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Original Article

Real Life Applications of Improper Integrals

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Abstract - *In Calculus , a function can grow infinitely large and yet give a finite area under its curve. The study of improper integrals revolves about this notion. These integrals cannot be computed applying the usual Riemann Integral. In this article, we discuss the two types of improper integrals and their evaluation methods. We illustrate the real life applications of improper integrals in different fields. We also discuss the significance, properties and applications of Beta and Gamma functions which are defined using the concept of improper integrals.*

Keywords - *Area under the infinite curve, Beta function, Convergence, Gamma function, Improper integral.*

1. Introduction

The needs and challenges of calculating lengths, areas and volumes of curvilinear geometric figures gave rise to the notion of definite integrals. A definite integral is said to be "proper" if the integrand is defined and finite on a closed and bounded interval. However, for practical purposes, it is often required to compute integrals over intervals of infinite lengths or evaluate integrals of functions with an infinite discontinuity at some point within the interval of integration. Such integrals are called improper integrals. Improper integrals are of immense importance in theory and applications of Mathematics.
Improper integrals are broadly classified into two types.

1.1. Improper Integrals of Type I

Definite integrals which have infinite limits of integration are called Improper integrals of type I. If $f(x)$ is a continuous function on the infinite interval $[a, \infty)$, we define

$$\int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx$$

If the above limit exists and is finite, we say that the improper integral is convergent. Otherwise the integral is called divergent.

Similarly, if f is a continuous function on the infinite interval $(-\infty, b]$, we define

$$\int_{-\infty}^b f(x)dx = \lim_{t\to-\infty} \int_t^b f(x)dx$$

If the above limit exists and is finite, then the improper integral is convergent, otherwise it is said to be divergent.

In the case of a continuous function $f(x)$ being integrated over the infinite interval $(-\infty, \infty)$, we choose an arbitrary point c and split the integral into two as follows:

$$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^c f(x)dx + \int_c^\infty f(x)dx$$

If both the improper integrals appearing on the right hand side of the above equation are convergent, we say that $\int_{-\infty}^\infty f(x)dx$ is convergent, otherwise it is said to be divergent.

Example (i). Compute $\int_{-\infty}^{\infty} \frac{dx}{x^2+16}$

Let $\quad I = \int_{-\infty}^{\infty} \frac{dx}{x^2+16}$

$$= \int_{-\infty}^{0} \frac{dx}{x^2+16} + \int_{0}^{\infty} \frac{dx}{x^2+16} = I_1 + I_2.$$

Then,

$$I_1 = \int_{-\infty}^{0} \frac{dx}{x^2+16} = \lim_{t\to-\infty} \int_{t}^{0} \frac{dx}{x^2+16}$$

$$= \lim_{t\to-\infty} \frac{1}{4}\left[tan^{-1}\frac{x}{4}\right]_t^0 = \frac{1}{4}\left[0-\left(-\frac{\pi}{2}\right)\right] = \frac{\pi}{8}$$

Similarly,

$$I_2 = \int_{0}^{\infty} \frac{dx}{x^2+16} = \lim_{t\to\infty} \int_{0}^{t} \frac{dx}{x^2+16}$$

$$= \lim_{t\to\infty} \frac{1}{4}\left[tan^{-1}\frac{x}{4}\right]_0^\infty = \frac{\pi}{8}$$

$$\Rightarrow \qquad I = \frac{\pi}{4}$$

1.2. Improper Integrals of Type II

Definite integrals of functions that become infinite at a point within the interval of integration are called improper integrals of type II.

If the function $f(x)$ is continuous on $[a, c) \cup (c, b]$, and is discontinuous at$c \in (a, b)$ then we define

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

If $f(x)$ is discontinuous at a but continuous on $(a, b]$, then

$$\int_a^b f(x)dx = \lim_{t\to a^+} \int_t^b f(x)dx$$

and if $f(x)$ is discontinuous at b and continuous on $[a, b)$, then

$$\int_a^b f(x)dx = \lim_{t\to b^-} \int_a^t f(x)dx$$

The improper integral $\int_a^b f(x)dx$ in each of the above cases is said to be convergent if the corresponding limit exists and is finite, otherwise the improper integral is said to be divergent.

Example (ii). The integral $\int_0^1 ln(x)dx$ is an improper integral of type II as $\lim_{x\to 0} ln(x) = -\infty$. Hence we evaluate the integral as follows:

$$\int_0^1 ln(x)dx = \lim_{t\to 0^+}\int_t^1 ln(x)dx = \lim_{t\to 0^+}[x\,ln(x) - x]_t^1$$

$= \lim_{t\to 0^+}[-1 - t\,ln(t) + t] = -1$

Example (iii). The improper integral $\int_{-1}^{2}\frac{1}{x^3}dx$ is also an improper integral of type II as $\lim_{x\to 0^+}\left(\frac{1}{x^3}\right) = \infty$.

Hence, we write

$$I = \int_{-1}^{2}\frac{1}{x^3}dx = \int_{-1}^{0}\frac{1}{x^3}dx + \int_{0}^{2}\frac{1}{x^3}dx = I_1 + I_2$$

Then,

$$I_1 = \int_{-1}^{0}\frac{1}{x^3}dx = \lim_{t\to o^-}\int_{-1}^{t}\frac{1}{x^3}dx = \lim_{t\to o^-} -\frac{1}{2}x^{-2}]_{-1}^{t} = \lim_{t\to o^-} -\frac{1}{2t^2}+\frac{1}{2} = -\infty.$$

Therefore, I_1 diverges. Consequently I diverges.

Note. One can easily fall into the trap and wrongly evaluate the integral *I* as

$$\int_{-1}^{2}\frac{1}{x^3}dx = -\frac{1}{2}x^{-2}]_{-1}^{2} = -\frac{1}{2}\left[\frac{1}{4}-1\right] = \frac{3}{8}.$$

whereas in reality this improper integral diverges as shown above.

The reason for this incorrect computation is due to incorrect application of the fundamental theorem of integral calculus to a discontinuous function.

2. Tests for Convergence of Improper Integrals

Sometimes an improper integral may not be directly integrable by using the above definitions. In such cases, we can apply a suitable test for convergence. The four basic tests are :

2.1. Comparison Test for Convergence

If $0 \le f \le g$ and $\int g(x)dx$ converges then $\int f(x)dx$ converges.

2.2. Comparison Test for Divergence

If $0 \le f \le g$ and $\int f(x)dx$ diverges, then $\int g(x)dx$ diverges.

2.3. Limit Comparison Test

If the functions $f(x)$ and $g(x)$ are positive and continuous on $[a,\infty)$ and if $\lim_{x\to\infty}\frac{f(x)}{g(x)} = L$, $0 < L < \infty$ then

$\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)dx$ both converge or diverge together.

2.4. Absolute Convergence Test

If the integral $\int |f(x)|\,dx$ converges then the integral $\int f(x)dx$ also converges.

We illustrate the limit comparison test with the help of an example.

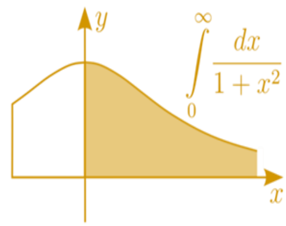
Example (iv). Show that $\int_1^\infty \frac{1-e^{-x}}{x} dx$ diverges.

For this, we take $f(x) = \frac{1-e^{-x}}{x}$ and $g(x) = \frac{1}{x}$. Then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty}(1 - e^{-x}) = 1 - 0 = 1.$

Also, $\int_1^\infty g(x)dx = \int_1^\infty \frac{1}{x} dx = \lim_{t\to\infty} ln(t)]_1^t = \infty.$ which gives that $\int_1^\infty g(x)dx$ diverges.

Therefore, by applying limit comparison test, we conclude that the integral $\int_1^\infty \frac{1-e^{-x}}{x} dx$ diverges.

3. Geometric Interpretation of Improper Integrals



If f is a non-negative function, the improper integral $\int_a^\infty f(x)dx$ is interpreted as the area of the region under the graph of f to the right of the line$x = a$.

Even though this region has an infinite extent, its area may be finite or infinite depending on whether the improper integral converges or diverges respectively.

4. Applications of Improper Integrals

Improper integrals are used in a number of applications in business, economics, biology, mathematical physics, probability, statistics and calculus. We now illustrate some of these applications.

4.1. Capital Value

The definite integral is used to determine the total income over a fixed number of years from a continuous income stream.

It is also used to find the present value of a continuous income stream that will be providing income in the future. When we extend the present value to an infinite time interval, the resulting quantity is called the capital value of the income stream and this is given by

$$\text{capital value} = \int_0^\infty f(t)e^{-rt}dt,$$

where $f(x)$ = annual rate of flow at a time *t,* and *r* = rate of interest compounded annually.

We illustrate this with an example.

Suppose that the alumni association of a college wants to pay back to the college by gifting an amount from which a sum of Rs.20,000 will be drawn every year to provide free text books to the needy students. Assuming that the annual rate of interest is 10% compounded continuously, find the amount that the alumni association must give to the college.

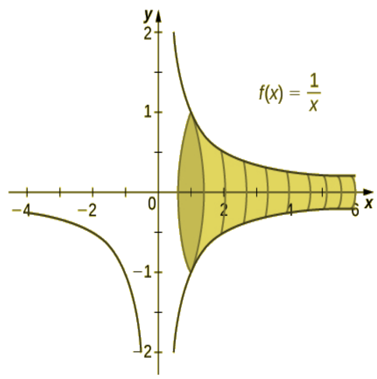
Here, we need to evaluate the

$$= \int_0^\infty f(t)e^{-rt}dt$$

$$Capital\ fund = \lim_{t\to\infty}\int_0^t (20000)e^{-0.10t}dt$$
$$= \lim_{t\to\infty}[-200000e^{-0.10t}]_0^t$$
$$= 200000.$$

Thus the fund needed for the purpose is Rs. 2,00,000.

4.2. Volume of Solid of Revolution



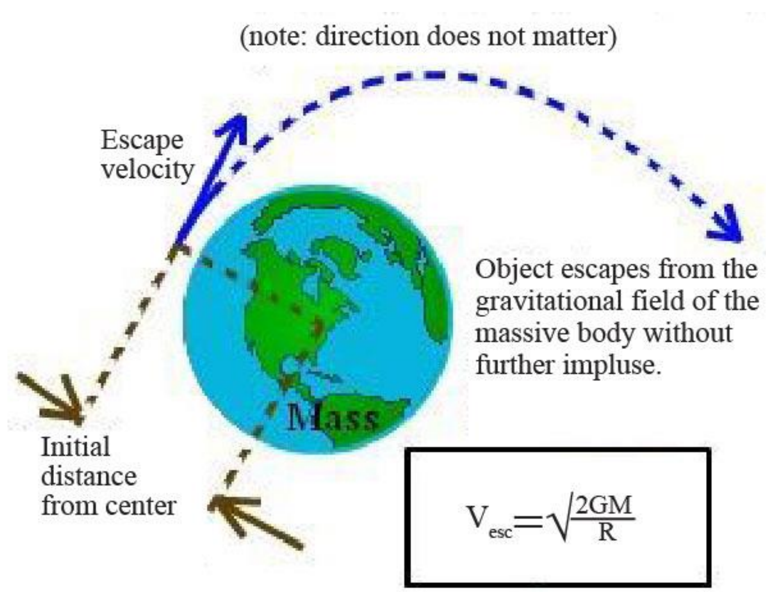
Since the process of integration has its roots in computing volumes, we can illustrate the use of improper integrals in finding volumes with the following example. Find the volume of the solid generated by revolving about the x-axis the region bounded by the graph of $f(x) = \frac{1}{x}$ and the x-axis for the interval $[1, \infty)$.

Here the solid of revolution is generated by revolving an infinite area about the x-axis.

The volume is given by

$$V = \pi\int_1^\infty \frac{1}{x^2}dx = \pi\lim_{t\to\infty}\int_1^t \frac{1}{x^2}dx = \pi\lim_{t\to\infty}\left[-\frac{1}{x}\right]_1^t = \pi\lim_{t\to\infty}\left(\frac{1}{t}+1\right) = \pi.$$

4.3. Escape Velocity



Taking $R \approx 6.37 \times 16^6 m$, $M \approx 5.98 \times 10^{24} kg$and $G \approx 6.67 \times 10^{-11} Nm^2/kg^2$ we obtain

$$m\frac{GM}{R} \approx m\frac{6.67\times10^{-11}Nm^2/kg^2 5.98\times10^{24}kg}{6.37\times10^6 m} \approx m \times 6.26 \times 10^7\,\frac{Nm}{kg}.$$

The energy required to escape the Earth's gravitational pull has to be kinetic energy which is given by $\frac{1}{2}mv^2$. Hence equating the kinetic energy to the above calculated energy we get

$$\frac{1}{2}mv^2 = m\frac{GM}{R}$$

$$\Rightarrow v = \sqrt{\frac{2GM}{R}} \approx \sqrt{2 \times 6.26 \times 10^7\,\frac{kg\frac{m}{s^2}m}{kg}} \approx 11{,}191\,\frac{m}{s}$$

Which is approximately equal to 11.2 km/s. This velocity is called the escape velocity and it is the minimum velocity required by an object on the earth to escape the earth's graviatation and enter the space. It may be noted though that other external factors like air friction, wind effect etc. are neglected in these calculations for the sake of simplicity.

4.4. Traffic Accidents at a Point

Consider a busy road intersection point where traffic accident occurs at an average of one in every three weeks. The traffic department looks into it and makes alterations in the traffic signals there. Consequently it has been noted that there has been no accident at the place in as long as eight weeks now. We investigate whether this is just a chance or the result of the steps taken by the traffic departmentFor this we use probability theory

If the average time between two consecutive accidents is *t*, then the probability density function is defined as

$$f(x) = \begin{cases} 0 \ldots x < 0 \\ te^{-tx} \ldots x \geq 0 \end{cases}.$$

And the probability that *X*, the time between two consecutive accidents, is between *a* and *b* weeks is given by

$$P(a \leq x \leq b) = \int_a^b f(x)dx.$$

We now calculate

$$P(X \geq 8) = \int_8^{+\infty} 3e^{-3x}dx$$

$$= \lim_{t\to+\infty}\left[\int_8^t 3e^{-3x}dx\right]$$

$$= \lim_{t\to+\infty} -e^{-3x}\Big|_8^t$$

$$= \lim_{t\to+\infty}(-e^{-3t} + e^{-24}) \approx 3.8 \times 10^{-11}$$

The value 3.8×10^{-11} represents the probability of no accidents in 8 weeks under the initial conditions. Since this value is very, very small, it is reasonable to conclude that the alterations made by the traffic department were effective to stop accidents.

5. Special Functions

Some of the convergent improper integrals have diverse applications in and outside the domain of mathematics. We now have a look at the two most significant improper integrals which converge under certain conditions. These are known as beta and gamma functions. As introduced by the Swiss mathematician Leonhard Euler in18th century, the gamma function is an extension of the factorial function to real numbers. It is a single variable function. Beta function (also known as Euler's integral of the first kind) is closely connected to gamma function. The Beta function was originally studied by Euler and Legendre while it was given its name by Jacques Binet. It is a two variable function. Both Beta and Gamma functions are very important in calculus as complex integrals can be moderated into simpler form using these functions. We now discuss the Beta and Gamma functions in detail.

5.1. Gamma Function

For $x > 0$, the improper integral given by $\boldsymbol{\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt}$converges and it is called the Gamma function. To establish the convergence of this improper integral, we note that

$t^{x-1}e^{-t} \leq t^{x-1}$, for all $t > 0$ and the integral $\int_0^1 t^{x-1}dt$ converges for $x > 0$.

Hence, by comparison test, $\int_0^1 t^{x-1}e^{-t}dt$ converges for $x > 0$. Also, we know that the improper integra$\int_1^\infty t^{-2}dt$ is convergent and $\lim\limits_{t\to\infty} \frac{t^{x-1}e^{-t}}{t^{-2}} = 0$. So by limit comparison test the improper integral $\int_1^\infty t^{x-1}e^{-t}dt$ also converges. Therefore we get that

$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt = \int_0^1 t^{x-1}e^{-t}dt + \int_1^\infty t^{x-1}e^{-t}dt$ converges for all $x > 0$.

Gamma function is also known as the Euler's integral of second kind.

5.2. Beta Function

For all $x, y > 0$, the improper integral

$\boldsymbol{\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt}$ converges

and it is called the Beta function.

It is clear that for $x, y \geq 1$, the integral is proper and finite.

However if $x, y < 1$, then the integrand becomes infinite at $t = 0$ and $t = 1$.

So we write.

$$\int_0^1 t^{x-1}(1-t)^{y-1}dt = \int_0^{1/2} t^{x-1}(1-t)^{y-1}dt + \int_{1/2}^1 t^{x-1}(1-t)^{y-1}dt$$

$$= I_1 + I_2$$

Now for $0 < t \leq \frac{1}{2}$, $(1-t)^{y-1} \leq 2^{1-y}$

$\Rightarrow \quad t^{x-1}(1-t)^{y-1} \leq 2^{1-y}t^{x-1}$

Since the integral $\int_0^{1/2} t^{x-1}dt$ converges,

$I_1 = \int_0^{1/2} t^{x-1}(1-t)^{y-1} dt$ also converges by comparison test.

Again, in I_2 we put $u = 1 - t \Rightarrow du = -dt$.

This gives that

$$I_2 = \int_{1/2}^{1} t^{x-1}(1-t)^{y-1} dt = \int_0^{1/2} u^{y-1}(1-u)^{x-1} du$$

which converges in the similar way as does I_1.

Hence $\beta(x, y) = I_1 + I_2$ converges for every $x, y > 0$.

The Beta function is also called Euler's integral of first kind.

Properties of the Gamma and Beta Function

1. $\Gamma(1) = 1$.
2. $\Gamma(n) = (n-1)\Gamma(n-1)$.
3. $\Gamma(n+1) = n!$.
4. $\beta(m, n) = \beta(n, m)$.
5. $\beta(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$.

Applications of Beta and Gamma Functions

(i) Many features of the strong nuclear force are described by the beta function. In time management problems, the beta function is used to determine the average time of performing chosen tasks. In the preferential attachment process, the stochastic scattering process and beta function are used. A preferential attachment process is one in which a particular amount of something is divided among persons based on how much of it they already have.

(ii) Calculus, differential equations, complex analysis, and statistics all use the gamma function in some way. While the gamma function behaves like a factorial when applied to natural numbers, which are a discrete set, its application to positive real numbers, which are a continuous set, makes it ideal for modelling scenarios involving continuous change.

6. Conclusion

The role of Improper integrals is highly significant in the development of various mathematical concepts. Also these integrals appear in numerous applications. The study of properties and behavior of the improper integrals is imperative. Special functions like the beta and gamma functions are defined with the help of improper integrals that converge under specific conditions. These functions are applicable in various branches of engineering and technology.

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