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Original Article

Common Neighbourhood Eccentric Dominating and Its Energy of Graphs

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Abstract *- In this paper, we introduce the concept of common neighbourhood eccentric domination in graphs. An eccentric dominating set D is called a common neighbourhood eccentric dominating set (CNED-set) if for all $v_j \in V - D$, there exists a vertex $v_i \in D$ such that $(v_i, v_j) \in E(G)$ and $\Gamma(v_i, v_j) \geq 1$. We calculate the common neighbourhood eccentric domination number for some standard graphs, and some results are stated and proved. The minimum common neighbourhood eccentric dominating energy $\mathbb{M}_{cned}(G)$ is the sum of the eigenvalues obtained from the minimum common neighbourhood eccentric dominating $n \times n$ matrix $\mathbb{M}_{cned}(G) = (m_{ij})$. $\mathbb{M}_{cned}(G)$ of standard graphs are computed. New properties, upper and lower bounds for $\mathbb{M}_{cned}(G)$ are established.*

Keywords *- Common neighbourhood, Eccentricity, Domination, Minimum common neighbourhood eccentric dominating set, Eigenvalues, Energy.*

1. Introduction

Spectral graph theory finds vast application in molecular biology and chemistry. Ore introduced the terms "dominating set" and "domination number" in his book on graph theory, which was published in 1962. There are numerous concepts of domination which have always intrigued mathematicians and have led to advances in the field of domination. Alwardi et al.[1] introduced the concept of common neighbourhood domination in 2011. Let $G = (V, E)$ be a graph. For any two vertices, $v_i, v_j \in V(G)$, the common neighbourhood denoted by $\Gamma(v_i, v_j)$ is the set of vertices different from v_i and v_j, which are adjacent to both v_i and v_j. A subset D of V is called a common neighbourhood dominating set (CN-dominating set) if, for every $v \in V - D$, there exists a vertex $u \in D$ such that $uv \in E(G)$ and $|\Gamma(u, v)| \geq 1$, where $|\Gamma(u, v)|$ is the number of elements in the common neighbourhood of the vertices u and v. Janakiraman et al.[6] introduced the concept of eccentric domination in graphs in 2010. The eccentricity $e(v)$ of v is the distance to a vertex farthest from v. Thus, $e(v) = \max\{d(u, v): u \in V\}$. For a vertex v, each vertex at a distance $e(v)$ from v is an eccentric vertex. The eccentric set of a vertex v is defined as $E(v) = \{u \in V(G): d(u, v) = e(v)\}$. A set $d \subseteq V(G)$ is an eccentric dominating set if D is a dominating set of G, and for every $v \in V - D$, there exists at least one eccentric vertex of v in D. The eccentric domination number $\gamma_{ed}(G)$ of a graph, G equals the minimum cardinality of an eccentric dominating set. That is, $\gamma_{ed}(G) = \min |D|$, where the minimum is taken over D in the set of all minimal eccentric dominating sets of G. Inspired by the above concepts, in this paper, we introduce a common neighbourhood eccentric dominating set, and we discuss and prove results related to this concept. In 1978, Gutman[4] introduced the concept of the energy of a graph. Inspired by Gutman, many authors have introduced and explored different types of energy in graph theory by means of different matrices. Kanna et al.[7] found the minimum dominating energy of a graph. For a graph $G = (V, E)$, let $A = (a_{ij})$ be the minimum dominating matrix defined by

$$a_{ij} = \begin{cases} 1, & if\ v_i v_j \in E, \\ 1, & if\ i = j\ and\ v_i \in D, \\ 0, & otherwise. \end{cases}$$

and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A. The minimum dominating energy is defined by $E_D(G) = \sum_{i=1}^{n} |\lambda_i|$. In 2023, Tejaskumar R, A Mohamed Ismayil and Ivan Gutman introduced 'Minimum eccentric dominating energy of graphs'. Inspired by Tejaskumar et al., we introduce minimum common neighbourhood eccentric dominating energy $\mathbb{E}_{cned}(G)$ of graphs. In this paper, we find $\mathbb{E}_{cned}(G)$ of some standard graphs. For undefined terminologies related to graphs, we refer to [5].

2. Common Neighbourhood Eccentric Domination in Graphs

We begin with some necessary notions:

Definition 2.1: For a graph $G=(V,E)$, a dominating set S is called a common neighbourhood eccentric dominating set (CNED-set) if, for every vertex $v \in V-S$, there exists an eccentric vertex $u \in S$ such that $E(v)=u$ and $(v,x) \in E(G)$ where $x \in S$ such that $|\Gamma(v,x)| \geq 1$.

Definition 2.2: A CNED-set S is called a minimal CNED-set if no proper subset of S is a CNED-set.

Definition 2.3: The CNED number $\gamma_{cned}(G)$ of a graph, G is the minimum cardinality among the minimal CNED sets of G.

Definition 2.4: The upper CNED number $\Gamma_{cned}(G)$ of a graph, G is the maximum cardinality among the minimal CNED sets of G.

Example 2.1: Consider the graph G in Fig.1 consisting of 6 vertices. Here, the dominating set is $\{v_1, v_4\}$, but it is not an eccentric dominating set. An eccentric dominating set is $\{v_1, v_6\}$, but it is not a CNED set. A CNED-set is $\{v_1, v_2, v_4\}$.

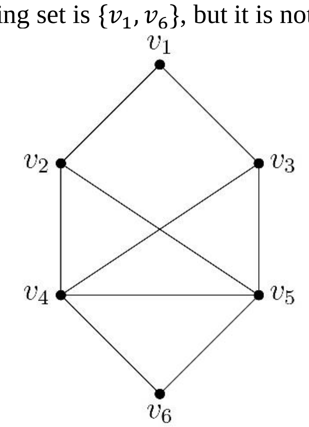


Fig. 1 The graph G in Example 2.1

Theorem 2.1: For complete graph K_n, $\gamma_{cned}(K_n)=1$ for all $n \geq 3$.
Proof: For a complete graph K_n, the eccentric vertices of any vertex $v_i \in V(K_n)$ are given by $E(v_i) \in V(K_n)-\{v_i\}$ and every single vertex dominates all other vertices. For any set $D=\{v_i\}\ \exists\ |\Gamma(v_i, v_m)| = n-2$ for all $v_m \in V-D$, and there exists an edge between every v_i and v_m for all $v_m \in V-D$. Hence $\gamma_{cned}(K_n)=1$.

Theorem 2.2: If a graph G does not contain at least one cycle C_3, then $\gamma_{cned}(G)=0$.
Proof: If a graph G does not contain at least one cycle C_3, then there is not exit a set D where we can find two vertices $v_i, v_j \in V(G)$ such that $|\Gamma(v_i, v_j)| \geq 1$ and $(v_i, v_j) \in E(G)$. To find a common adjacent vertex between them is not possible.

Remark 2.1:
(i) The $\gamma_{cned}(G)=0$ for star S_n, path P_n and tree graphs T_n.
(ii) Due to the fact that $C_3=K_3$, $\gamma_{cned}(C_3)=1$.
(iii) The $\gamma_{cned}(C_n)=0$ for $n>3$.

Theorem 2.3: For the wheel graph W_n with $n \geq 5$, we have

$$\gamma_{cned}(W_n)=\begin{cases} 2, & if\ n=5,7 \\ 3, & otherwise. \end{cases}$$

Proof: Case(i): Let $n=5,7$. For W_5, any two non-central vertices $v_i, v_j \in D$ forms a CNED-set since $V-D$ contains two non-central vertices and a central vertex, which satisfies the condition of a CNED-set. Similarly for W_7, choose any two non-central vertices v_i, v_j such that $d(v_i, v_j) = 3$ and the path between v_i and v_j should not pass through the central vertex. They form a CNED-set $D = \{v_i, v_j\}$ such that for any vertex $v_k \in V - D$, the condition that $|\Gamma(v_i, v_k)| \geq 1$ holds and $(v_i, v_k) \in E(G)$.

Case(ii): For W_n, where $n \neq 5,7$. A set $D = \{v_i, v_j, v_c\}$ such that v_c is a central vertex and v_i, v_j are non-central adjacent vertices forms CNED-set since $|\Gamma(v_i, v_k)| \geq 1$ and $(v_i, v_k) \in E(G)$ for any vertex $v_k \in V - D$.

Remark 2.2: (i) For the wheel graph W_4, $\gamma_{cned}(W_4) = 1 = \gamma_{cned}(K_4)$.

Theorem 2.4: Every CNED set contains all the pendant vertices of the graph.
Proof: To form a CNED set, we need a C_3 in the graph by theorem-2.2. A pendant vertex can not satisfy the C_3 property, i.e., a pendant vertex u does not have a common neighbour v such that $(u, v) \in E(G)$ and $|\Gamma(u, v)| \geq 1$ where $v \in V - D$ and $u \in D$.

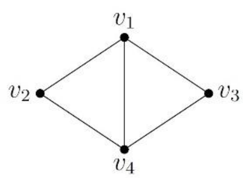
Theorem 2.5: A CNED-set D is a minimal CNED-set if one of the following conditions holds:
1. For every vertex u in $V - D$, there does not exist a vertex v in D such that $E(u) = \{v\}$, i.e., u has no eccentric vertex in D.
2. There exists some $u \in V - D$ such that $N(u) \cap D = \{v\}$ and $N(u) \cap N(v) \neq \emptyset$, i.e., $|\Gamma(u, v)| \geq 1$ where $(u, v) \in E(G)$.

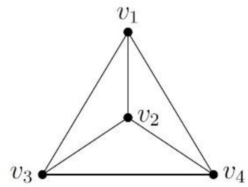
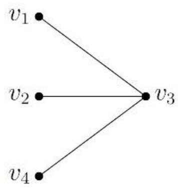
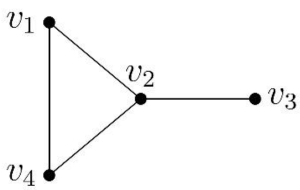
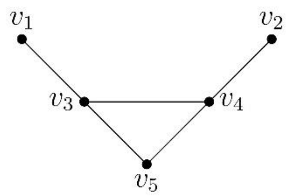
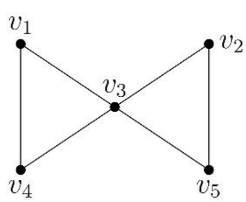
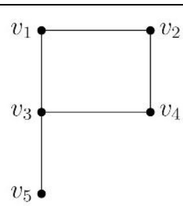
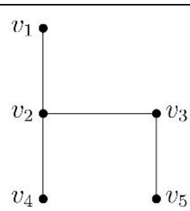
Proof: Suppose D is a minimal CNED set of G. Then, for every vertex v in D, $D - \{v\}$ is not a CNED-set. Thus there exists some vertex u in $V - D \cup \{v\}$ which is not dominated by any vertex in $D - \{v\}$ or there exists $u \in V - D \cup \{v\}$ such that u does not have an eccentric vertex in $D - \{v\}$, i.e., $E(u) \neq D - \{v\}$ or $N(u) \cap N(v) = \emptyset$, i.e. $|\Gamma(u, v)| = \emptyset$ where $(u, v) \in E(G)$, since the intersection of the open neighbourhoods of u and v is empty, $|\Gamma(u, v)| \ngeq 1$. Therefore, the common neighbourhood does not exist.

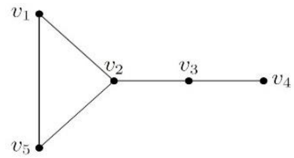
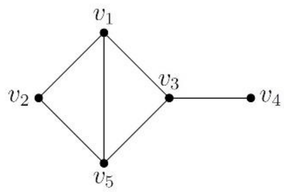
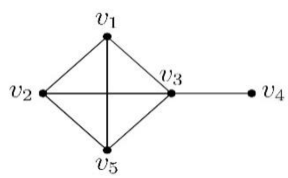
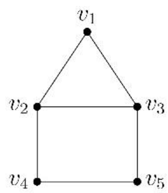
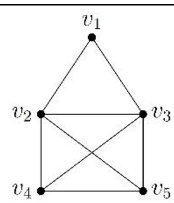
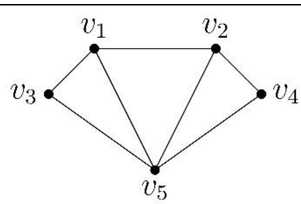
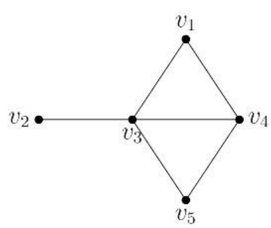
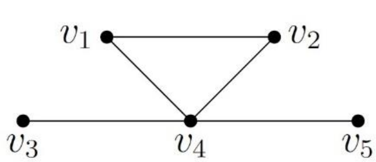
Case(i) If $v = u$, then u does not have an eccentric vertex in D, i.e., $E(u) \neq D$.
Case(ii) (a) If $u \in V - D$ and u are not dominated by $D - \{v\}$, but dominated by D, then u is adjacent to only v in D, i.e., $N(u) \cap D = \{v\}$.
(b) If $u \in V - D$ and u does not have an eccentric vertex in $D - \{v\}$ but u has an eccentric vertex in D, then v is the only eccentric vertex of u in D, i.e. $E(u) \cap D = \{v\}$.
(c) If $u \in V - D$ and $N(u) \cap N(x) = \emptyset$ where $x \in D - \{v\}$ but $N(u) \cap N(v) \neq \emptyset$, i.e. $|\Gamma(u, v)| \geq 1$ where $(u, v) \in E(G)$.

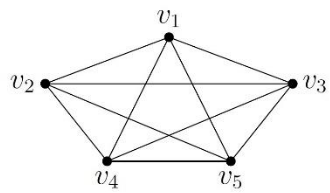
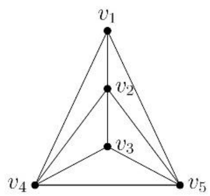
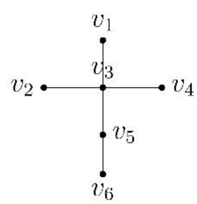
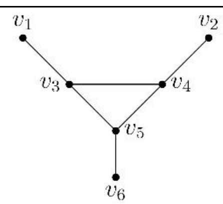
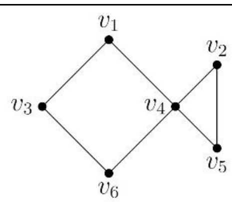
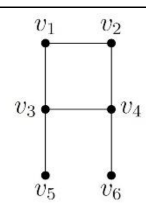
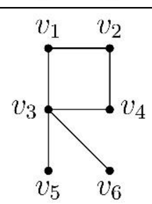
Conversely, suppose D is a CNED-set, and for each $v \in D$, one of the two conditions holds. Now, we show that D is a minimal CNED set. Suppose D is not a minimal CNED-set, i.e. there exists a vertex $v \in D$ such that $D - \{v\}$ is a CNED-set. Hence v is adjacent to at least one vertex x in $D - \{v\}$, v has an eccentric vertex in $D - \{v\}$ i.e. $E(v) \in D - \{v\}$ and $N(v) \cap N(x) \neq \emptyset$. Hence $|\Gamma(v, x)| \geq 1$ where $(v, x) \in E(G)$. Therefore, a CNED set exists. Therefore, condition-(1) does not hold. Also, if $D - \{v\}$ is a CNED-set, then every vertex u in $V - D$ is adjacent to at least one vertex x in $D - \{v\}$, u has an eccentric vertex in $D - \{v\}$ i.e. $E(u) \in D - \{v\}$ and $N(u) \cap N(x) \neq \emptyset$ i.e. $|\Gamma(u, x)| \geq 1$ where $(u, x) \in E(G)$. Therefore, condition-(2) does not hold. Hence, neither condition-(1) nor (2) holds, which is a contradiction to our assumption that for each $v \in D$, one of the conditions holds. This proves the theorem.

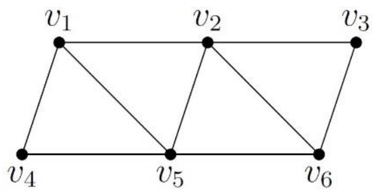
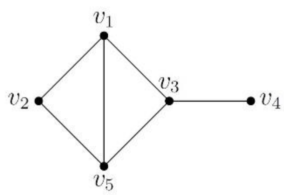
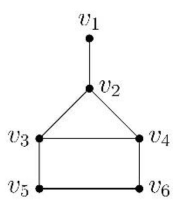
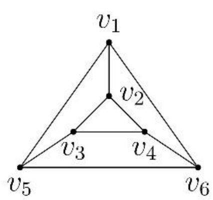
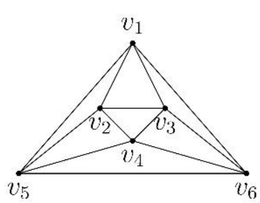
Table 1. The common neighbourhood eccentric dominating set, $\gamma_{cned}(G)$, upper common neighbourhood eccentric dominating set and $\Gamma_{cned}(G)$ of standard graphs are tabulated

Graph	Figure	D – Minimum CNED-set. $\lvert D\rvert = \gamma_{cned}(G)$	$\gamma_{cned}(G)$	S – Upper CNED-set. $\lvert S\rvert = \Gamma_{cned}(G)$	$\Gamma_{cned}(G)$
Diamond graph		$\{v_1, v_2\}$, $\{v_1, v_3\}$, $\{v_2, v_3\}$, $\{v_2, v_4\}$, $\{v_3, v_4\}$.	2	$\{v_1, v_2, v_4\}$, $\{v_1, v_3, v_4\}$.	3

Tetrahedral graph		$\{v_1\}$, $\{v_2\}$, $\{v_3\}$, $\{v_4\}$.	1	$\{v_1\}$, $\{v_2\}$, $\{v_3\}$, $\{v_4\}$.	1
Claw graph		Does not exist.	0	Does not exist.	0
Paw graph		$\{v_1, v_3\}$, $\{v_2, v_3\}$, $\{v_3, v_4\}$.	2	$\{v_1, v_2, v_3, v_4\}$.	4
Bull graph		$\{v_1, v_2, v_3\}$, $\{v_1, v_2, v_4\}$, $\{v_1, v_2, v_5\}$.	3	$\{v_1, v_2, v_3, v_4, v_5\}$.	5
Butterfly graph		$\{v_1, v_2\}$, $\{v_1, v_5\}$, $\{v_2, v_4\}$, $\{v_4, v_5\}$.	2	$\{v_1, v_2, v_3\}$, $\{v_1, v_2, v_4\}$, $\{v_1, v_2, v_5\}$, $\{v_1, v_3, v_4\}$, $\{v_1, v_3, v_5\}$, $\{v_1, v_4, v_5\}$, $\{v_2, v_3, v_4\}$, $\{v_2, v_3, v_5\}$, $\{v_2, v_4, v_5\}$, $\{v_3, v_4, v_5\}$.	3
Banner graph		Does not exist.	0	Does not exist.	0
Fork graph		Does not exist.	0	Does not exist.	0

(3,2)-Tadpole graph		$\{v_1, v_3, v_4\}$, $\{v_2, v_3, v_4\}$, $\{v_3, v_4, v_5\}$.	3	$\{v_1, v_2, v_3, v_4, v_5\}$.	5
Kite graph		$\{v_1, v_2, v_4\}$, $\{v_1, v_3, v_4\}$, $\{v_2, v_3, v_4\}$, $\{v_2, v_4, v_5\}$, $\{v_3, v_4, v_5\}$.	3	$\{v_1, v_2, v_3, v_4, v_5\}$.	5
(4,1)-Lollipop graph		$\{v_1, v_4\}$, $\{v_2, v_4\}$, $\{v_3, v_4\}$, $\{v_4, v_5\}$.	2	$\{v_1, v_2, v_3, v_4, v_5\}$.	5
House graph		$\{v_1, v_4, v_5\}$, $\{v_2, v_4, v_5\}$, $\{v_3, v_4, v_5\}$.	3	$\{v_1, v_2, v_3, v_4, v_5\}$.	5
House X graph		$\{v_1, v_2\}$, $\{v_1, v_3\}$, $\{v_1, v_4\}$, $\{v_1, v_5\}$.	2	$\{v_1, v_2, v_3, v_4\}$, $\{v_1, v_2, v_3, v_5\}$, $\{v_2, v_3, v_4, v_5\}$.	4
Gem graph		$\{v_1, v_2\}$, $\{v_3, v_4\}$.	2	$\{v_1, v_2, v_3, v_5\}$, $\{v_1, v_2, v_4, v_5\}$, $\{v_1, v_3, v_4, v_5\}$, $\{v_2, v_3, v_4, v_5\}$.	4
Dart graph		$\{v_2, v_3\}$, $\{v_2, v_4\}$.	2	$\{v_1, v_2, v_3, v_4, v_5\}$.	5
Cricket graph		$\{v_1, v_3, v_5\}$, $\{v_2, v_3, v_5\}$, $\{v_3, v_4, v_5\}$.	3	$\{v_1, v_2, v_3, v_4, v_5\}$.	5

Pentatope graph		$\{v_1\}$, $\{v_2\}$, $\{v_3\}$, $\{v_4\}$.	1	$\{v_1\}$, $\{v_2\}$, $\{v_3\}$, $\{v_4\}$.	1
Johnson solid skeleton 12 graph		$\{v_1, v_2\}$, $\{v_1, v_3\}$, $\{v_1, v_4\}$, $\{v_1, v_5\}$, $\{v_2, v_3\}$.	2	$\{v_1, v_2, v_4, v_5\}$, $\{v_2, v_3, v_4, v_5\}$.	4
Cross graph		Does not exist.	0	Does not exist.	0
Net graph		$\{v_1, v_2, v_3, v_6\}$, $\{v_1, v_2, v_4, v_6\}$, $\{v_1, v_2, v_5, v_6\}$.	4	$\{v_1, v_2, v_3, v_6\}$, $\{v_1, v_2, v_4, v_6\}$, $\{v_1, v_2, v_5, v_6\}$.	4
Fish graph		$\{v_1, v_2, v_3, v_6\}$, $\{v_1, v_3, v_4, v_6\}$, $\{v_1, v_3, v_5, v_6\}$.	4	$\{v_1, v_2, v_3, v_6\}$, $\{v_1, v_3, v_4, v_6\}$, $\{v_1, v_3, v_5, v_6\}$.	4
A graph		Does not exist.	0	Does not exist.	0
R graph		Does not exist.	0	Does not exist.	0

4-polynomial graph		$\{v_3, v_4\}$.	2	$\{v_1, v_2, v_3, v_5, v_6\}$, $\{v_1, v_2, v_4, v_5, v_6\}$.	5
(2,3)-King graph		$\{v_1, v_3\}$, $\{v_1, v_6\}$, $\{v_3, v_4\}$, $\{v_4, v_6\}$.	2	$\{v_1, v_2, v_4, v_5\}$, $\{v_2, v_3, v_5, v_6\}$.	4
Antenna graph		$\{v_1, v_2, v_5, v_6\}$, $\{v_1, v_3, v_5, v_6\}$, $\{v_1, v_4, v_5, v_6\}$.	4	$\{v_1, v_2, v_5, v_6\}$, $\{v_1, v_3, v_5, v_6\}$, $\{v_1, v_4, v_5, v_6\}$.	4
3-prism graph		$\{v_1, v_2, v_5, v_6\}$, $\{v_1, v_3, v_5, v_6\}$, $\{v_1, v_4, v_5, v_6\}$.	4	$\{v_1, v_2, v_5, v_6\}$, $\{v_1, v_3, v_5, v_6\}$, $\{v_1, v_4, v_5, v_6\}$.	4
Octahedral graph		$\{v_1, v_2, v_3\}$, $\{v_1, v_2, v_5\}$, $\{v_1, v_3, v_6\}$, $\{v_1, v_5, v_6\}$, $\{v_2, v_3, v_4\}$, $\{v_2, v_4, v_5\}$, $\{v_3, v_4, v_6\}$, $\{v_4, v_5, v_6\}$.	3	$\{v_1, v_2, v_3\}$, $\{v_1, v_2, v_5\}$, $\{v_1, v_3, v_6\}$, $\{v_1, v_5, v_6\}$, $\{v_2, v_3, v_4\}$, $\{v_2, v_4, v_5\}$, $\{v_3, v_4, v_6\}$, $\{v_4, v_5, v_6\}$.	3

3. The Minimum Common Neighbourhood Eccentric Dominating Energy-$\mathbb{E}_{cned}(G)$

In this section, the minimum common neighbourhood eccentric dominating matrix and its energy are defined. Minimum common neighbourhood eccentric dominating energy of some standard graphs is obtained.

Definition 3.1: Let $G = (V, E)$ be a simple graph where $V(G) = \{v_1, v_2, \dots v_n : n \in \mathbb{N}\}$ is the set of vertices, and E is the set of edges. Let D be a minimum CNED set of G. Then, the minimum CNED matrix of G is a $n \times n$ matrix defined by $\mathbb{M}_{cned}(G) = (m_{ij})$, where

$$m_{ij} = \begin{cases} 1, & if\ \left|\Gamma(v_i, v_j)\right| \geq 1, (v_i, v_j) \in E(G)\ and\ v_i \in E(v_j)\ or\ v_j \in E(v_i), \\ 1, & if\ i = j\ and\ v_i \in D, \\ 0, & otherwise \end{cases}$$

Definition 3.2: The characteristic polynomial of $\mathbb{M}_{cned}(G)$ is defined by $Q_n(G, \beta) = \det\left(\mathbb{M}_{cned}(G) - \beta I\right)$.

Definition 3.3: The minimum CNED eigenvalues of G are the eigenvalues of $\mathbb{M}_{cned}(G)$. Since $\mathbb{M}_{cned}(G)$ is symmetric and real, the eigenvalues are real. We label the eigenvalues in non-increasing order $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n$.

Definition 3.4: The minimum CNED energy of G is defined by $\mathbb{E}_{cned}(G) = \sum_{i=1}^{n} |\beta_i|$.

Remark 3.1: The trace of $\mathbb{M}_{cned}(G)$= CNED number.

Example 3.1:

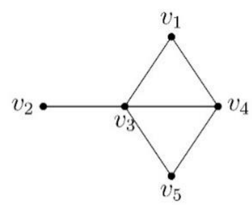


Fig. 3.1. Dart graph

Vertex	Eccentricity $e(v)$	Eccentric vertex $E(v)$
v_1	2	v_2, v_5
v_2	2	v_1, v_4, v_5
v_3	1	v_1, v_2, v_4, v_5
v_4	2	v_2
v_5	2	v_1, v_2

The minimum CNED sets of dart graphs are $D_1 = \{v_2, v_3\}$, and $D_2 = \{v_2, v_4\}$.
1. $D_1 = \{v_2, v_3\}$,

$$\mathbb{M}_{cned}(G) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Hence, the characteristic polynomial is $Q_n(G, \beta) = -5\beta^5 + 2\beta^4 + 2\beta^3 - 3\beta^2$. The minimum CNED eigenvalues are $\beta_1 \approx 2.3028$, $\beta_2 \approx 1$, $\beta_3 \approx 0$, $\beta_4 \approx 0.618$, $\beta_5 \approx$-1.618. Minimum CNED energy $\mathbb{E}_{cned}(G) \approx 4.6056$.

2. $D_2 = \{v_2, v_4\}$,

$$\mathbb{M}_{cned}(G) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

The characteristic polynomial is $Q_n(G, \beta) = -5\beta^5 + 2\beta^4 + 2\beta^3 - 5\beta^2 + 2\beta$. Hence, the minimum CNED eigenvalues are $\beta_1 \approx 2$, $\beta_2 \approx 1$, $\beta_3 \approx 0$, $\beta_4 \approx 0.618$, $\beta_5 \approx$-1.618, implying that the minimum CNED energy is $\mathbb{E}_{cned}(G) \approx 5.236$.

Remark 3.2: The minimum CNED energy depends on the CNED set.

Theorem 3.1: For a complete graph K_n where $n > 2$, the minimum CNED energy of a complete graph is
$\mathbb{E}_{cned}(K_n) = (n-2) + \left|\frac{(n-1)+\sqrt{(n-1)^2+4}}{2}\right| + \left|\frac{(n-1)-\sqrt{(n-1)^2+4}}{2}\right|$.

Proof: Let K_n be a complete graph with the vertex set $V = \{v_1, v_2, \ldots v_n\}$. The minimum CNED set is $D = \{v_1\}$. Then

$$\mathbb{M}_{cned}(K_n) = \begin{pmatrix} 1 & 1 & 1 & & 1 & 1 & 1 \\ 1 & 0 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 1 & 0 & & 1 & 1 & 1 \\ & \vdots & & \ddots & & \vdots & \\ 1 & 1 & 1 & & 0 & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 0 & 1 \\ 1 & 1 & 1 & & 1 & 1 & 0 \end{pmatrix}_{n\times n}$$

Its characteristic polynomial is $Q_n(K_n,\beta) = \det(\mathbb{M}_{cned}(K_n) - \beta I) =$

$$= \begin{vmatrix} 1-\beta & 1 & 1 & & 1 & 1 & 1 \\ 1 & -\beta & 1 & \cdots & 1 & 1 & 1 \\ 1 & 1 & -\beta & & 1 & 1 & 1 \\ & \vdots & & \ddots & & \vdots & \\ 1 & 1 & 1 & & -\beta & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & -\beta & 1 \\ 1 & 1 & 1 & & 1 & 1 & -\beta \end{vmatrix}$$

The characteristic equation is $Q_n(K_n,\beta) = (-1)^{n-2}(\beta+1)^{n-2}(\beta^2 - (n-1)\beta - 1)$. Then, the minimum CNED eigenvalues are

$$\beta = -1 \ (n-2 \ times),$$
$$\beta = \frac{(n-1)+\sqrt{(n-1)^2+4}}{2} \text{ and}$$
$$\beta = \frac{(n-1)-\sqrt{(n-1)^2+4}}{2}.$$

The minimum CNED energy of the complete graph K_n is given by

$$\mathbb{E}_{cned}(K_n) = |(-1)|(n-2) + \left|\frac{(n-1)+\sqrt{(n-1)^2+4}}{2}\right| + \left|\frac{(n-1)-\sqrt{(n-1)^2+4}}{2}\right|.$$

$$\mathbb{E}_{cned}(K_n) = (n-2) + \left|\frac{(n-1)+\sqrt{(n-1)^2+4}}{2}\right| + \left|\frac{(n-1)-\sqrt{(n-1)^2+4}}{2}\right|.$$

Theorem 3.2: For cocktail party graph G where $n \geq 4$, the minimum CNED energy of the cocktail party graph is $\mathbb{E}_{cned}(G) = \frac{n}{2}$.

Proof: Let G be a cocktail party graph with the vertex set $V = \{v_1, v_2. \dots v_n\}$. The minimum CNED set is $D = \{v_1, v_2, \dots v_{\frac{n}{2}}\}$ so that $|D| = \frac{n}{2}$, then

$$\mathbb{M}_{cned}(G) = \begin{pmatrix} 1 & 0 & 0 & & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & & 0 & 0 & 0 \end{pmatrix}_{n\times n}$$

Then, the characteristic polynomial is $Q_n(G,\beta) = \det(\mathbb{M}_{cned}(G) - \beta I) =$

$$= \begin{vmatrix} 1-\beta & 0 & 0 & & 0 & 0 & 0 \\ 0 & 1-\beta & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1-\beta & & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & & -\beta & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -\beta & 0 \\ 0 & 0 & 0 & & 0 & 0 & \beta \end{vmatrix}$$

Hence, the characteristic equation is $Q_n(G,\beta) = (\beta)^{\frac{n}{2}}(\beta-1)^{\frac{n}{2}}$ and therefore, the minimum CNED eigenvalues are

$$\beta = 0,$$

$$\beta = 1 \left(\frac{n}{2}\ times\right).$$

The minimum CNED energy of the cocktail party graph G is given by $\mathbb{E}_{cned}(G) = 0 + |1|\frac{n}{2} = \frac{n}{2}$.

4. Properties of Minimum Common Neighbourhood Eccentric Dominating Eigenvalues

In this section, we discuss the properties of eigenvalues of $\mathbb{M}_{cned}(G)$ for complete and cocktail party graphs. Bounds for the energy of common neighbourhood eccentric dominating energy of some standard graphs are obtained.

Theorem 4.1: If D is the minimum CNED set and $\beta_1, \beta_2, \dots \beta_n$ are the eigenvalues of the minimum CNED matrix $\mathbb{M}_{cned}(G)$, then

1. For any graph G, $\sum_{i=1}^{n} \beta_i = |D|$,
2. For a complete graph K_n, $\sum_{i=1}^{n} \beta_i^2 = |D| + (n)(n-1)$,
3. For a cocktail party graph G, $\sum_{i=1}^{n} \beta_i^2 = |D|$.

Proof:

1. We know that the sum of eigenvalues of $\mathbb{M}_{cned}(G)$ is the trace of $\mathbb{M}_{cned}(G)$. Hence $\sum_{i=1}^{n} \beta_i = \sum_{i=1}^{n} m_{ii} = |D|$.

2. Similarly, for a complete graph K_n, the sum of the square of eigenvalues of $\mathbb{M}_{cned}(K_n)$ is trace of $[\mathbb{M}_{cned}(K_n)]^2$

$$\text{Now } \sum_{i=1}^{n} \beta_i^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij} m_{ij}$$

$$\sum_{i=1}^{n} \beta_i^2 = \sum_{i=1}^{n} (m_{ii})^2 + \sum_{i \neq j} m_{ij} m_{ij}$$

$$\sum_{i=1}^{n} \beta_i^2 = \sum_{i=1}^{n} (m_{ii})^2 + 2 \sum_{i<j} \left(m_{ij}\right)^2$$

$$\sum_{i=1}^{n} \beta_i^2 = |D| + (n)(n-1)$$

since for a complete graph K_n, $2\sum_{i<j} \left(m_{ij}\right)^2 = (n)(n-1)$.

3. Similarly, for a cocktail party graph G sum of the square of eigenvalues of $\mathbb{M}_{cned}(G)$ is trace of $[\mathbb{M}_{cned}(G)]^2$

$$\text{Now } \sum_{i=1}^{n} \beta_i^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij} m_{ij}$$

$$\sum_{i=1}^{n} \beta_i^2 = \sum_{i=1}^{n} (m_{ii})^2 + \sum_{i \neq j} m_{ij} m_{ij}$$

$$\sum_{i=1}^{n} \beta_i^2 = \sum_{i=1}^{n} (m_{ii})^2 + 2 \sum_{i<j} \left(m_{ij}\right)^2$$

$$\sum_{i=1}^{n} \beta_i^2 = |D| + 0$$

since for a cocktail party graph G, $2\sum_{i<j} \left(m_{ij}\right)^2 = 0$.

Theorem 4.2: For cocktail party graph G where $n \geq 6$, if D be the minimum CNED-set and $W = |\det \mathbb{M}_{cned}(G)|$, then $\sqrt{|D| + n(n-1)W^{2/n}} \leq \mathbb{E}_{cned}(G) \leq \sqrt{n(|D|)}$.

Proof: By Cauchy Schwarz inequality, we have

$$\left(\sum_{i=1}^{n} g_i h_i\right)^2 \leq \left(\sum_{i=1}^{n} g_i^2\right)\left(\sum_{i=1}^{n} h_i^2\right).$$

If $g_i = 1$ and $h_i = \beta_i$ then

$$\left(\sum_{i=1}^{n} |\beta_i|\right)^2 \leq \left(\sum_{i=1}^{n} 1\right)\left(\sum_{i=1}^{n} \beta_i^2\right)$$

$$(\mathbb{E}_{cned}(G))^2 \leq n(|D|)$$

$$\Rightarrow \mathbb{E}_{cned}(G) \leq \sqrt{n(|D|)}$$

Since the arithmetic mean is not smaller than the geometric mean, we have

$$\frac{1}{n(n-1)}\sum_{i\neq j}|\beta_i||\beta_j| \geq \left[\prod_{i\neq j}|\beta_i||\beta_j|\right]^{\frac{1}{n(n-1)}}$$
$$\frac{1}{n(n-1)}\sum_{i\neq j}|\beta_i||\beta_j| = \left[\prod_{i=1}^{n}|\beta_i|^{2(n-1)}\right]^{\frac{1}{n(n-1)}}$$
$$\frac{1}{n(n-1)}\sum_{i\neq j}|\beta_i||\beta_j| = \left[\prod_{i=1}^{n}|\beta_i|\right]^{\frac{2}{n}}$$
$$\frac{1}{n(n-1)}\sum_{i\neq j}|\beta_i||\beta_j| = \left[\prod_{i=1}^{n}\beta_i\right]^{\frac{2}{n}}$$
$$\frac{1}{n(n-1)}\sum_{i\neq j}|\beta_i||\beta_j| = |\det \mathbb{M}_{cned}(G)|^{\frac{2}{n}} = W^{\frac{2}{n}}$$
$$\sum_{i\neq j}|\beta_i||\beta_j| \geq n(n-1)W^{\frac{2}{n}}$$

Now consider

$$(\mathbb{E}_{cned}(G))^2 = \left(\sum_{i=1}^{n}|\beta_i|\right)^2$$
$$(\mathbb{E}_{cned}(G))^2 = \left(\sum_{i=1}^{n}|\beta_i|\right)^2 + \sum_{i\neq j}|\beta_i||\beta_j|$$
$$(\mathbb{E}_{cned}(G))^2 = |D| + n(n-1)W^{\frac{2}{n}}$$
$$\mathbb{E}_{cned}(G) \geq \sqrt{(|D|) + n(n-1)W^{\frac{2}{n}}}$$

Theorem 4.3: For a complete graph K_n where $n > 2$, if D be the minimum CNED-set and $W = |\det \mathbb{M}_{cned}(K_n)|$ then $\sqrt{|D| + n(n-1) + n(n-1)W^{2/n}} \leq \mathbb{E}_{cned}(K_n) \leq \sqrt{n(n(n-1) + |D|)}$.
Proof: The proof follows the similar lines of Theorem-4.2.

Theorem 4.4: If $\beta_1(G)$ is the largest minimum CNED eigenvalue of $\mathbb{M}_{cned}(G)$ then
1. For a complete graph K_n, $\beta_1(K_n) \geq \frac{|D|+n(n-1)}{n}$,
2. For a cocktail party G, $\beta_1(G) \geq \frac{|D|}{n}$.
Proof:

1. Let Y be a non-zero vector, then by [2], we have
$$\beta_1\big(\mathbb{M}_{cned}(G)\big) = \max_{Y\neq 0} \frac{Y^T\mathbb{M}_{cned}(G)Y}{Y^TY}.$$
$$\beta_1\big(\mathbb{M}_{cned}(G)\big) \geq \frac{U^T\mathbb{M}_{cned}(G)U}{U^TU} = \frac{|D| + n(n-1)}{n}$$
where U is the unit matrix.

2. Let Y be a non-zero vector, then by [2], we have
$$\beta_1\big(\mathbb{M}_{cned}(G)\big) = \max_{Y\neq 0} \frac{Y^T\mathbb{M}_{cned}(G)Y}{Y^TY}.$$
$$\beta_1\big(\mathbb{M}_{cned}(G)\big) \geq \frac{U^T\mathbb{M}_{cned}(G)U}{U^TU} = \frac{|D|}{n}$$

where U is the unit matrix.

Table 2. Characteristic equation $Q_n(G,\beta)$, roots $\beta(G)$ and energy $\mathbb{E}_{cned}(G)$ of minimum CNED sets of various standard graphs are tabulated

Graph	Minimum CNED set	Characteristic equation $Q_n(G,\beta)$	Roots $\beta(G)$	Energy $\mathbb{E}_{ined}(G)$
Paw	$\{v_1,v_3\}$	$\beta^4-2\beta^3-\beta^2+3\beta-1$	$\beta_1=1.8019$, $\beta_2=1$, $\beta_3=0.445$, $\beta_4=-1.247$.	4.4939
	$\{v_2,v_3\}$	$\beta^4-2\beta^3-\beta^2+2\beta$	$\beta_1=2$, $\beta_2=1$, $\beta_3=0$, $\beta_4=-1$.	4
	$\{v_3,v_4\}$	$\beta^4-2\beta^3-\beta^2+3\beta-1$	$\beta_1=1.8019$, $\beta_2=1$, $\beta_3=0.445$, $\beta_4=-1.247$.	4.4939
Bull	$\{v_1,v_2,v_3\}$	$-\beta^5+3\beta^4-3\beta^3+\beta^2$	$\beta_1=1$, $\beta_2=1$, $\beta_3=1$, $\beta_4=0$.	4
	$\{v_1,v_2,v_4\}$	$-\beta^5+3\beta^4-3\beta^3+\beta^2$	$\beta_1=1$, $\beta_2=1$, $\beta_3=1$, $\beta_4=0$.	4
	$\{v_1,v_2,v_5\}$	$-\beta^5+3\beta^4-3\beta^3+\beta^2$	$\beta_1=1$, $\beta_2=1$, $\beta_3=1$, $\beta_4=0$.	4
(3,2)Tadpole	$\{v_1,v_3,v_4\}$	$-\beta^5+3\beta^4+\beta^3-5\beta^2+2\beta$	$\beta_1=2.8136$, $\beta_2=1$, $\beta_3=0.5293$, $\beta_4=0$, $\beta_5=-1.3429$.	5.6858
	$\{v_2,v_3,v_4\}$	$-\beta^5+3\beta^4+\beta^3-5\beta^2+2\beta$	$\beta_1=2.8136$, $\beta_2=1$, $\beta_3=0.5293$, $\beta_4=0$, $\beta_5=-1.3429$.	5.6858
	$\{v_3,v_4,v_5\}$	$-\beta^5+3\beta^4+\beta^3-5\beta^2+2\beta$	$\beta_1=2.8136$, $\beta_2=1$, $\beta_3=0.5293$, $\beta_4=0$, $\beta_5=-1.3429$.	5.6858
House	$\{v_1,v_4,v_5\}$	$-\beta^5+3\beta^4-3\beta^3+\beta^2+3\beta-1$	$\beta_1=1$, $\beta_2=1$, $\beta_3=1$, $\beta_4=0$.	3
	$\{v_2,v_4,v_5\}$	$-\beta^5+3\beta^4-3\beta^3+\beta^2+3\beta-1$	$\beta_1=1$, $\beta_2=1$,	3

			$\beta_3 = 1$, $\beta_4 = 0$.	
	$\{v_3, v_4, v_5\}$	$-\beta^5 + 3\beta^4 - 3\beta^3 + \beta^2 + 3\beta - 1$	$\beta_1 = 1$, $\beta_2 = 1$, $\beta_3 = 1$, $\beta_4 = 0$.	3
Gem	$\{v_1, v_2\}$	$-\beta^5 + 2\beta^4 + 3\beta^3 - 6\beta^2 + 2\beta$	$\beta_1 = 2.3429$, $\beta_2 = 1$, $\beta_3 = 0.4707$, $\beta_4 = 0$, $\beta_5 = -1.8136$.	5.6272
	$\{v_3, v_4\}$	$-\beta^5 + 2\beta^4 + 3\beta^3 - 6\beta^2 + 2\beta$	$\beta_1 = 2.3429$, $\beta_2 = 1$, $\beta_3 = 0.4707$, $\beta_4 = 0$, $\beta_5 = -1.8136$.	5.6272
Dart	$\{v_2, v_3\}$	$-\beta^5 + 2\beta^4 + 2\beta^3 - 3\beta^2$	$\beta_1 = 2.3028$, $\beta_2 = 1$, $\beta_3 = 0$, $\beta_4 = -1.3028$.	4.6056
	$\{v_2, v_4\}$	$-\beta^5 + 2\beta^4 + 2\beta^3 - 5\beta^2 + 2\beta$	$\beta_1 = 2$, $\beta_2 = 1$, $\beta_3 = 0.618$, $\beta_4 = 0$, $\beta_5 = -1.618$.	5.236
Cricket	$\{v_1, v_3, v_5\}$	$-\beta^5 + 3\beta^4 - \beta^3 - 4\beta^2 + 4\beta - 1$	$\beta_1 = 1.8019$, $\beta_2 = 1$, $\beta_3 = 1$, $\beta_4 = 0.445$, $\beta_5 = -1.247$.	5.4939
	$\{v_2, v_3, v_5\}$	$-\beta^5 + 3\beta^4 - \beta^3 - 4\beta^2 + 4\beta - 1$	$\beta_1 = 1.8019$, $\beta_2 = 1$, $\beta_3 = 1$, $\beta_4 = 0.445$, $\beta_5 = -1.247$.	5.4939
	$\{v_3, v_4, v_5\}$	$-\beta^5 + 3\beta^4 - \beta^3 - 4\beta^2 + 4\beta - 1$	$\beta_1 = 1.8019$, $\beta_2 = 1$, $\beta_3 = 1$, $\beta_4 = 0.445$, $\beta_5 = -1.247$.	5.4939
Net	$\{v_1, v_2, v_3, v_6\}$	$\beta^6 - 4\beta^5 + 6\beta^4 - 4\beta^3 + \beta^2$	$\beta_1 = 1$, $\beta_2 = 1$, $\beta_3 = 1$, $\beta_4 = 1$, $\beta_5 = 0$.	4
	$\{v_1, v_2, v_4, v_6\}$	$\beta^6 - 4\beta^5 + 6\beta^4 - 4\beta^3 + \beta^2$	$\beta_1 = 1$, $\beta_2 = 1$,	4

			$\beta_3 = 1,$ $\beta_4 = 1,$ $\beta_5 = 0.$	
	$\{v_1, v_2, v_5, v_6\}$	$\beta^6 - 4\beta^5 + 6\beta^4 - 4\beta^3 + \beta^2$	$\beta_1 = 1,$ $\beta_2 = 1,$ $\beta_3 = 1,$ $\beta_4 = 1,$ $\beta_5 = 0.$	4
Fish	$\{v_1, v_2, v_3, v_6\}$	$\beta^6 - 4\beta^5 + 6\beta^4 - 4\beta^3 + \beta^2$	$\beta_1 = 1,$ $\beta_2 = 1,$ $\beta_3 = 1,$ $\beta_4 = 1,$ $\beta_5 = 0.$	4
	$\{v_1, v_3, v_4, v_6\}$	$\beta^6 - 4\beta^5 + 6\beta^4 - 4\beta^3 + \beta^2$	$\beta_1 = 1,$ $\beta_2 = 1,$ $\beta_3 = 1,$ $\beta_4 = 1,$ $\beta_5 = 0.$	4
	$\{v_1, v_3, v_5, v_6\}$	$\beta^6 - 4\beta^5 + 6\beta^4 - 4\beta^3 + \beta^2$	$\beta_1 = 1,$ $\beta_2 = 1,$ $\beta_3 = 1,$ $\beta_4 = 1,$ $\beta_5 = 0.$	4
4-polynomial	$\{v_3, v_4\}$	$\beta^6 - 4\beta^5 + 6\beta^4 - 4\beta^3 + \beta^2$	$\beta_1 = 1,$ $\beta_2 = 1,$ $\beta_3 = 1,$ $\beta_4 = 1,$ $\beta_5 = 0.$	4
Antenna	$\{v_1, v_2, v_5, v_6\}$	$\beta^6 - 4\beta^5 + 6\beta^4 - 4\beta^3 + \beta^2$	$\beta_1 = 1,$ $\beta_2 = 1,$ $\beta_3 = 1,$ $\beta_4 = 1,$ $\beta_5 = 0.$	4
	$\{v_1, v_3, v_5, v_6\}$	$\beta^6 - 4\beta^5 + 6\beta^4 - 4\beta^3 + \beta^2$	$\beta_1 = 1,$ $\beta_2 = 1,$ $\beta_3 = 1,$ $\beta_4 = 1,$ $\beta_5 = 0.$	4
	$\{v_1, v_4, v_5, v_6\}$	$\beta^6 - 4\beta^5 + 6\beta^4 - 4\beta^3 + \beta^2$	$\beta_1 = 1,$ $\beta_2 = 1,$ $\beta_3 = 1,$ $\beta_4 = 1,$ $\beta_5 = 0.$	4
	$\{v_1, v_2, v_5, v_6\}$	$\beta^6 - 4\beta^5 + 6\beta^4 - 4\beta^3 + \beta^2$	$\beta_1 = 1,$ $\beta_2 = 1,$ $\beta_3 = 1,$ $\beta_4 = 1,$ $\beta_5 = 0.$	4

3-prism	$\{v_1, v_3, v_5, v_6\}$	$\beta^6 - 4\beta^5 + 6\beta^4 - 4\beta^3 + \beta^2$	$\beta_1 = 1,$ $\beta_2 = 1,$ $\beta_3 = 1,$ $\beta_4 = 1,$ $\beta_5 = 0.$	4
	$\{v_1, v_4, v_5, v_6\}$	$\beta^6 - 4\beta^5 + 6\beta^4 - 4\beta^3 + \beta^2$	$\beta_1 = 1,$ $\beta_2 = 1,$ $\beta_3 = 1,$ $\beta_4 = 1,$ $\beta_5 = 0.$	4

5. Conclusion

We have introduced the concept of CNED-set. CNED-number of standard graphs is calculated. The minimum CNED energy $\mathbb{E}_{cned}(G)$ of family of graphs and their properties are stated and proved. The concept can be extended to other distances.

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