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Original Article

Exact Doubling the Cube with Straightedge and Compass by Euclidean Geometry

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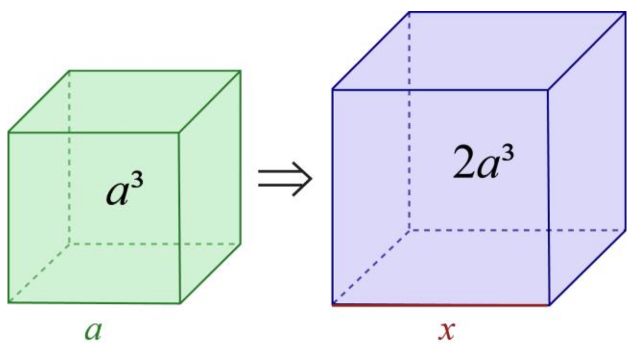
Abstract - *This independent research shows an exact precision and accurate solution for the ancient Greek challenge – "Doubling The Cube" using a straightedge and a compass only. Mathematics tools and propositions used in this solution are all in Geometry and algebraic geometry at the 6th Form level of today's United Kingdom. The methodology of the solution includes geometrical methods to arrange the given cube and its double volume cube into a concentric position of which side x of the double cube can be calculated accurately by algebra then in terms of geometrical length the side is constructive. And then, use the straightedge and the compass to construct the double-volume cube.*

Keywords - *Cube duplicating, Double a cube, Double cube volume, Doubling the cube, Geometrical duplicating cube.*



1. Introduction

Before 2022, there are no exact precision and accurate solutions for all three of the ancient Greek mathematics challenges: "Squaring The Circle", "Trisecting An Angle", and "Doubling The Cube" using only a compass and a straightedge. It was proven by French mathematician Pierre Wantzel in 1837 that it is impossible to solve these problems using only a compass and a straightedge, except for certain specific cases. Approximate solutions for these problems do exist. However, in 2022 and 2023, I did solve the "Trisecting An Angle" problem & "Squaring The Circle" problem with exact precision and accurate solutions using only the geometry method with straightedge and compass [3] & [5]. These exact solutions were published in the International Journal Of Mathematics Trends And Technology in 2023 (International Journal of Mathematics Trends and Technology Volume 69 Issue 5, 9-24, May 2023 ISSN: 2231-5373/https://ijmttjournal.org/Volume-69/Issue-5/IJMTT-V69I5P502.pdf © Seventh Sense Research Group® and International Journal of Mathematics Trends and Technology Volume 69 Issue 6, 39-47, June 2023 ISSN: 2231-5373/ https://ijmttjournal.org/Volume-69/Issue-6/IJMTT-V69I6P506.pdf © 2023 Seventh Sense Research Group®).

These classical challenge problems have been extremely important in the development of geometry. Three such problems stimulated so much interest among later geometers that they have come to be known as the "classical problems": Trisecting An Angle, Squaring The Circle, and Doubling The Cube (i.e., constructing a cube of which volume is twice that of a given cube). This problem is described in detail as follows: *Given a cube of side a & volume a^3 then use a straightedge & a compass to construct a cube of volume $2a^3$.* The challenge of "doubling the cube" refers to constructing a cube with twice the volume of a given cube using only a straightedge and compass. In classical Euclidean geometry, it has been proven that doubling the cube using only these two tools is not possible. This impossibility is known as the Doubling The Cube Problem, and it is rooted in

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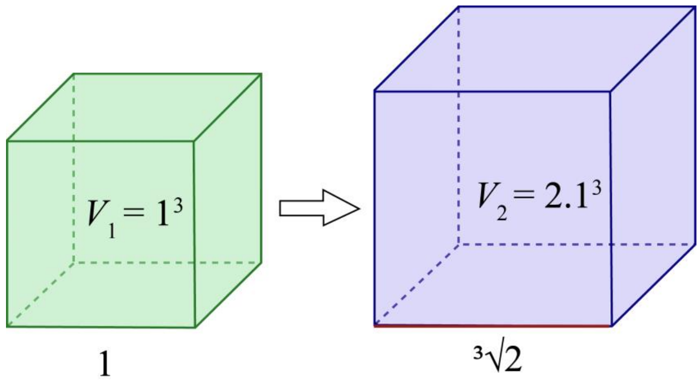
the fact that the cube root of 2 (which is necessary for doubling the cube) is not constructible using only a straightedge and compass. The construction requires finding a length equal to the cube root of 2, which is a transcendental number. Various attempts have been made throughout history to solve the problem, but they involve more advanced mathematical techniques beyond classical constructions. These methods typically involve algebraic or geometric concepts that go beyond the scope of the traditional straightedge and compass constructions. Until 2022, there was no exact precision and accurate solution for the challenge of doubling the cube using only a straightedge and compass, based on classical Euclidean geometry. It is fair to say that although the problem of squaring the circle was to become the most famous in more modern times, certainly the problem of "Doubling The Cube" was the more famous in the time of the ancient Greeks. The "Doubling The Cube" challenge asks for a method to construct a cube which has double volume of a given cube. That means if the given cube volume is 1 unit then we have to construct a cube with side $\sqrt[3]{2}$ from this given unit cube, using only a compass and a straightedge. Cube duplication is believed to be impossible under the stated restrictions of Euclidean geometry, because the Delian constant is classified as an irrational number, which was stated to be geometrically irreducible (Pierre Laurent Wantzel, 1837). Contrary to the impossibility consideration, the solution for this ancient problem is theorem, in which an elegant approach is presented, as a refute to the cube duplication impossibility statement. Geogebra software as one of the interactive geometry software is used to illustrate the accuracy of the obtained results, at higher accuracies which cannot be perceived using the idealized platonic straightedge and compass construction [1]. Despite the efforts of many mathematicians, the problem remained unsolved for more than two thousand years, and it became one of the most famous and intriguing unsolved problems in the history of mathematics. Today, the "doubling the cube" problem is considered to be a classic example of a difficult mathematical problem that was finally solved through the application of mathematical methods and techniques that were not available to the ancient Greeks. It is still studied in mathematics courses as a historical and challenging problem, and its solution continues to inspire and influence mathematicians and students alike. The French mathematician Pierre Wantzel, 1837, proved that it is impossible to double a cube using only a straightedge and compass. This proof by Wantzel raised the issue that the problem cannot be solved with the traditional methods of ancient Greek geometry. However, if we do not limit the use of only a straightedge and compass, we can use more modern mathematical tools to solve the problem. For example, we can use functions on the number line to solve equations and calculate the dimensions of the necessary cubes to double a given cube. But this calculation method is not considered within the scope of ancient Greek geometry.

In this article paper, a groundwork proof for solving the ancient problem of doubling the volume of a given cube, to a certain accuracy/precision, is presented. My obtained/earned results indicated that, algebraic irrationalities should be extended to plane geometric constructions, since subject to application, the desired degree of precision could be possible for compass straightedge construction. Through the presented discussion, it can be concluded that the Wantzel's statement of impossibility is not geometrically valid, since it does not give the geometrical relationship between the quadratic and the cubic extensions used/employed in the proof of cube duplication impossibility, with respect to the formal framework of classical geometric constructions. The impossibility proof simply justify a statement and not a concept. The focus of my Core Theorem in PART II below is to convert the problem from the complex 3D consideration as presented in the impossibility proof, into a simpler 2D problem, and its solution found following the formal Greek‟s rules of geometry. For centuries, the problem of doubling a cube has been a subject to pseudo mathematical approaches, which do not reach the set limits of accuracy. It can therefore be affirmed that, by following the revealed approach, it is geometrically certainty to solve the coefficient $\sqrt[3]{2}$, which is the magnitude of the given cube.

This article objectively presents a provable construction of generating a length of magnitude; as the geometrical solution for the ancient classical problem of doubling the volume of a cube. I follow strictly the constraint use of straightedge & compass to develop a method to solve accurately the "doubling the cube" problem by geometry and algebraic geometry with a special technique that was developed by geometers and called "analysis". Geometers assumed the problem to have been solved and then, by investigating the properties of this solution, worked back to find an equivalent problem that could be solved, based on the givens. To obtain the formally correct solution of the original problem, then, geometers reversed the procedure: firstly, the data were used to solve the equivalent problem derived in the analysis, and, from the solution obtained, the original problem was then solved. In contrast to analysis, this reversed procedure is called "synthesis" [2]. I adopted the technique "ANALYSIS" to solve precisely the "Doubling The Cube" problem with only a straightedge & compass, using only classic Euclidian Geometry.

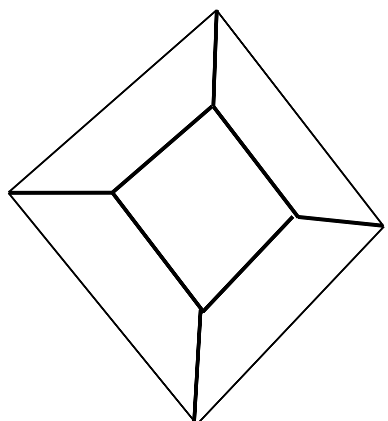
For a Unit Cube, side = 1, then its duplicated cube has side $\sqrt[3]{2}$ and this research shows how to construct, precisely, a straight-line segment $\sqrt[3]{2}$ geometrically.



2. Proposition

2.1. Definition 01: "HEAD-CUT PYRAMID"

Given a 5-facet pyramid. A plane paralleled to the pyramid base will cut and divides the pyramid into 2 parts: one is the smaller pyramid and the other is the "Head-cut Pyramid", described in the following figure:



From the definition above, the head-cut pyramid has the top and bottom bases paralleled and squared. The top square is smaller than the bottom square. Its other 4 side faces are the 4 equal isosceles trapezoids.

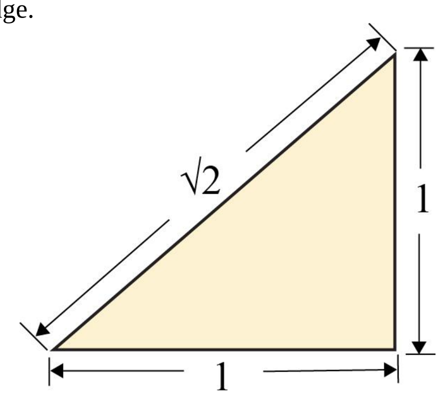
2.2. Theorem 01:

Given a UNIT LENGTH, then

a.-) the exact lengths of $\sqrt{2}$ and $\sqrt{3}$ are constructive in algebraic geometry with a compass and a straightedge.

and,

b.-) the exact length $\sqrt{84}$ is also constructive in algebraic geometry with a compass and a straightedge.



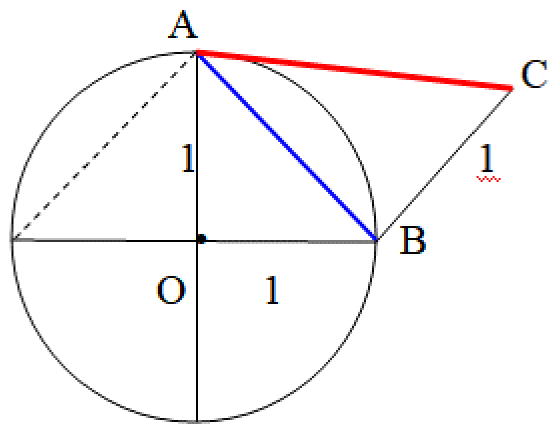


Fig. 1 Length AB(blue) and AC (red) are the exact geometrical lengths of √2 and √3

Proof

a.-) Use the given unit length U and a compass & a straightedge to draw a circle (O, U=1) and 2 perpendicular diameters of the circle (Figure 01). Then, in the right-angle triangle AOB, Pythagoras' theorem gives:

$$AB^2 = 1^2 + 1^2 = 2$$

Length AB $\equiv \sqrt{2}$ *(Figure 01)*

And then, use the straightedge & the compass to construct the right-angle triangle ABC, of which BC = Unit = 1. Similarly to the above:

$$AC^2 = AB^2 + 1^2 = (\sqrt{2})^2 + 1^2 = 3$$

Length AC $\equiv \sqrt{3}$ *(Figure 01 above).*

b.-) To generate one more step, we draw a straight line perpendicular to AC at C then mark D, CD = 1 (unit), using the straightedge and the compass (Figure 2 below). From the right-angle triangle ACD we get:

$$AD^2 = AC^2 + 1^2 = (\sqrt{3})^2 + 1^2 = 4 \Rightarrow \text{length } AD = \sqrt{4}$$

Then generate 80 more steps until we get the length AZ = $\sqrt{84}$.

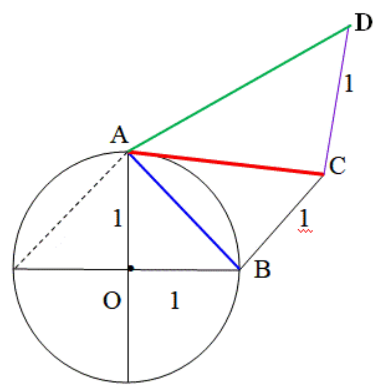
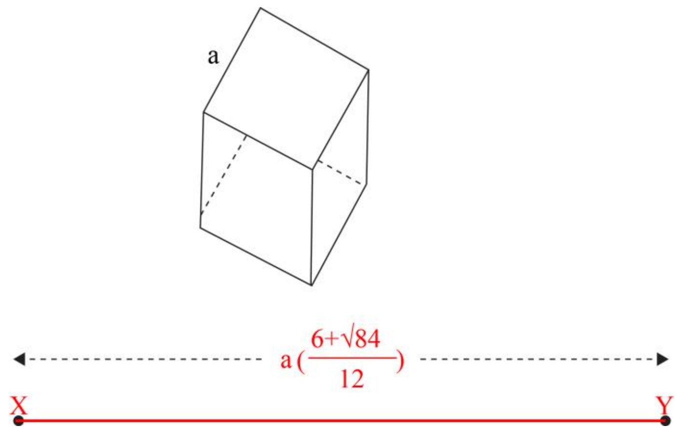


Fig. 2 Length AD (green) is the exact length of √4.

2.3. Theorem 02

Given a cube side a, a⊂R (rational number), then value the length XY = a$(\frac{6+\sqrt{84}}{12})$ is rational and XY is constructive with a compass & a straightedge.



Proof

By Theorem 01 (section b.-), length $\sqrt{84}$ is constructive by straightedge & compass, despite it is algebraically an irrational number. Therefore, [length $\sqrt{84}$ + length 6] is also constructive by the straightedge & the compass. Let the length [length $\sqrt{84}$ + length 6] be MN then use the same straightedge & compass to construct the length PQ = a(MN) by adding MN a times to itself, as follows (PQ is shown in Figure 04 below):

$$\underbrace{MN + MN + \ldots + MN}_{a\ times}$$

Then length $\underbrace{MN + MN + \ldots + MN}_{a\ times}$ be a straight line segment with length PQ = a[length $\sqrt{84}$ + length 6] = a($\sqrt{84}$ + 6),

and then, we can construct the length XY = a($\frac{6+\sqrt{84}}{12}$), in Figure 03 above, by the following procedure:

In a straight line segment starting from P, use compass & straightedge to mark 11 units consecutively to point R, CR = 11 (coloured black in Figure 04 below) starting from point C, PC =1 and PR =12. Connect R to point Q to get the line segment QR, which is the standard line (coloured green in Figure 04 below). From C, draw a line that parallels QR and meet PQ at Y (green colour in Figure 04 below), then:

Consider two congruent triangles PCY & PRQ to get the following equation:

$$\frac{1}{PY} = \frac{12}{\mathrm{PQ}} \quad or \quad \frac{\mathrm{PQ}}{12} = PY \quad or\ PY = a(\frac{6+\sqrt{84}}{12})$$

Therefore the geometrical length PY = a($\frac{6+\sqrt{84}}{12}$) is constructive by straightedge & compass as required

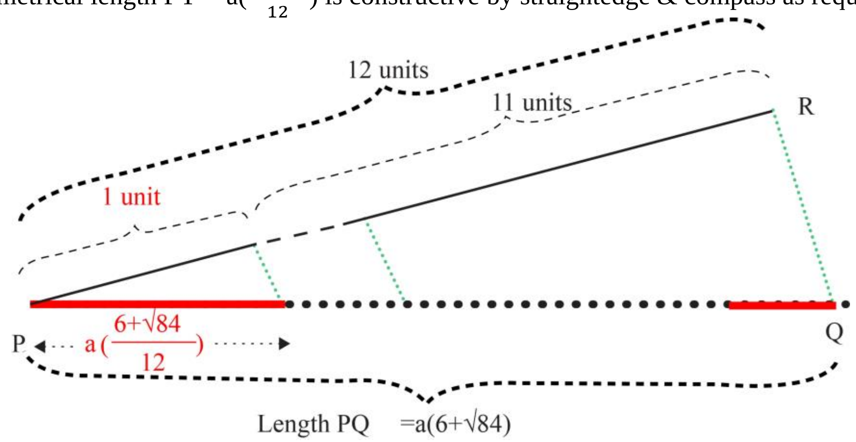


Fig. 4 Contructed exact length XY = a($\frac{6+\sqrt{84}}{12}$)

2.4. *Core Theorem: "Doubling The Cube"*

Given a cube side a, $a \subset \mathbb{R}$, volume a^3 and, assume there exists a double cube $2a^3$, then the irrational expression $a(\frac{6+\sqrt{84}}{12})$ is a side of the double cube $2a^3$, and this side certainly be constructive by a straightedge & a compass.
PROOF:
Let's locate the given cube with volume a^3 inside the double cube with volume $2a^3$ & side $a\sqrt[3]{2}$ concentrically, then the volume of space around the given cube is (Figure 05, below):

$$2a^3 - a^3 = a^3 \qquad (1)$$

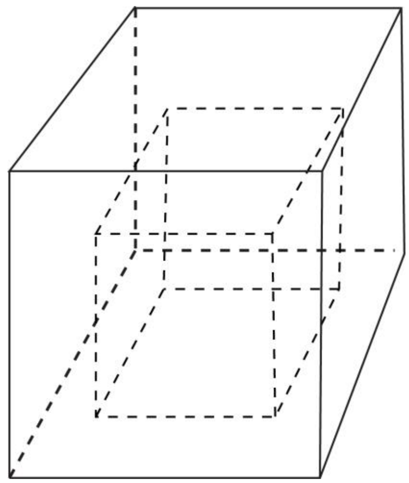


Fig. 5 Concentric image of the given cube and the double cube.

Because of the concentric property of these cubes, the 8 straight line segments connected the centre to the 8 vertices of the double cube go through the 8 vertices of the given cube. This causes the inner space surrounding the given cube within the double cube to be divided into 6 equal head-cut pyramids (Definition 01, above). One of the 6 head-cut pyramids is illustrated as follows:

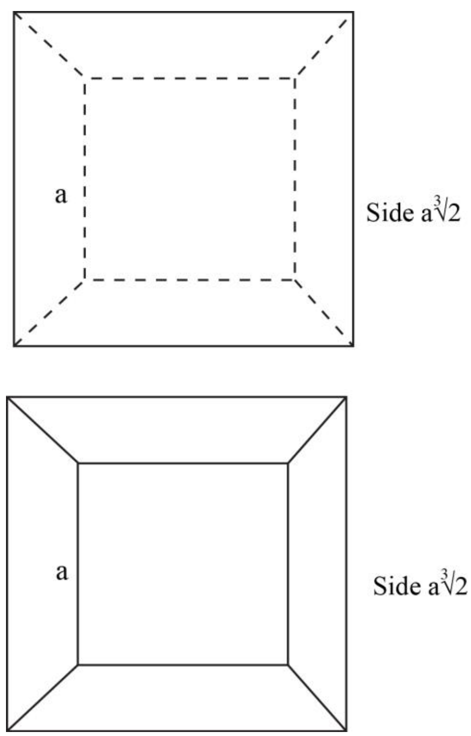


Fig. 6 The 2 square bases of the head-cut pyramid are given by the concentric location/arrangement/position of the given cube & the double cube. (The 6 head-cut pyramids surrounding the given cube a^3 can be also called 6 regular isosceles trapezoid cuboids)

Each head-cut pyramid consists of a small base with side a and a larger base with side a$\sqrt[3]{2}$. These 2 bases are parallel. The distance H of these bases is (side a$\sqrt[3]{2}$. – side a) divided by 2, or

$$H = \frac{a\sqrt[3]{2}-a}{2} = a(\frac{\sqrt[3]{2}-1}{2}) \tag{2}$$

The 4 sided facets of the head-cut pyramid are the 4 regular isosceles trapezoids.

By expression (1) above, volume V of one of the above 6 equal head-cut pyramids is as follows:

$$V = \frac{1}{6}a^3 \tag{3}$$

To establish an equation for the expression (3) above, we use the following facts:

- The side of the small square base of the above head-cut pyramid (Figure 6, above) is a.
- The side of the larger square base of the above head-cut pyramid (Figure 6, above) is a$\sqrt[3]{2}$.
- The distance H of these bases is a($\frac{\sqrt[3]{2}-1}{2}$), given by (2) above.
- The area of the square small base is a^2.
- The area of the square larger base is $(a\sqrt[3]{2})^2 = a^2(\sqrt[3]{2})^2$.
- The average area of the above bases areas is $\frac{a^2+a^2(\sqrt[3]{2})^2}{2}$.

Therefore, we get the details of volume V of the head-cut pyramid (Definition 01) in expression (3) above as follows:
The volume of the head-cut pyramid = Average area of its two square bases multiplied by the distance H of these bases.

$$(\frac{a^2+a^2(\sqrt[3]{2})^2}{2})(\frac{\sqrt[3]{2}-1}{2})a = \frac{1}{6}a^3$$

$$(\frac{1+(\sqrt[3]{2})^2}{2})a^2(\frac{\sqrt[3]{2}-1}{2})a = \frac{1}{6}a^3$$

$$(\frac{(1+\left(\sqrt[3]{2}\right)^2)(\sqrt[3]{2}-1)}{4})a^3 = \frac{1}{6}a^3$$

$$\frac{(\sqrt[3]{2})^3-(\sqrt[3]{2})^2+\sqrt[3]{2}-1}{4} = \frac{1}{6}$$

Because $(\sqrt[3]{2})^3$ is the volume 2 of the double cube with side $\sqrt[3]{2}$, the above equation becomes:

$$\frac{2-(\sqrt[3]{2})^2+\sqrt[3]{2}-1}{4} = \frac{1}{6}$$

$$\frac{-\left(\sqrt[3]{2}\right)^2+\sqrt[3]{2}+1}{4} = \frac{1}{6}$$

$$6(-(\sqrt[3]{2})^2 + \sqrt[3]{2} + 1) = 4$$

$$-6(\sqrt[3]{2})^2 + 6\sqrt[3]{2} + 6 = 4$$

$$-6(\sqrt[3]{2})^2 + 6\sqrt[3]{2} + 2 = 0$$

then this is a quadratic equation in term of the unknown $x = \sqrt[3]{2}$.

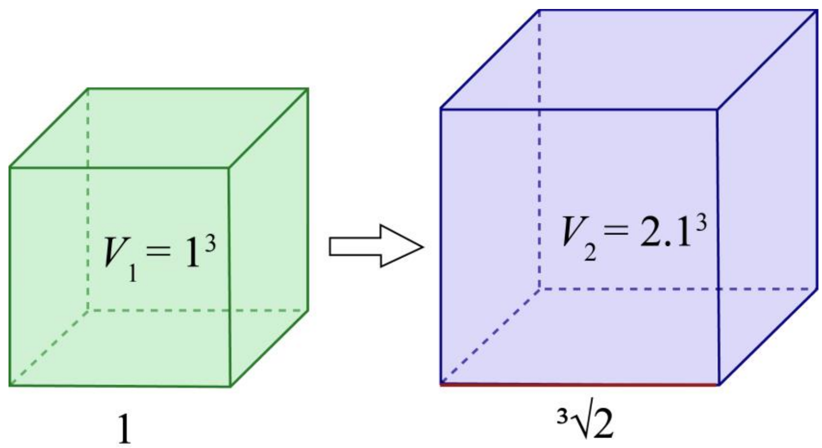
Solve the above quadratic equation $-6(\sqrt[3]{2})^2 + 6\sqrt[3]{2} + 2 = 0$, in term of the unknown $x = \sqrt[3]{2}$, to get:

$x = \sqrt[3]{2} = \frac{6+\sqrt{84}}{12}$ which is an algebraic expression coefficient of the side a for the double cube with volume $2a^3$, as required. Then the side of the double cube is $a\frac{6+\sqrt{84}}{12}$, as required. Thus, by Theorem 02 above, side $a\frac{6+\sqrt{84}}{12}$ of the double cube $2a^3$ is constructive by a straightedge & a compass.

2.5. Method for Doubling The Cube

Apply all the proved Theorems in PART II above to use a straightedge and a compass to construct a double cube of volume $2a^3$ from a given cube of side a, volume a^3.

If a = 1, then it is a special case of the double cube with volume 2 and then the side $\sqrt[3]{2}$ of this cube, which is $\sqrt[3]{2} = \frac{6+\sqrt{84}}{12}$, is easier to construct by Theorems 01, 02, and the Core Theorem with straightedge & compass, as follows:



3. Materials and Methods

The materials and methods include straightedge, compass, ANALYSIS Method and SYNTHESIS Method, within the scope of Algebraic Geometry and Pure Geometry.

4. Discussion and Conclusion

The "Doubling The Cube" problem refers to the ancient Greek problem of constructing a cube with double the volume of a given cube, using only a straightedge and compass. The problem dates back to at least the 5th century BC and was one of the three famous unsolved problems of ancient Greek mathematics, alongside the "Trisecting An Angle" and the "Squaring The Circle" problem.

This research result objectively presents a provable construction of generating a length of magnitude; as the geometrical solution for the ancient classical problem of doubling the volume of a given cube. The "Doubling The Cube" problem, which has challenged mathematicians since the time of the ancient Greeks, is precisely solved by the ANALYTICS method to concentrically locate a given cube of volume a^3 in its double cube with volume $2a^3$, side $a\sqrt[3]{2}$. In other words, I did succeed the concentric location for the given cube a^3 inside the goal cube $2a^3$ to solve exactly the problem with a straightedge and a compass. Such positioning creates six regular head-cut pyramids that take up the space around the given cube. These six 3D shapes are six specific cuboids. From there calculate the volume V of 1 of the 6 head-cut pyramids, and then set up the equation $V = \frac{1}{6}a^3$. In that cubic equation, the cubic term will be equal to the volume of the double cube itself which is $2a^3$, therefore the equation is reduced from a cubic equation to a quadratic equation. Solving this quadratic equation will have a root of the irrational number that is the algebraic value of the edge of the double cube with volume $2a^3$. Then apply the theorems in Part II to use a straightedge & compass to construct the exact lengths for edges/sides of the double cube.

The problem of doubling a cube is a well-known millenary problem that mathematicians stated as impossible to geometrically resolve because the unknown x in the expression $x^3 = 2$ is classified as an irrational number. The incomprehensible proof of impossibility concerned showing that the cubic equation $x^3 = 2$ is unsolved, which is not reducible; and thus geometrically unsolvable. Irrational numbers are mathematically defined as being not a finite solution from a division.

However, this is not a fashionable definition, as most number divisions are open-loop operations that can never be ended. The Cube Duplication Solution of Kimuya. M. Alex and Josephine Mutembei from Meru University in 2017 is much more complicated than my simple & exact solution proved above [1]. The impossibility proof of doubling a cube was based on three-dimensional cubic extensions in abstract algebra, an approach that entirely shifted the problem to solid geometry from its Greek"s definition in plane geometry, and therefore the algebraic statement of impossibility has no geometrical validity. This is evident from the fact that no two facets of a cube can share all four vertices from two different planes. However, according to this study result, the impossible imprecise classification should not be extended to geometry so that the irrationality definition was stated as "algebraic irrationality is not a constructible number of the geometry". The possibility to solve geometrically the coefficient constant $\sqrt[3]{2}$ to an exact precision is proved. This study paper also presents a geometrically certain method under the set restrictions of Euclidean geometry (in the sense that, all presented constructions have been reduced to the Euclidean postulates of practical geometry), by the construction of the relation as depicted in the justification section in PART II above.

4.1. An Open Area for Research

The Core Theorem "Doubling The Cube", applied for doubling the cube of volume a^3 into the cube with volume $2a^3$, certainly converted from "its cubic equation to its quadratic equation successfully", in order to have a precise geometrical length constructed by straightedge & compass. Therefore, the new problem of *"converting a cubic equation to a quadratic equation, equivalently"* is a possible open research area in algebraic geometry.

Acknowledgments

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References

[1] Kimuya M. Alex, and Josephine Mutembei, "The Cube Duplication Solution (A Compassstraightedge (Ruler) Construction)," *International Journal of Mathematics Trends and Technology*, vol. 50, no. 5, 2017. [CrossRef] [Google Scholar] [Publisher Link]

[2] The Three Classical Problems, Britannica. [Online]. Available: https://www.britannica.com/science/mathematics/The-three-classical-problems

[3] Tran Dinh Son, "Exact Angle Trisection with Straightedge and Compass by Secondary Geometry," *International Journal of Mathematics Trends and Technology*, vol. 69, no. 5, pp. 9-24, 2023. [CrossRef] [Publisher Link]

[4] Benjamin Bold, *Famous Problems of Geometry and How to Solve Them*, New York, Dover, chapter. 4, pp. 29-31, 1982. [Google Scholar] [Publisher Link]

[5] Tran Dinh Son, "Exact Squaring the Circle with Straightedge and Compass by Secondary Geometry," *International Journal of Mathematics Trends and Technology*, vol. 69, no. 6, pp. 39-47, 2023. [CrossRef] [Publisher Link]

[6] G. Cornelli, R. McKirahn, and C. Macris, "Archytas and the Duplication of the Cube," *On Pythagoreanism*, Berlin: De Gruyter; pp. 203-33, 2013.

[7] L. Wantzel, "Research on Ways to Recognize if a Geometry Problem can be Solved with the Ruler and the Compass," *Journal of Pure and Applied Mathematics*, vol. 2, pp. 366-372, 1837. [Google Scholar] [Publisher Link]

[8] M. L. Wantzel, "Research on the Means of Knowing Whether a Problem in Geometry can be Solved with Ruler and Compass," *Bit-Player*, pp. 366-372, 1837. [Publisher Link]

[9] Henk J. M. Bos, "The Legitimation of Geometrical Procedures Before 1590," *Redefining Geometrical Exactness*, pp. 23-36. 2001. [CrossRef] [Google Scholar] [Publisher Link]

[10] Wilbur Richard Knorr, *The Ancient Tradition of Geometric Problems*, Boston: Dover Publications, 1986. [CrossRef] [Publisher Link]

[11] J Delattre, and R Bkouche, "Why Ruler and Compass?," *History of Mathematics: History of Problems*, pp. 89-113, 1997. [Google Scholar]

[12] Lucye Guilbeau, "The History of the Solution of the Cubic Equation," *Mathematics News Letter*, vol. 5, no. 4, pp. 8-12, 1930. [CrossRef] [Google Scholar] [Publisher Link]

[13] Bulmer-Thomas, and Ivor, *Selections Illustrating the History of Greek Mathematics*, Thales to Euclid, London, vol. 1, 1967. [Google Scholar] [Publisher Link]

[14] Wilbur Richard Knorr, *The Ancient Tradition of Geometric Problems*, Dover Books on Mathematics, Courier Dover Publications, 1986. [Google Scholar] [Publisher Link]

[15] Henryk Fukś, "Adam Adamandy Kochański's Approximations of π: Reconstruction of the Algorithm," *The Mathematical Intelligencer*, vol. 34, no. 4, pp. 40-45, 2012. [CrossRef] [Google Scholar] [Publisher Link]

[16] John Leech, "An Impossible Constructions," *The Mathematical Gazette*, vol. 38, no. 324, pp. 117-118, 1954. [CrossRef] [Google Scholar] [Publisher Link]

[17] Tom Davis, "Classical Geometric Construction," *Geometer*, 2002. [Google Scholar] [Publisher Link]

[18] The Three Impossible Constructions of Geometry, University of Toronto, 1997. [Online]. Available: https://www.math.toronto.edu/mathnet/questionCorner/impossconstruct.html

[19] Jesper Lützen, "The Algebra of Geometric Impossibility: Descartes and Montucla on the Impossibility of the Duplication of the Cube and the Trisection of the Angle," *Centaurus*, vol. 52, no. 1, pp. 4-37, 2010. [CrossRef] [Google Scholar] [Publisher Link]

[20] Richard Courant, and Herbert Robbins, *What is Mathematics?Aan Elementary Approach to Ideas and Methods,* New York Oxford University Press, pp. 134-135, 1996. [Google Scholar] [Publisher Link]

[21] Lucye Guilbeau**,** "The History of the Solution of the Cubic Equation," *Mathematics News Letter*, vol. 5, no. 4, pp. 8-12, 1930. [CrossRef] [Google Scholar] [Publisher Link]

[22] Heinrich Dörrie, *100 Great Problems of Elementary Mathematics,* Dover Publications, 2013. [Google Scholar] [Publisher Link]

[23] Felix Klein**,** *Famous Problems of Elementary Geometry: The Duplication of the Cube, the Trisection of the Angle, and the Quadrature of the Circle*, Palala Press, 2018. [Publisher Link]

[24] Wilbur Richard Knorr, "Pappus' Texts on Cube Duplication," *Textual Studies in Ancient and Medieval Geometry*, pp. 63-76, 1989. [CrossRef] [Google Scholar] [Publisher Link]

[25] Francois Rivest, and Stephane Zafirov**,** "Duplication of the Cube," *Zafiroff*, 1998. [Google Scholar] [Publisher Link]

[26] University of Wisconsin, Duplicating the Cube, Wisconsin-Green Bay, 2001.

[27] Ramon Masià, "A New Reading of Archytas' Doubling of the Cube and its Implications," *Archive for History of Exact Sciences*, vol. 70, pp. 175-204, 2016. [CrossRef] [Google Scholar] [Publisher Link]