Every multiplication N-group over local N is cyclic and localized N-group of a multiplication N-group is also multiplication N-group, every localized multiplication N-group over local N is cyclic.

Theorem 3.5 If E is a generated finitely, then E is multiplication if and only if E_P is multiplication N-group, $\forall$ $P \in Max(N)$.
Proof. Let E be multiplication. So, by theorem 3.4, E_P is a multiplication N-group. Conversely, let E_P be multiplication N-group. Let $X \leq_N E$. Then $X_P = I_P.E_P$, for some ideal I_P of N_P. $\therefore X_P = S^{-1}I.S^{-1}E = (SS)^{-1}(IE) = S^{-1}(IE) = (IE)_P$ [since $SS = S$]. So, by corollary 2.1, $X = IE$. Hence E is a multiplication N-group.

Theorem 3.6 If $Ann(E) \subseteq P_i$ only, $P_i \in Max(N)$ such that each principal N-subgroup is an ideal and E_{P_i} is cyclic, then E_P is a multiplication N-group for $i = 1,2 \ldots n$.

Proof. Since E_{P_i} is cyclic, $E_{P_i} = (Ne_i)_{P_i}$, where $e_i \in E$, $i = 1,2,3, \ldots n$. Let us choose $b_i \in (\bigcap_{i \neq j} P_i) \backslash P_i, i \neq j$, $i = 1,2 \ldots n$. Let $X \leq_N E$ be cyclic and generated by $x = \sum_{i=1}^{n} b_i e_i$. Now, $E_{P_1} = (Ne_1)_{P_1} \Rightarrow (N \backslash P_1)^{-1}E = (N \backslash P_1)^{-1}(Ne_1) \Rightarrow (N \backslash P_1)E = Ne_1$ [since $(N \backslash P_1)N \subseteq N$] $\Rightarrow s_i e_i = n_i e_1$, for some $s_i \in N \backslash P_1, n_i \in N$. Let $s = s_1 s_2 s_3 \ldots s_n$ and $s'_i = s_1 s_2 s_3 \ldots s_{i-1} s_{i+1} \ldots s_n$ such that $s = s_i s'_i$. $\therefore sx = s(b_1 e_1 + b_2 e_2 + \cdots b_n e_n)$. Now, $s(b_1 e_1 + b_2 e_2 + \cdots b_n e_n) - s(b_2 e_2 + \cdots b_n e_n) \in Sb_1 e_1$ $\Rightarrow sx - s(b_2 e_2 + \cdots b_n e_n) = s' b_1 e_1$, for some $s' \in S$ $\Rightarrow (ss)^{-1}[sx - s(b_2 e_2 + \cdots b_n e_n)] = (ss)^{-1}(s' b_1 e_1)$ $\Rightarrow (ss)^{-1}[sx - s(b_2 e_2 + \cdots b_n e_n)] = s^{-1}(s' b_1 e_1)$ $\Rightarrow s^{-1}(x) - s^{-1}(b_2 e_2 + \cdots b_n e_n)] = (ss)^{-1}(s' b_1 e_1)$. Since $s' b_1 e_1 \in E, ss \in S, x \in X, b_2 e_2 + \cdots b_n e_n \in J(N)E$, therefore $E_P = X_P - (J(N)E)_P$. Since $Ann(E) \subseteq P_i$ only, $Ann(E) \not\subseteq P$, $\forall$ $P \in Max(N)$. So, $\exists$ $s \in Ann(E)$, but $s \not\subseteq P \Rightarrow sE = 0$, for $s \in N$, but $s \not\subseteq P \Rightarrow E_P = 0$[since $s \in N \backslash P = S$]. So, $X_P = (J(N)E)_P \Rightarrow X_P = S^{-1}(J(N)E) \Rightarrow X_P = (SS)^{-1}(J(N)E)$[since SS=S] $\Rightarrow X_P = (S^{-1}J(N)).(S^{-1}E) \Rightarrow X_P = J(N)_P.E_p$. By lemma 2.5, $J(N)_P \lhd N_P$[since $J(N) \lhd N$]. This shows that E_P is a multiplication N-group.

Definition 3.7 E is called locally cyclic if E_P is cyclic, $\forall$ $P \in Max(N)$.

Theorem 3.7 If E generated finitely on local N, then E is multiplication iff it is locally cyclic N-group.
Proof. If E is a multiplication ideal N-group which generated finitely on local N, then by theorem 3.5, E_P is multiplication N-group, $\forall$ $P \in Max(N)$. Since N is local, E_P is cyclic N-group $\forall P \in Max(N)$[by theorem3.3]. So, by definition E is locally cyclic N-group. Conversely, suppose E is locally cyclic N-group. Then E_P is cyclic, $\forall$ $P \in Max(N)$. So, E_P is multiplication

N-group, $\forall$ $P \in Max(N)$[by theorem 3.2] and so E is multiplication [by theorem 3.5].

Definition 3.8 Every N-subgroup of E is referred to as a principal ideal N-group if it is both principal and ideal.

Theorem 3.8 If E is principal DN-group over local N and every localized DN-group over local N is Uniserial, then E is multiplication N-group.

Proof. Since E is a principal DN-group, every N-subgroup is principal and so generated finitely. Let $M \leq_N E$ generated finitely. Since by lemma 1.2, N-subgroups of a DN-group are also ideal DN-groups, M is a DN-group. ∴ By theorem 2.2, M_P is a DN-group. Since N is a local, M_P is an Uniserial N-group[by hypothesis]. Since M generated finitely, M_P is also generated finitely[by proposition 2.1]. So, for $m_P \in M_p$ we have, $m_P = n_{1p}e_{1p} + n_{2p}e_{2p} + n_{3p}e_{3p} + \cdots + n_{np}e_{np}$, where $n_{iP} \in N_P, e_{iP} \in M_P \Rightarrow m_P \in N_p e_{1P} + N_p e_{2P} + N_p e_{3P} + \cdots + N_p e_{nP}$. Since M_P is Uniserial, so any two of its N-subgroups are comparable, we may assume $N_p e_{1P} \subseteq N_p e_{2P} \subseteq N_p e_{3P} \subseteq \cdots \subseteq N_p e_{nP}$. $\therefore m_P \subseteq N_P e_{nP} \Rightarrow M_P \subseteq N_P e_{nP}$. Since M_P is N-subgroup and $e_{nP} \in M_P, N_P e_{nP} \subseteq M_P$. $\therefore M_P = N_P e_{nP} \Rightarrow M_P$ is cyclic $\Rightarrow$ M is locally cyclic So, by theorem 3.7, M is a multiplication N-group and hence E is multiplication.

Definition 3.9 A local near ring is referred to as convey if it is strongly regular.

Theorem 3.9 If E is an ideal DN-group which generated finitely over a convey N with inverse property and every ideal DN-group over a strongly regular near ring is Bezout, then E is a multiplication N-group.

Proof. Let $M \leq_N E$. Since E generated finitely, M generated finitely. Since N-subgroups of an ideal DN-group are also ideal DN-group, M is an ideal DN-group. So, by theorem 2.2, M_P is also DN-group. Since M generated finitely, M_P is also generated finitely [by proposition 2.1]. Since N is convey, M_P is an ideal DN-group over a strongly regular near ring. By hypothesis, M_P is a Bezout N-group. So, every generated finitely N-subgroup is cyclic. Since M_P generated finitely, M_P is cyclic. So, by definition M is locally cyclic. Thus by theorem 3.7, M is a multiplication N-group. Hence E is multiplication.

Proposition 3.1 If N is an arithmetical, then $\forall$ $Q \in Max(N)$, $\frac{N_Q}{I_Q}$ is an Uniserial N-group.

Proof. Let N be an arithmetical. Then by definition, N_Q is Uniserial, $\forall$ $Q \in Max(N)$. Now, to show for any sub factors (ideals of $\frac{N_Q}{I_Q}$) $\overline{X}_Q = \frac{N_Q}{I_{1p}}$ and $\overline{Y}_Q = \frac{N_Q}{I_{2p}}, \overline{X}_Q \subseteq \overline{Y}_Q$ or $\overline{Y}_Q \subseteq \overline{X}_Q$. Since I_{1p}, I_{2p} are ideals of the Uniserial N-group N_Q, $I_{1p} \subseteq I_{2p}$ or $I_{2p} \subseteq I_{1p}$. Let $\overline{a} \in \overline{X}_Q$. Then $a \in I_{1p} \Rightarrow a \in I_{2p} \Rightarrow \overline{a} \in \overline{Y}_Q$. $\therefore \overline{X}_Q \subseteq \overline{Y}_Q$. Thus if $I_{1p} \subseteq I_{2p}$, then $\overline{X}_Q \subseteq \overline{Y}_Q$. Similarly, if $I_{2p} \subseteq I_{1p}$, then $\overline{Y}_Q \subseteq \overline{X}_Q$. This shows the result

Theorem 3.10 If Eis multiplication ideal N-group which generated finitely and N is arithmetical local near ring, then E is a DN-group.

Proof. E_P is also multiplication N-group as E is a multiplication N-group[by theorem 3.4]. Also, by theorem 3.3, E_P is cyclic. So, by lemma 2.6, $\frac{N_P}{I_P} \cong E_P$ $\forall P \in Max(N)$. Since N is a arithmetical local, $\frac{N_P}{I_P}$ is an Uniserial N-group[by proposition 3.1]. So, E_P is an Uniserial N-group $\Rightarrow E_P$ is a DN-group[since Uniserial N-group is DN-group] $\Rightarrow$ E is a DN-group[by theorem 2.2].

4. Conclusion

Near ring theory is a domain of Algebra with many applications. Multiplication N-groups have a wide range to study. By this work this structure will become familiar in near ring theory. This study describes the Uniserial N-groups, Bezout N-groups, multiplication N-groups and their relations under certain conditions. Although, these works will not study their direct application and societal benefit, other science communities may use these structures for their different works.

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