International Journal of Mathematics Trends and Technology
ISSN: 2231-5373/ https://doi.org/10.14445/22315373/IJMTT-V69I9P503
Volume 69 Issue 9, 13-22, September 2023

Original Article

𝕋-Relative Fuzzy Linear Programming

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Received: 13 July 2023 Revised: 19 August 2023 Accepted: 04 September 2023 Published: 27 September 2023

Abstract - *We introduce and develop in this article linear programming in the context of the 𝕋-Relative fuzzy sets that were introduced by Osawaru, Olaleru and Olaoluwa [3]. Here, the objective function and (or) its set of constraints expressed as a function of a parameter (or variable), say time, is considered. By relaxing the pretenses of optimization using a subjective gradation relative to the parameter, we model and obtain optimal solutions by employing the tools of the relative fuzzy membership functions. Thus, fuzzy optimal values obtained are expressed relative to the parameter of control defining the dynamics of the fuzziness. The results of this study generalize the results obtained for fuzzy linear programming in literature.*

Keywords - 𝕋-*Relative fuzzy sets,* 𝕋-*Relative fuzzy mapping,* 𝕋-*Relative fuzzy linear programming, Fuzzy sets.*

1. Introduction

Complexities due to uncertainty in the form of ambiguity, uncertainty related to errors, vagueness, sparsity of data, subjectivity of expert judgements, and chance or incomplete knowledge characterize most real-world problems. Thus, fuzzy methods are best suited to define, express and solve such problems. So, formulating linear programming problems using fuzzy membership maps as well as employing fuzzy concept tools to obtain optimal solutions, has yielded better and richer results. This approach provides better results as the manager's knowledge is considered a fuzzy one.

Fuzzy set was introduced by L. A. Zadeh to represent these complexities; several authors have extended and applied it in different fields. Our interest in this study is the Fuzzy Linear Programming (FLP). The concept of FLP as a traditional Linear Programming (LP) in a fuzzy environment was introduced in 1978 by Zimmermann [28] following "Decision Making in Fuzzy Environment" and "On Fuzzy Mathematical Programming" proposed by Bellman R. E., Zadeh L. A. [1] and Tanaka H. et al. respectively. Gasimov and Yenilmez [24] studied mathematical programming with a single objective expressed by fuzzy parameters. Since the introduction of the concept, FLP has been constructed in a number of directions with several positive applications, including the proposed parametric programming approach for solving FLP. The authors of [29] formulated an FLP problem for a fuzzy system endowed with a fuzzy Boolean coverage, and recently, the authors of [31] simplified with certain substitutions on nonlinear terms the formulated FLP of [29] to a $0-1$ integer program. The model was used to determine optimality conditions for a set covering problem in a fuzzy environment.

Recently, [3] introduced a more profound generalization of the fuzzy set. It is called 𝕋-Relative fuzzy set (TRFS). The authors gave its characterization and showed that they extended the concept of the fuzzy set by taking into account the variability of membership grades of elements of the set. The applications of the TRFS sets were also buttressed by [3]. [23] defined Relative fuzzy maps in the Heilpern. [2] sense and studied fixed point results for Relative Fuzzy Maps. This further enriched the fuzzy map definition with the capacity to represent and express dynamic fuzzy maps in this sense. Thus, many fuzzy maps results were deductions from the relative fuzzy maps results.

This article focused on the introduction of 𝕋-Relative Fuzzy Linear Programming (RFLP) as a Fuzzy Linear Programming (FLP) whose objective function and (or) constraints are defined as 𝕋-Relative fuzzy membership functions. It represents the fuzzy version of an LP problem whose constraints and or objective functions are expressed 𝕋-Relative to a variable. As a generalization of the fuzzy point, the concept of a 𝕋-Relative fuzzy point and relative fixed fuzzy point was defined by [23], which is suitable for an RFLP and is useful in the sequel. Their results also provide sufficient theoretical background for this study.

2. Preliminaries

The following definitions and results from [3,23] will suffice in this article.

Definition 2.1. A $\mathbb{T}$-Relative Fuzzy Set (TRFS) R of X $\mathbb{T}$ is a set with the membership growth function $\mu_R: X \times \mathbb{T} \to [0,1]$ such that

$$\mu_R(x, \sigma(t)) = \begin{cases} 0 & \text{if } x \notin R \\ k \in (0,1] & \text{if } x \in R \text{ and } t \in \mathbb{T} \end{cases}$$

$\forall x \in X$ and $t \in \mathbb{T}$, where $\mathbb{T}$ is any time scale, $R \subset X$ and $\sigma: \mathbb{T} \to \mathbb{T}$ a forward difference operator.

Definition 2.2. Let $\{R_i\}_{i=1}^{n}$ be TRFSs of X, (X, any nonempty set) with membership functions $\{\mu_{R_i}\}_{i=1}^{n}$ respectively. Then
(i) the union of $\{R_i\}_{i=1}^{n}$ over $\mathbb{T}$ (respectively for any $t \in \mathbb{T}$) is the TRFS membership function defined as

$$\mu_{(\cup_{i=1}^{n} R_i)}(x,t)_{\mathbb{T}} = \max_{x \in X, t \in \mathbb{T}} \{\mu_{R_i}(x,t)\}$$

(respectively $\mu_{(\cup_{i=1}^{n} R_i)}(x,t)_t = \max_{x \in X} \{\mu_{R_i}(x,t)\}$ for any $t \in \mathbb{T}$)

(ii) the intersection of $\{R_i\}_{i=1}^{n}$ over $\mathbb{T}$ (respectively for any $t \in \mathbb{T}$) is the TRFS with $\mathbb{T}$- membership function defined as

$$\mu_{(\cap_{i=1}^{n} R_i)}(x,t)_{\mathbb{T}} = \min_{x \in X, t \in \mathbb{T}} \{\mu_{R_i}(x,t)\}$$

(respectively $\mu_{(\cap_{i=1}^{n} R_i)}(x,t)_t = \min_{x \in X} \{\mu_{R_i}(x,t)\}$ for any $t \in \mathbb{T}$)

(iii) $R_i \subset R_j$ over $\mathbb{T}$ (respectively for any $t \in \mathbb{T}$) if

$$\mu_{R_i}(x,t)_{\mathbb{T}} \leq \mu_{R_j}(x,t)_{\mathbb{T}} \forall x \in U \text{ and } \forall t \in \mathbb{T}$$

(respectively $\mu_{R_i}(x,t)_t \leq \mu_{R_j}(x,t)_t \forall x \in U$ and for any $t \in \mathbb{T}$) for any i, j with $i \neq j$.

Definition 2.3. For a metric space (X, d), from [23] the level sets of a TRFS with $\alpha \in (0,1]$ is defined by

$$R(\alpha)_{\mathbb{T}} = \{x \in X: \mu_R(x,t) \geq \alpha, \text{ for all } t \in \mathbb{T}\}$$

(respectively $R(\alpha)_t = \{x \in X: \mu_R(x,t) \geq \alpha, \text{ for any } t \in \mathbb{T}\}$)

$\phi - \mathbb{T}$-Relative fuzzy point for all members of $\mathbb{T}$ denoted $x_{\phi_{\mathbb{T}}}$ is called a $\phi - \mathbb{T}$-Relative fuzzy fixed point of $T_{\mathbb{T}}$ if $x_{\phi_{\mathbb{T}}} \subset T_{\mathbb{T}}x$. That is $x \in T_{\mathbb{T}}x(\phi)$ for all $t \in \mathbb{T}$, where ϕ is a function on $X \times \mathbb{T}$ into 0,1].

Definition 2.4. For a metric space (X, d), let $\pi(X)_{\mathbb{T}}$ be a family of approximate quantities and $\alpha \in (0,1]$. Then the $\mathbb{T}$-Relative fuzzy map $T_{\mathbb{T}}: X \to \pi(X)_{\mathbb{T}}$ for all members of $\mathbb{T}$ is said to be a $\alpha - \mathbb{T}$-Relative fuzzy contraction map if

$$D_\alpha(T_{\mathbb{T}}x, T_{\mathbb{T}}y)_{\mathbb{T}} \leq ad(x,y)$$

for all $x, y \in X$, and $t \in \mathbb{T}$.

Definition 2.5. For a metric space (X, d), let $\pi(X)_{\mathbb{T}}$ be a family of approximate quantities and $\alpha \in (0,1]$. Then for $T_{\mathbb{T}}: X \to \pi(X)_{\mathbb{T}}$ for all members of $\mathbb{T}$,

(i) $F_T(\alpha) = \{x^* \in X: x^* \in T_{\mathbb{T}}x^*(\alpha)\}$ is the set of $\alpha - \mathbb{T}$-Relative fuzzy fixed points of $T_{\mathbb{T}}$ for all members of $\mathbb{T}$.
(ii) for any $x_0 \in X$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by $x_{n+1} \in T_{\mathbb{T}}x_n(\alpha)$ where $n \geq 0$, is called the $\alpha - \mathbb{T}$-Relative fuzzy Picard iterative scheme iff for each $x \in X$ and any $y \in T_{\mathbb{T}}x$ the sequence $\{x_n\}_{n=0}^{\infty}$ converges to a fixed point of $T_{\mathbb{T}}$ with $x_0 = x$ and $x_1 = y$.

Theorem 2.1. The ϕ - $\mathbb{T}$-Relative fuzzy contraction function, where X is a complete metric space. Then
(i) there exists $x^* \in X$ (called the $\phi - \mathbb{T}$-Relative fuzzy fixed point) i.e., $x^* \in T_{\mathbb{T}}x^*(\phi)$.

(ii) $x^* is$ a unique if $d(x^*, y^*) \leq p_\phi(T_{\mathbb{T}}x^*, T_{\mathbb{T}}y^*)_{\mathbb{T}}$ for any $y^* \in F_T(\phi)$.
(ii) the scheme, $x_{n+1} \in T_{\mathbb{T}}x_n(\phi)$ converges to $x^*_{\phi_{\mathbb{T}}}$ of $T_{\mathbb{T}}$ strongly.

3. Main Result

In optimization theory, problems are modelled in the form of optimizing objective(s) at the instance of some constraints. Obtaining a feasible solution for the modeled problem is usually challenging; hence, the task of a manager making a decision is usually complex, especially in multi-criteria problems. In operations research, linear programming techniques are robust tools used to solve such problems. In real-life problems, it is usually difficult to determine precise targets for set goals. Fuzzy membership functions are introduced to deal with such difficult situation. The goal is to relax the pretenses of optimization by means of a subjective gradation, which can be modelled into fuzzy membership functions. μ_i. So, that if $F = \cap \mu_i$ the objective will be to search for x such that max$F = F(x)$. If max$F = 1$, then there exists x such that $F(x) = 1$, but if max$F = \alpha \in (0,1)$, the solution of the multiobjective optimization is a fuzzy point. x_α and $F(x) = \alpha$. In a more general sense than the one given by Heilpern [2], a mapping $F: X \to I^X$ is a fuzzy function over X and $(F(x))(x)$ is the fixed degree of x for F. Now if an objective function and or its set of constraints is expressed as a function of a parameter, say time, then relaxing the pretenses of optimization by means of a subjective gradation, which can be modelled into fuzzy membership functions is not possible. The way out would be to resolve it using TRS tools. The stages are as summarized below.

4. $\mathbb{T}$-Relative Fuzzy Optimization

Here we propose in the sense of [28] and [30] $\mathbb{T}$-Relative Fuzzy Linear Programming (RFLP) problem.

(i) The set of objective functions that maximizes (or minimizes) a problem may change with respect to some factors. Thus, such problems are presented as optimizing some goal function $\mathbb{T}$-Relative to the factors (for example time) given certain constraints. Such set of objective functions can be modeled into $\mathbb{T}$-Relative fuzzy membership function $\mu_{\mathbb{T}}$. It is achieved by relaxing the pretenses of optimization by means of a subjective gradation with respect to a time scale value.

(ii)Examine the following standard form of the symmetric time scale dependent LP problem:

$$\begin{aligned} \max\ z &= tcx \\ \text{s.t. } tAx &\lesssim \mathrm{b} \\ x &\geq 0 \end{aligned}$$

where, $A \in \mathbb{R}^{m\times n}$, $\mathrm{b} \in \mathbb{R}^m$, $c^T \in \mathbb{R}^n$, $x \in \mathbb{R}^n$, $t \in \mathbb{T}$ and $\lesssim$ and $\gtrsim$ are the fuzzy versions of $\leq$ and $\geq$ respectively.

To represent the $\mathbb{T}$-Relative fuzzy goal, we impose that the objective function ctx with respect to t be essentially greater than or equal to an aspiration level b_0, chosen by the manager. The following problems will be adequate:

$$\begin{aligned} \text{find} &\quad x \\ \text{s.t.tcx} &\quad \gtrsim b_0 \\ tAx &\quad \lesssim \mathrm{b} \\ x &\quad \geq 0. \end{aligned}$$

(iii) For treating fuzzy inequalities, we propose a $\mathbb{T}$-Relative linear membership function as follows:

$$\mu_A^o(x,t) = \begin{cases} 1 & \text{if} \quad tcx > b_0 \\ 1 - \dfrac{b_0 - tcx}{p_0} & \text{if} \quad b_0 - p_0 \leq tcx \leq b_0 \\ 0 & \text{if} \quad tcx < b_0 - p_0 \end{cases}$$

$$\mu_A^j(x,t) = \begin{cases} 1 & \text{if} \quad \left(tAx_j\right) < b_j \\ 1 - \dfrac{\left(tAx_j\right) - b_j}{p_j} & \text{if} \quad b_j\left(tAx_j\right) < b_j + p_j j = 1,2,\dots,m \\ 0 & \text{if} \quad \left(tAx_j\right) > b_j - p_j \end{cases}$$

where for $j=1,\ldots,m(tAx_j)$ is the jth row of Atx, b_j is the jth element of b and for $j=0,1,\ldots,m$, p_j is a subjectively chosen constant by the manager expressing the limit of the admissible violation of the jth inequality.

(iv) Suppose $\{\mu_{\mathbb{T}}\}_j^m$ is the m set of $\mathbb{T}$-Relative fuzzy membership functions maximizing the set of objective functions $\mathbb{T}$-Relative to the $\mathbb{T}$. Using the "min" operator of Bellman and Zadeh (1970) together with the above $\mathbb{T}$-Relative linear membership functions then

(a) the optimal region (solution space) for any fixed $t\in\mathbb{T}$ is given by the $\mathbb{T}$-Relative fuzzy membership function

$$\mu_D(x,t)=\min_{j=1,\ldots,m}\{\mu^j(x,t)\} \text{ or } F_t=\min\{\{\mu_t\}_j^m\}.$$

If $\max F_t=1$, then there exists x, the optimal solution of the multi objective optimization with respect to t is a fuzzy point x_{1_t} such that $F(x,t)_t=1$ for a fixed $t\in\mathbb{T}$. More generally if $\max F_t=\alpha_t, \alpha_t\in(0,1]$ then the optimal solution of the multi objective optimization with respect to t is a fuzzy point x_{α_t} such that $F_t(x,t)=\alpha_t$ i.e. there exists x such that $F(x,t)_t=\alpha$ for a fixed $t\in\mathbb{T}$.

(b) the optimal region (solution space) for all $t\in\mathbb{T}$ is given by the $\mathbb{T}$-Relative fuzzy membership function

$$\mu_D(x,t)=\min_{t\in\mathbb{T}}\left\{\min_{j=1,\ldots,m}\{\mu^j(x,t)\}\right\} \text{ or } F_{\mathbb{T}}=\min_{t\in\mathbb{T}}\left\{\min\{\{\mu_t\}_j^m\}\right\}=\min\{\{\mu_{\mathbb{T}}\}_j^m\}.$$

If $\max F_{\mathbb{T}}=1$, then there exists x the optimal solution of the multi objective optimization with respect to $\mathbb{T}$ is a fuzzy point $x_{1_{\mathbb{T}}}$ such that $F(x,t)_{\mathbb{T}}=1$ for all $t\in\mathbb{T}$. More generally, if $\max F_{\mathbb{T}}=\alpha_{\mathbb{T}}$, $\alpha_{\mathbb{T}}\in(0,1]$ then the optimal solution of the multi-objective optimization with respect to $\mathbb{T}$ is a fuzzy point $x_{\alpha_{\mathbb{T}}}$ such that $F_{\mathbb{T}}(x,t)=\alpha_{\mathbb{T}}$ i.e there exists x such that $F(x,t)_{\mathbb{T}}=\alpha$ for all $t\in\mathbb{T}$.

(v) Applying Theorem 2.1 of [26], we can determine the existence of an α relative fixed point. Thus, we have to determine if

(a) $D_\alpha(\{x\in X:\mu_D(x,t)=\alpha$, for any $t\in\mathbb{T}\},\{y\in X:\mu_D(y,t)=\alpha$, for any $t\in\mathbb{T}\})_t\le ad(x,y)$, for all $x,y\in X$

(b) $D_\alpha(\{x\in X:\mu_D(x,t)=\alpha$, for all $t\in\mathbb{T}\},\{y\in X:\mu_D(y,t)=\alpha$, for all $t\in\mathbb{T}\})_{\mathbb{T}}\le ad(x,y)$, for all $x,y\in X$

(vi) The α relative fixed point if it exists coincide with the α-optimal solution of the multi-objective optimization problem. So

(a) If $\max F_t=1$, then there exists x, the optimal solution of the multi objective optimization with respect to t is a fuzzy point x_{1_t} such that $F(x,t)_t=1$ for a fixed $t\in\mathbb{T}$. More generally if $\max F_t=\alpha_t,\alpha_t\in(0,1]$ then the optimal solution of the multi objective optimization with respect to t is a fuzzy point x_{α_t} such that $F_t(x,t)=\alpha_t$ i.e. there exists x such that $F(x,t)_t=\alpha$ for a fixed $t\in\mathbb{T}$.

(b) If $\max F_{\mathbb{T}}=1$, then there exists x the optimal solution of the multi objective optimization with respect to $\mathbb{T}$ is a fuzzy point $x_{1_{\mathbb{T}}}$ such that $F(x,t)_{\mathbb{T}}=1$ for all $t\in\mathbb{T}$. More generally, if $\max F_{\mathbb{T}}=\alpha_{\mathbb{T}}$, $\alpha_{\mathbb{T}}\in(0,1]$ then the optimal solution of the multi objective optimization with respect to $\mathbb{T}$ is a fuzzy point $x_{\alpha_{\mathbb{T}}}$ such that $F_{\mathbb{T}}(x,t)=\alpha_{\mathbb{T}}$ i.e. there exists x such that $F(x,t)_{\mathbb{T}}=\alpha$ for all $t\in\mathbb{T}$. (vii) Alternatively, the RFLP problem can be transformed into an equivalent LP problem by introducing an auxiliary variable λ as follows:

$$\begin{aligned} \max\quad & \lambda \\ \text{s.t. } \lambda\quad & \le t\mu_A^j(x,t) \\ x\quad & \ge 0, \lambda\in[0,1] \end{aligned}$$

which gives

$$\begin{aligned} \max\quad & \lambda \\ \text{s.t. } ctx\quad & \ge b_0-(1-\lambda)p_0 \\ (\text{ Atx })_j\quad & \ge b_0+(1-\lambda)p_j \\ x\quad & \ge 0, \lambda\in[0,1] \end{aligned}$$

Thus, we have the equivalent LP problem for each $t\in\mathbb{T}$. So the optimal solution for $3.1(iv)$ is also optimal for $3.1(v)$

Example 4.5.2. Suppose a firm's problem is to design an attractive dividend structure depending on the volume of shares units held by its shareholders limited by the government regulation of what a modest dividend should be. If the interval of dividend x and share units u are $X = [0,5.8]$ and $\mathbb{U} = [0.1,2.0]$ respectively. The fuzzy set of the objective function attractive dividend $\mathbb{T}$-Relative to units of shares could, for instance, be defined by:

$$\mu_X^o(x,u) = \begin{cases} 1 & x \geq 5, u \geq 1.5 \\ 0.9 & x \geq 5, 0.1 \leq u < 1.5 \\ -\left[\dfrac{u+0.9}{2100}(29x^3 - 366x^2 - 877x + 540)\right] & 0.5 < x < 5, 0.1 \leq u \leq 2 \\ 0.05 & x \leq 0.5, 0.2 < u \leq 2 \\ 0 & x \leq 0.5, u \leq 0.2 \end{cases}$$

The fuzzy set (constraint) "modest dividend" could be represented by

$$\mu_X^c(x) = \begin{cases} 1 & x \geq 5 \\ \dfrac{1}{2100}[-29x^3 - 243x^2 + 16x + 2388] & 0.5 < x < 4.9 \\ 0 & x \leq 0.5 \end{cases}$$

The $\mathbb{T}$-Relative fuzzy set "decision" at any u is then characterized by its membership function $\mu_X^D(x,u) = \min\{\mu_X^o(x,u), \mu_X^c(x)\}$ i.e. the fixed fuzzy point is x_{α_u} such that $\max\mu_X^D(x,u) = \alpha$ for a fixed u For $u = 1.5, \mu_X^D(x,u) = \min\left\{-\left[\frac{u+0.9}{2100}(29x^3 - 366x^2 - 877x + 540)\right], 0.05, \mu_X^c(x)\right\}$ and the $\mathbb{T}$-Relative fuzzy set "decision" for all u is then characterized by its membership function

$$\mu_X^D(x) = \min_{u \in \mathbb{U}}\{\min\{\mu_X^o(x,u), \mu_X^c(x)\}\}$$

for all $u \in \mathbb{U}$ i.e. the fixed fuzzy point is $x_{\alpha_{\mathbb{U}}}$ such that $\max\mu_X^D(x) = \alpha$ for all $u \in \mathbb{U}$

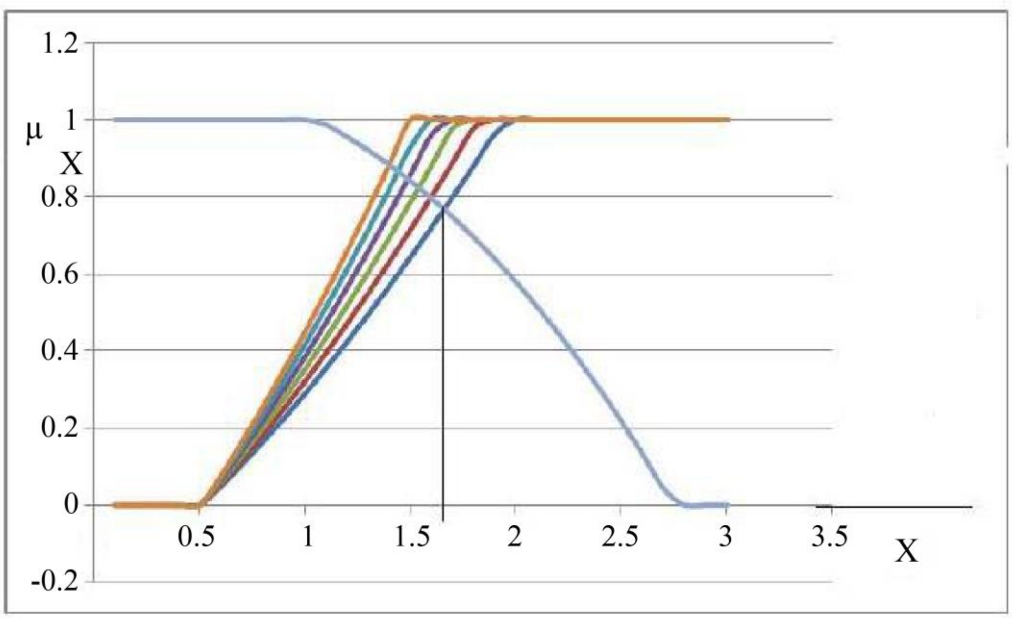

Fig. 1 The $\mathbb{T}$-Relative fuzzy set "decision"

The graph in the figure below shows the optimal region. Thus $\max\mu_X^D(1.7) \approx 0.7$. The next example considers the application of the fuzzy cover problem in [29] in the context of a relative fuzzy set and a fuzzy Boolean coverage setting. A generalized mathematical programming problem is formulated and used to obtain relative fuzzy optimality conditions. The results are compared with results for the fuzzy set covering problem and (or) Boolean coverage.

Example 4.5.3. Suppose there are five goals $I = \{1,2,3,4,5\}$ to be attained (reached) by levels (degrees) of the combination of four means $J = \{1,2,3,4\}$ in three seasons $K = \{1,2,3\}$. Let $\chi_j(i,k) \in [0,1], i \in I, j \in J, k \in K$, with $\chi_j: I \times K \to [0,1]$ such that

$$\chi_j(i,k) = \begin{cases} p \in (0,1] & \text{if mean } j \text{ is used to achieve Goal } i \text{ in Season } k \\ 0 & \text{if mean } j \text{ is not used to achieve Goal } i \end{cases}$$

be the measure of application of mean j to achieve goal i in Season k. Thus, the grade of attained goal i by using mean $j = j'$ is more than the grade of attained goal i at any season k if $\chi_{j'}(i,k) \geq \chi_{j''}(i,k)$ and the grade of attained goal i is full at any season k if $\chi_{j'''}(i,k) = 1$.

Let c_j be the cost of using mean j and $x_j, j = 1,2,3,4$, the fuzzy (partial usage) means usable defined as follows;

$$x_j = \begin{cases} p \in (0,1] & \text{if mean } j \text{ is used} \\ 0 & \text{if mean } j \text{ is not used} \end{cases}$$

Thus, the usage of mean $j = j^*$ is more than the usage of mean j^{**} if $x_{j^*} \geq x_{j^{**}}$ and any mean j^{**} is fully used if $x_{j^{***}} = 1$.

The values for $\chi_j(i,k), c_j$ for the seasons are summarized in the Table below:

Table 1. The Array of $\chi_j(i,k)$

	Goals (I)	**1**	**2**	**3**	**4**	**5**
Means (J)	**Seasons (K)**					
$1(c_1 = 4)$	1	0.4	0.1	0.5	0.7	0.8
	2	0.2	0.2	0.4	0.8	0.7
	3	0.1	0.3	0.3	0.9	0.6
$2(c_2 = 2)$	1	0.1	0.3	0.8	0.2	0.6
	2	0.05	0.4	0.7	0.3	0.4
	3	0.05	0.5	0.6	0.4	0.4
$3(c_3 = 5)$	1	0.3	0.7	0.2	0.9	0.4
	2	0.2	0.8	0.1	0.9	0.3
	3	0.1	0.9	0.05	1.0	0.02
$4(c_4 = 2)$	1	0.5	0.9	0.4	0.1	0.2
	2	0.4	0.9	0.3	0.2	0.1
	3	0.3	1.0	0.2	0.3	0.05

Table 1. The Array of $\chi_j(i,k)$

We need to find

(i) a system of fuzzy means which ensures that the goal i is reached at each k season with α - level and the minimum total cost of mean usage.

(ii) a system of fuzzy means which ensures that the goal i is reached for all k season with α - level and the minimum total cost of mean usage for all k seasons.

5. Remark 4.5.4.

(i) We refer to $\chi_j(i,k)$ defined above as the Relative fuzzy Boolean coverage.

(ii) x_j defined above is called the fuzzy system [29].

(iii) If for each i and $j, \chi_j(i,k)$ is constant for all $k \in K$ then the problem becomes a fuzzy Boolean coverage problem. Thus, we have the following relative fuzzy sets,

$P_1 =$ {(1,1,0.4), (1,2,0.2), (1,3,0.1), (1,1,0.1), (1,2,0.2), (1,3,0.3), (1,1,0.5), (1,2,0.4), (1,3,0.3), (1,1,0.7), (1,2,0.8), (1,3,0.9), (1,1,0.8), (1,2,0.7), (1,3,0.6)}
$P_2 =$ {(1,1,0.1), (1,2,0.05), (1,3,0.05), (1,1,0.3), (1,2,0.4), (1,3,0.5), (1,1,0.8), (1,2,0.7), (1,3,0.6), (1,1,0.2), (1,2,0.3), (1,3,0.4), (1,1,0.6), (1,2,0.4), (1,3,0.4)}
$P_3 =$ {(1,1,0.3), (1,2,0.2), (1,3,0.1), (1,1,0.7), (1,2,0.8), (1,3,0.9), (1,1,0.2), (1,2,0.1), (1,3,0.05), (1,1,0.9), (1,2,0.9), (1,3,1.0), (1,1,0.4), (1,2,0.3), (1,3,0.02)}
$P_4 =$ {(1,1,0.5), (1,2,0.4), (1,3,0.3), (1,1,0.9), (1,2,0.9), (1,3,1.0), (1,1,0.4), (1,2,0.3), (1,3,0.2), (1,1,0.1), (1,2,0.2), (1,3,0.3), (1,1,0.2), (1,2,0.1), (1,3,0.05)}

Then the RFLP problem can be expressed as a FLP problem for each k as follows:

$$\text{Min} \sum_{j=1}^{n} c_j x_j$$
$$\text{s.t} \quad \left[1 - \prod_{j=j}^{n} \left(1 - \chi_j(i,k)x_j\right)\right] \geq \alpha, i = 1.2 \dots, m$$
$$x_j \in [0,1], j = 1,2, \dots, n, \alpha \in (0,1]$$

So, the problem in example above can be formulated as follows

$$\text{Min} \sum_{j=1}^{n} c_j x_j = 4x_1 + 3x_2 + 5x_3 + 2x_4$$
$$\text{s.t} \ 1 - [(1 - 0.4x_1)(1 - 0.1x_2)(1 - 0.3x_3)(1 - 0.5x_4)] \geq \alpha,$$
$$1 - [(1 - 0.1x_1)(1 - 0.3x_2)(1 - 0.7x_3)(1 - 0.9x_4)] \geq \alpha,$$
$$1 - [(1 - 0.5x_1)(1 - 0.8x_2)(1 - 0.2x_3)(1 - 0.4x_4)] \geq \alpha,$$
$$1 - [(1 - 0.7x_1)(1 - 0.2x_2)(1 - 0.9x_3)(1 - 0.1x_4)] \geq \alpha,$$
$$1 - [(1 - 0.8x_1)(1 - 0.6x_2)(1 - 0.4x_3)(1 - 0.2x_4)] \geq \alpha,$$
$$x_j \in [0,1], j = 1,2, \dots, n, k = 1.$$
$$\text{Min} \sum_{j=1}^{n} c_j x_j = 4x_1 + 3x_2 + 5x_3 + 2x_4$$
$$\text{s.t} \ 1 - [(1 - 0.2x_1)(1 - 0.05x_2)(1 - 0.2x_3)(1 - 0.4x_4)] \geq \alpha,$$
$$1 - [(1 - 0.2x_1)(1 - 0.4x_2)(1 - 0.8x_3)(1 - 0.9x_4)] \geq \alpha,$$
$$1 - [(1 - 0.4x_1)(1 - 0.7x_2)(1 - 0.1x_3)(1 - 0.3x_4)] \geq \alpha,$$
$$1 - [(1 - 0.8x_1)(1 - 0.3x_2)(1 - 0.9x_3)(1 - 0.2x_4)] \geq \alpha,$$
$$1 - [(1 - 0.7x_1)(1 - 0.4x_2)(1 - 0.3x_3)(1 - 0.1x_4)] \geq \alpha,$$
$$x_j \in [0,1], j = 1,2, \dots, n, k = 2.$$
$$\text{Min} \sum_{j=1}^{n} c_j x_j = 4x_1 + 3x_2 + 5x_3 + 2x_4$$
$$\text{s.t} \ 1 - [(1 - 0.1x_1)(1 - 0.05x_2)(1 - 0.1x_3)(1 - 0.3x_4)] \geq \alpha,$$
$$1 - [(1 - 0.3x_1)(1 - 0.5x_2)(1 - 0.9x_3)(1 - 1.0x_4)] \geq \alpha,$$
$$1 - [(1 - 0.3x_1)(1 - 0.6x_2)(1 - 0.05x_3)(1 - 0.2x_4)] \geq \alpha,$$
$$1 - [(1 - 0.9x_1)(1 - 0.4x_2)(1 - 1.0x_3)(1 - 0.3x_4)] \geq \alpha,$$
$$1 - [(1 - 0.6x_1)(1 - 0.4x_2)(1 - 0.02x_3)(1 - 0.05x_4)] \geq \alpha,$$
$$x_j \in [0,1], j = 1,2, \dots, n, k = 3.$$

At an $\alpha = 0.5$-level degree, the following optimal solutions are obtained using LINGO 18.0. See Appendix 1-3 for the output values in the Table below. They were obtained from the LINGO 18.0 program written to execute this problem.

Seasons	x_1^*	x_2^*	x_3^*	x_4^*	**Min *Z***	**Remark**
k_1	1	0	0	1	6	Crisp
	0.7	0	0	0.5	3.8	Fuzzy
k_2	1	0	0	1	6	Crisp
	0.7	0	0	0.5	3.8	Fuzzy
k_3	1	0	1	1	11	Crisp
	0.7	0	0.2	0.5	4.8	Fuzzy

From the Table above, a system of fuzzy means which ensures that the goal i is reached at each k season with 0.5 – level, the minimum total cost of mean usage smaller with the FLP than that obtained with the traditional LP. This shows the advantage of the FLP over LP. Also, a system of fuzzy means which ensures that the goal i is reached for all k season with 0.5 – level, the minimum total cost of mean usage for all k seasons is as summarized in the Table below;

	x_1^*	x_2^*	x_3^*	x_4^*	**Min *Z***	**Remark**
Seasons	$k_1, k_2, k_3(1)$	-	$k_3(1)$	$k_1, k_2k_3(1)$	11	Crisp
	$k_1, k_2, k_3(0.7)$	-	$k_3(0.2)$	$k_1, k_2k_3(0.5)$	4.8	Fuzzy

From the Table above, 4.8 is the minimum total cost of mean usage for all k seasons. This shows that a general minimal cost can be ascertained. This cannot be obtained with the LP and FLP problem formulation and solution approach. Thus, the RFLP is best suited for problems exhibiting membership value changes.

6. Conclusion

RFLP, a generalization of FLP is studied through the approaches of [28, 30], where the objective and (or) constraints are subjectively graded, incorporating manager's input and [29, 31] which represent the objective and (or) constraints crisply but with a Relative fuzzy Boolean coverage and fuzzy systems. The problem formulation and solution for problems characterized by dynamic membership values are achievable and the formulations and results shows the strength of the RFLP over FLP and LP in literature. One can also explore other approaches e.g. Tanaka's approach [25] where the variables are fuzzy numbers. Our future work is the application of RFLP to a generalized target coverage problem formulation.

Appendix

Appendix 1

```
Model Class:                              PILP

  Total variables:                  3
  Nonlinear variables:              0
  Integer variables:                3

  Total constraints:                3
  Nonlinear constraints:            0

  Total nonzeros:                   8
  Nonlinear nonzeros:               0


                    Variable           Value        Reduced Cost
                         TOM        1.000000            4.000000
                        DICK        1.000000            3.000000
                       HARRY        1.000000            2.000000

                         Row    Slack or Surplus      Dual Price
                           1        9.000000           -1.000000
                           2       0.6000000            0.000000
                           3       0.3000000            0.000000
```

Appendix 2

```
Global optimal solution found.
Objective value:                              6.000000
Objective bound:                              6.000000
Infeasibilities:                              0.000000
Extended solver steps:                               0
Total solver iterations:                             0
Elapsed runtime seconds:                          0.12

Model Class:                                      PILP

Total variables:                      4
Nonlinear variables:                  0
Integer variables:                    4

Total constraints:                    6
Nonlinear constraints:                0

Total nonzeros:                      24
Nonlinear nonzeros:                   0


                       Variable           Value        Reduced Cost
                             X1        1.000000            4.000000
                             X2        0.000000            3.000000
                             X3        0.000000            5.000000
                             X4        1.000000            2.000000

                            Row    Slack or Surplus      Dual Price
                              1        6.000000           -1.000000
                              2       0.1000000            0.000000
                              3       0.6000000            0.000000
                              4       0.2000000            0.000000
                              5       0.5000000            0.000000
                              6       0.3000000            0.000000
```

Appendix 3

```
Global optimal solution found.
Objective value:                              11.00000
Objective bound:                              11.00000
Infeasibilities:                              0.000000
Extended solver steps:                               0
Total solver iterations:                             0
Elapsed runtime seconds:                          0.11

Model Class:                                      PILP

Total variables:                      4
Nonlinear variables:                  0
Integer variables:                    4

Total constraints:                    6
Nonlinear constraints:                0

Total nonzeros:                      24
Nonlinear nonzeros:                   0


                       Variable           Value        Reduced Cost
                             X1        1.000000            4.000000
                             X2        0.000000            3.000000
                             X3        1.000000            5.000000
                             X4        1.000000            2.000000

                            Row    Slack or Surplus      Dual Price
                              1        11.00000           -1.000000
                              2        0.000000            0.000000
                              3        1.700000            0.000000
                              4     0.5000000E-01          0.000000
                              5        1.700000            0.000000
                              6       0.1700000            0.000000
```

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