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Original Article

Unstability of Zero Solution of a Class Nonautonomous Differential Equation

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Abstract - *In this paper, we consider a class of one-dimensional nonautonomous differential equation. Under certain conditions, we construct the general solution and prove that the zero solution x = 0 are unstable. The differential equations of mechanics arises from the variational problems of the laws of nature, thus, the study about stability or the unstability of the special solutions, such as the zero solution, attracts many researchers. The novelty of this paper is that the system we consider is nonautonimous with the vector field being the polynomial of any order and prove that the zero solution is unstable. Our result will throw light on the research of the dynamics of celestial mechanics and enrich the differential equation theory.*

Keywords - *Nonautonomous differential equation, General solution, Unstability, Zero solution.*

1. Introduction

Differential equations are the mathematical models of various branches, such as Physical, Chemistry, Biology and describe the law of the nature. The dynamic behavior of a system refers to the motion or changes that occur in the system over time. Studying the dynamic behavior of systems can increase our understanding of natural laws, promote the development of science and technology, and improve our standard of living. There are fruitful results about the study about the dynamics of autonomous differential equation, we refer the readers to [1–4] and the reference therein for the further study about autonomous differential equations.

The research on nonautonomous differential equation is relatively limited, even though in the one-dimensional "polynomial" case

$$\begin{cases}\frac{dx}{dt} = a_n(t)x^n + a_{n-1}(t)x^{n-1} + \cdots + a_2(t)x^2 + a_1(t)x, \\ x(0) = x_0 \in R.\end{cases} \tag{1}$$

In the case $n = 1, 2, 3$, equation (1) is Linear, Riccati and the Abel type first-order differential equation, respectively. These three typical differential equations play an very important role in many physical and technical applications, we refer the readers to [5–9].

As for the dynamics of dynamical system, the stability and unstability of the special solution, such as the equilibria, invariant tori, limit tori, is a hot topic and attract the interest of the researchers most. At present, there are many research results on the stability of low order differential equations, however, as far as we know, few results on unstability [10–11]. For this reason, we consider the Lyapunov unstability of equation (1) under suitable hypotheses on the vector fields.

More concretely, we restrict ourselves to the cases $n > 3$ and $a_i(t)(i = 1, \cdots, n)$ are continuous functions. We will formulate the hypotheses on the coefficients $a_i(t)$ that ensure us construct the general solutions of equation (1), then with more hypotheses, we prove that all the solutions approach infinity as t goes to infinity, thereby that the zero solutions $x = 0$ is unstable.

2. Unstability of Zero Solution

In this section, we will prove that the zero solution $x = 0$ of equation (1) is unstable though construct general solution under the certain hypotheses.

Theorem 2.1. Consider the equation (1), where $a_n(t)(\neq 0), a_{n-1}(t)$ are first-order differentiable with t, set

$$d(t) = -\frac{a_{n-1}(t)}{na_n(t)}, \tag{2}$$

and assume

(H1) $a_{n-k}(t) = (-1)^k C_n^k a_n(t) d^k(t), (C_n^k = \frac{n!}{k!(n-k)!})$ $(0 \leq k \leq n-2)$;

(H2) $(-1)^{n-1}(1-n)a_n(t)d^n(t) + a_1(t)d(t) - d'(t) = 0$;

hold, then equation (1) possesses general solution and special solution defined by equations (6) and (7), respectively. Moreover, if

(H3) $a_n(t) \equiv 1, a_{n-1}(t) = \tilde{c}e^{\int_0^t (\xi+g(s))ds}$ with $\xi > 0, \tilde{c}(> 0)$ small enough, and $|g(t)| \leq M_1, \left|\int_0^{+\infty} g(s)ds\right| \leq M_2$ also hold, then, the zero solution $x = 0$of the equation (1) is unstable provided initial value $x_0(> 0)$ small enough.
Proof: Under the hypotheses (H1)-(H2), equation (1) is reduced to

$$\frac{d}{dt}(x - d(t)) = a_n(t)(x - d(t))^n + m_1(t)(x - d(t)), \tag{3}$$

where $m_1(t) = \sum_{k=1}^{n} C_k^{k-1} a_k(t) d^{k-1}(t)$. Set

$$x(t) = y(t) + d(t),$$

where $d(t)$ is the one defined by (2). Then equation (3) can be transformed to

$$\frac{dy}{dt} = a_n(t)y^n + m_1(t)y, \tag{4}$$

which is the Bernoulli equation, whose general solution is given by

$$y(t) = \left[m(t)(\bar{C} + (1-n)\int_0^t m^{-1}(s)ds)\right]^{\frac{1}{1-n}},$$

where $m(t) = e^{(1-n)\int_0^t m_1(s)ds}$, $\bar{C}$ is a constant depend on initial value:

$$\bar{C} = \left(x_0 + \frac{a_{n-1}(t)}{na_n(t)}\right)^{1-n}. \tag{5}$$

Thus we can get the general solution of (1)

$$x(t) = d(t) + \left[m(t)(\bar{C} + (1-n)\int_0^t m^{-1}(s)ds)\right]^{\frac{1}{1-n}}. \tag{6}$$

Moreover, equation (4) shows that (1) possesses a special solution

$$x(t) = -d(t). \tag{7}$$

Once we get the general solution and the special solution defined by (6) and (7), we will investigated the stability of zero solution of equation (1). To this end, under the condition (H3), we give the quantities explicitly:

$$\begin{cases} d(t) = -\frac{1}{n}a_{n-1}(t), \\ m(t) = e^{(1-n)\int_0^t m_1(s)ds}, \\ m_1(t) = \frac{a'_{n-1}(t)}{a_{n-1}(t)} - \frac{1}{n^{n-1}}a_{n-1}^{n-1}(t). \end{cases} \tag{8}$$

We consider the limit of $d(t), m_1(t)$ and $m(t)$ as t goes to infinity first. By the hypotheses (H3), we get

$$\lim_{t\to+\infty} d(t) = -\lim_{t\to+\infty}\frac{1}{n}a_{n-1}(t) = -\lim_{t\to+\infty}\frac{\tilde{c}}{n}e^{\int_0^t(\xi+g(s))ds} = -\infty, \tag{9}$$

and

$$\lim_{t\to+\infty} m_1(t) = \lim_{t\to+\infty}\left(\frac{a_{n-1}'(t)}{a_{n-1}(t)} - \frac{1}{n^{n-1}}a_{n-1}^{n-1}(t)\right)$$

$$= \lim_{t\to+\infty}\left(\xi + g(t) - \left(\frac{\tilde{c}}{n}\right)^{n-1} e^{(n-1)\int_0^t(\xi+g(s))ds}\right)$$
$$= -\infty. \tag{10}$$

Thus

$$\lim_{t\to+\infty} m(t) = \lim_{t\to+\infty} e^{(1-n)\int_0^t m_1(s)ds} = +\infty, \tag{11}$$

thus the equation (11) yields

$$\lim_{t\to+\infty} m^{\frac{1}{1-n}}(t) = 0. \tag{12}$$

By L'Hospital rule and equation (8)

$$\lim_{t\to+\infty}\frac{m^{-1}(t)}{t^{-2}} = \lim_{t\to+\infty}\frac{t^2}{m(t)} = \lim_{t\to+\infty}\frac{2t}{(1-n)m_1(t)m(t)}$$
$$= \frac{2}{1-n}\lim_{t\to+\infty}\frac{1}{m_1(t)}\cdot\frac{t}{m(t)}$$
$$= 0, \tag{13}$$

Where the last equality is by equation (10) and

$$\lim_{t\to+\infty}\frac{t}{m(t)} = \lim_{t\to+\infty}\frac{1}{(1-n)m_1(t)\cdot e^{(1-n)\int_0^t m_1(s)ds}} = 0.$$

Then we estimate the upper bound of integral of the $m^{-1}(t)$ as t goes to infinity. By the equation (13), there exists $p \in N^+$, such that for all $t > p$,

$$\left|\frac{m^{-1}(t)}{t^{-1}} - 0\right| < \frac{1}{2}. \tag{14}$$

We divide the term $\int_0^{+\infty} m^{-1}(s)ds$ into two parts as follows

$$\int_0^{+\infty} m^{-1}(s)ds = \int_0^p m^{-1}(s)ds + \int_p^{+\infty} m^{-1}(s)ds, \tag{15}$$

We consider the first term in (15). By equation (8) and (H3), we get

$$|m_1(t)| = \left|\xi + g(t) - \left(\frac{\tilde{c}}{n}\right)^{n-1} e^{(n-1)\int_0^t(\xi+g(s))ds}\right|$$
$$\leq \xi + M_1 + \frac{\tilde{c}^{n-1}}{n^{n-1}}e^{(n-1)p(\xi+M_1)}$$
$$:= \bar{M},$$

which yields

$$\int_0^p m^{-1}(s)ds \leq \int_0^p |m^{-1}(s)|ds \leq \bar{M}p.$$

Thus

$$\int_0^p m^{-1}(s)ds = \int_0^p e^{(n-1)\int_0^t m_1(s)ds}dt \le e^{(n-1)\bar{M}p}p.$$

(16)

As for the second term in equation (15), by equation (14)

$$\int_p^{+\infty} m^{-1}(s)ds \le \int_p^{+\infty} \frac{1}{2}s^{-2}ds = -\frac{1}{2}s^{-1}\Big|_p^{+\infty} = \frac{1}{2p}. \tag{17}$$

Then by the above equations (16) and (17), we have the following estimate

$$\int_0^{+\infty} m^{-1}(s)ds = \int_0^p m^{-1}(s)ds + \int_p^{+\infty} m^{-1}(s)ds \le \frac{1}{2p} + e^{(n-1)\bar{M}p}p.$$

The fact $\tilde{c}(> 0)$ small enough, and definition of $\bar{C}$ in (5) enable us to get

$$\bar{C} = \left(x_0 + \frac{\tilde{c}}{n}\right)^{1-n} > \frac{3(n-1)}{2}\left(\frac{1}{2p} + e^{(n-1)\bar{M}p}p\right) > 0,$$

which yield

$$\bar{C}^{\frac{1}{1-n}} < \left[\bar{C} + (1-n)\int_0^t m^{-1}(s)ds\right]^{\frac{1}{1-n}} < \left[\frac{n-1}{2}\left(\frac{1}{2p} + e^{(n-1)\bar{M}p}p\right)\right]^{\frac{1}{1-n}}. \tag{18}$$

Thus by equations (11) and (18), we get the second term in equation (6)

$$\lim_{t\to+\infty} \left[m(t)\left(\bar{C} + (1-n)\int_0^t m^{-1}(s)ds\right)\right]^{\frac{1}{1-n}} = 0. \tag{19}$$

By equations (9) and (19), we get

$$\lim_{t\to+\infty} |x(t)| \ge \lim_{t\to+\infty} \left\{|d(t)| - \left|\left[m(t)\left(\bar{C} + (1-n)\int_0^t m^{-1}(s)ds\right)\right]^{\frac{1}{1-n}}\right|\right\} = +\infty.$$

the zero solution $x = 0$ is unstable. This completes the proof.

3. Conclusion

In this article, we obtain the general solution and a special solution for a class of first-order nonautonmous polynomial differential equation under the coefficient functions satisfy certain specific relationship. And most importantly, we proved that the zero olution of the equation (1) is unstable under more conditions.

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