On Most Generalized Topologies on a Non Empty Set

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Abstract

The purpose of this paper is to introduce a notion of continuity called (G ,D)- continuity between two non empty sets X and Y, and its relationships with other functions are studied.

Key words : (G, D)-*Continuity*, G-*interior*, G-*closure*.

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1 Introduction

Semi-open sets, pre-open sets, α-open sets and semi-pre-open sets play a major role in the generalization of continuous functions. By using these sets several authors introduced and studied various types of non-continuous functions. Generally continuity is defined between two topological spaces. But recently functions between topological spaces are studied by using nearly open sets and generalized open sets. In this paper functions between two non empty sets will be studied by using arbitrary collections of subsets of domains and co-domains of such functions. Throughout this paper X and Y denote non empty sets and f: $X\rightarrow Y$ is a function. In this paper we introduce a notion of a (G, D)-continuous function defined between two non empty sets. We obtain some characterizations and properties of such functions.

2 Preliminaries

Let X be a topological space and A a subset of X. The closure of A and the interior of A are denoted by $cl(A)$ and $int(A)$ respectively.

Definition 2.1 Let (X,τ) be a topological space and $A\subseteq X$. Then A is

(i) regular open [15] if $A=int(cl(A))$ and regular closed if $A = cl(int(A))$.

(ii) semi-open [5] if $A \subseteq cl(int(A))$ and semi-closed if $int(cl(A)\subseteq A$.

(iii) α-open [13] if $A \subseteq int(cl(int(A))$ and α-closed if $cl(int(cl(A))\subseteq A$.

(iv) pre-open [11] if $A \subseteq int(cl(A))$ and pre-closed if $cl(int(A))\subseteq A$.

(v) A is semi-pre-open[2] if $A\subseteq cl(int(cl(A)))$ and semi-pre-closed if $int(cl(int(A)))\subseteq A$.

Semi–pre-open sets are also called β-open sets in sense of [1]. RO(X,τ), SO(X,τ), αO(X,τ), PO(X,τ) and βO(X,τ) denote the collection of all regular open, semi-open, α-open, pre-open and β-open sets respectively in (X,τ). Also $RC(X,\tau)$, $SC(X,\tau)$, $\alpha C(X,\tau)$, $PC(X,\tau)$ and $\beta C(X,\tau)$ denote the collection of all regular closed, semi- closed, α- closed, pre- closed and β-closed sets respectively in (X,τ). The semi-interior of A, denoted by $sint(A)$, is the union of all semi-open sets that are contained in A. The concepts α-interior, pre-interior and β-interior of a set A can be analogously defined and are respectively denoted by $\alpha int(A)$, $pint(A)$ and $\beta int(A)$. The semi-closure of A, denoted by $scl(A)$, is the intersection of all semi-closed sets containing A. The concepts α-closure, pre-closure and β-closure of a set A can be analogously defined and are respectively denoted by $\alpha cl(A)$, $pclA)$ and $\beta cl(A)$.

Definition 2.2 Let (X,τ) be a topological space and $A\subseteq X$. Then A is

(i) generalized closed (briefly g-closed)[9] if $cl(A)\subseteq U$ whenever $A\subseteq U$ and U is open in X.

(ii) regular generalized closed (briefly rg-closed)[14] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is regular open in X.

The intersection of all g-closed sets containing B is called the g-closure of B and is denoted by $cl^*(B)$ [5]. The rg-closure of A is analogously defined. The complement of a g-closed set is g-open and that of an rg-closed set is rg-open.

Definition 2.3 A subset A of a topological space (X,τ) is called

(i) pre-generalized closed (briefly pg-closed)[10] if $pcl(A)\subseteq U$ whenever $A\subseteq U$ and U is pre-open in X.

(ii) semi-generalized closed (briefly sg-closed)[4] if $scl(A)\subseteq U$ whenever $A\subseteq U$ and U is semi-open in X.

(iii) generalized pre-regular-closed (briefly gpr-closed)[6] if $pcl(A)\subseteq U$ whenever $A\subseteq U$ and U is regular open,

(iv) pre-generalized pre-regular-closed (briefly pgpr-closed)[3] if $pcl(A)\subseteq U$ whenever $A\subseteq U$ and U is rg-open.

The intersection of all gpr-closed sets containing A is called the gpr-closure of A and is denoted by $gpr\text{-}cl(A)$. The pgpr-closure of A is analogously defined and is denoted by $pgpr\text{-}cl(A)$. The complement of a gpr-closed set is gpr-open and that of a pgpr-closed set is pgpr-open. The collections of g-open sets, rg-open sets, pg-open sets, sg-open sets,gpr-open sets and pgpr-open sets in (X,τ) are respectively denoted by $gO(X,\tau)$, $rgO(X,\tau)$, $pgO(X,\tau)$, $sgO(X,\tau)$, $gprO(X,\tau)$ and $pgprO(X,\tau)$.

Definition 2.4 A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is called

(i) semi-continuous[8] if $f^{-1}(V)$ is semi-open in X for every open sub set V of Y.

(ii) regular continuous[14] if $f^{-1}(V)$ is regular open (X,τ) for every open subset V in (Y,σ).

(iii) pre -continuous[11] if $f^{-1}(V)$ is pre-open in X for every open subset V of Y.

(iv) semi-pre-continuous[2] if $f^{-1}(V)$ is semi-pre-open in X for every open subset Vof Y.

(v) α-continuous[12] if $f^{-1}(V)$ is α-open in X for every open subset V of Y.

3 G-space

A topology τ on a non empty X is a collection of subsets of X having certain restrictions on the collection τ. But we don't have any restriction on G-structure where G is a non empty collection of subsets of X. More generally if G is a non empty collection of subsets of X then the pair (X,G) is called a G-space and the structure G is called a G-topology on X. For example if X has just n elements then there are 2^{2^n} -1 such topologies on X. If (X,G) is a G-space then we take $G' = \{ X \setminus A : A \in G\}$ where $X \backslash A$ denotes the complement of A in X in which case the members of G are called G-open sets in (X,G) and the members of G' are called G-closed sets in (X,G). In fact, this type of topology on X is finer than not only a general topology on X but also any generalization of topology on X.

Throughout this section, G is a collection of subsets of X and D is a collection of subsets of Y. We use the following notations for any subset A of X. $LOW(A) = \{B \subseteq A : B \in G\}$ and $UPP(A) = \{ B \supseteq A : B \in G'\}$. Note that the collections $LOW(A)$ and $UPP(A)$ may be empty. We call the members of $LOW(A)$ as the lower sets of A in G and the members of $UPP(A)$ are called the upper sets of A in G'.

Proposition 3.1 If (X,G) is a G-space and if A is a subset of X then

$$LOW(A) = \phi \Leftrightarrow UPP(X \setminus A) = \phi.$$

Proof. Suppose (X,G) is a G-space and A is a subset of X. Then $LOW(A) = \phi \Leftrightarrow$ whenever $B \subseteq A$, $B \notin G \Leftrightarrow$ whenever $X \backslash B \supseteq X \setminus A$, $B \notin G \Leftrightarrow$ whenever $X \backslash B \supseteq X \setminus A$, $X \backslash B \notin G' \Leftrightarrow UPP(X \setminus A) = \phi$.

Corollary 3.2 If (X,G) is a G-space and if A is a subset of X then

$$LOW(A) \neq \phi \Leftrightarrow UPP(X \setminus A) \neq \phi.$$

Corollary 3.3 If (X,G) is a G-space and if A and B are subsets of X then

$$B \in LOW(A) \Leftrightarrow X \setminus B \in UPP(X \setminus A).$$

The above proposition and its corollary motivate us to introduce the concepts of G-interior and G-closure of a subset A of X.

Definition 3.4 Suppose (X,G) is a G-space and A is a subset of X. For a subset A of X, G-interior of A and G-closure of A are defined as

$Gint(A)$ =

$$\begin{cases} \bigcup\{B : B \in LOW(A)\} & \text{if } LOW(A) \neq \varphi \\ \varphi & \text{if } LOW(A) = \varphi \end{cases}$$

$Gcl(A)$ =

$$\begin{cases} \bigcap\{B : B \in UPP(A)\} & \text{if } UPP(A) \neq \varphi \\ X & \text{if } UPP(A) = \varphi \end{cases}$$

If (X,τ) is a topological space and if G= τ (resp. SO(X,τ), resp.αO(X,τ), resp.PO(X,τ), resp. βO(X,τ)) then $Gint(A)$ coincides with $int(A)$ (resp.$sint(A)$, resp.$\alpha int(A)$, resp.$pint(A)$, resp.$\beta int(A)$) and $Gcl(A)$ coincides with $cl(A)$ (resp.$scl(A)$, resp.$\alpha cl(A)$, resp. $pcl(A)$, resp.$\beta cl(A)$).

Proposition 3.5 Suppose (X, G) is a G-space and A is a subset of X. Then

(i) X\ $Gint(A)$ = Gcl(X\A).

(ii) X\ $Gcl(A)$ = $Gint$(X\A).

(iii) X\($Gint$(X\A)) = $Gcl(A)$.

(iv) X \ (Gcl(X\A)) = $Gint(A)$.

Proof. Suppose LOW(A) = ϕ. Then by using Definition 3.4, $Gint$(A) = ϕ that implies X\ $Gint(A)$ = X. Again by using Proposition 3.1, UPP(X\A) = ϕ that implies Gcl(X\A)= X.

Therefore X\ $Gint(A)$ =X= Gcl(X\A) when LOW(A) = ϕ. Now suppose LOW(A) $\neq \phi$. Then

$x \notin Gint(A) \Rightarrow x \notin \cup\{B: B \in LOW(A)\} \Rightarrow$

$x \notin$ B whenever B$\in$LOW(A)

$\Rightarrow x \in$ X\ B whenever X\ B $\in$UPP(X\A)}

$\Rightarrow x \in Gcl$(X\A).

Therefore X\ $Gint(A) \subseteq Gcl$(X\A).

Now x$\in$Gcl(X\A) $\Rightarrow$ x$\in$B whenever B$\in$UPP(X\A) $\Rightarrow$ x$\in$B whenever X\B$\in$LOW(A).

$\Rightarrow x \notin$X\B whenever X\B$\in$LOW(A)

$\Rightarrow x \notin$C for every C$\in$LOW(A).

$\Rightarrow x \in$X\ $Gint(A)$.

Therefore Gcl(X\A)$\subseteq$ X\ $Gint(A)$.

This proves (i). The proof for (ii) is analogous. (iii) and (iv) follow from (i) and (ii). This completes the proof of the proposition.

Proposition 3.6 Suppose (X,G) is a G-space. Let A and B be any two subsets of X with A $\subseteq$B.. Then

(i) $Gint(\phi) = \phi$ and $Gcl(\phi)$ need not be equal to ϕ.

(ii) Gcl(X) = X and $Gint$(X) need not be equal to X.

(iii) $Gint(A) \subseteq A \subseteq Gcl(A)$.

(iv) $Gint(A) \subseteq Gint(B)$

(v) $Gcl(A) \subseteq Gcl(B)$

Proof. (iii), (iv) , (v) and the first parts of (i) and (ii) follow from Definition 3.3. Examples can be constructed to show the second parts of (i) and (ii). Take X={a,b,c} and G={{a}, {b},ϕ}. Then G′ = {{a,c}, {b,c}, X}. Then $Gint$(X) = {a,b} and $Gcl(\phi)$ ={c}. This completes the proof.

Proposition 3.7 Suppose (X,G) is a G-space. Let A and B be any two subsets of X.. Then

(i) $Gint(A \cap B) \subseteq Gint(A) \cap Gint(B)$.

(ii) $Gcl(A \cap B) \subseteq Gcl(A) \cap Gcl(B)$.

(iii) $Gint(A) \cup Gint(B) \subseteq Gint(A \cup B)$.

(iv) $Gcl(A) \cup Gcl(B) \subseteq Gcl(A \cup B)$.

(v) The equality need not hold in the above inclusions.

Proof. The proofs for (i), (ii), (iii) and (iv) follow from Proposition 3.6 (iv) and (v).

The following examples show that the equality is not hold in (i), (ii), (iii) and (iv).

Example 3.8 Let X = {a,b,c}, G={{a,c}, {a,b}}, G′ = {{b}, {c}}. It can be proved that $Gint(A \cap B) = \phi$ and $Gint(A) \cap Gint(B)$ = {a} where A={a,c} and B={a,b}. Further it is easy to verify that $Gcl(C \cup D)$ = {b,c} and $Gcl(C) \cup Gcl(D)$ = X where C= {c} and D = {b}. This shows that the equality is not hold in (i) and (iv).

Example 3.9 Let X = {a,b,c}, G={{a}, {a,b},ϕ }, G′ = {{b,c}, {c}, X}. Take A= {a,c} and B = {b,c}. Then we see that $Gcl(A \cap B)$ = {c} and $Gcl(A) \cap Gcl(B)$ = {b,c}. This shows that the inclusion in (ii) is proper.

Example 3.10 Let X = {a, b}, G ={ ϕ, X} , G′ = {X, ϕ}. It is not difficult in verifying $Gint(A \cup B)$ = X and $Gint(A) \cup Gint(B) = \phi$ where A={a} and B={b}. This shows that the proper inclusion is hold in (iii).

This completes the proof of the proposition.

Proposition 3.11 Suppose (X,G) is a G-space and A is a subset of X. If A is G-open (resp. G-closed) then $Gint(A)$ (resp.$Gcl(A)$)=A.

Proof. Follows from Definition 3.3.

Examples can be constructed to show that $Gint(A)$ need not be G-open and that $Gcl(A)$ need not be G-closed.

Example 3.12 Let X = {a,b,c}, G={{a,c}, {a,b}}, G′ = {{b}, {c}}. It is easy to see that $Gint(\{a\})$ is not G-open and $Gcl(\{a,b\})$ is not G-closed.

Definition 3.13 Let (X,G) be G-space. Then G is said to be a strongly G-space if $Gint(A)$ is G-open in (X,G) for every subset A of X.

Example 3.14 Let X = {a,b,c}, G={{a}, {a,b},ϕ }, G′ = {{b,c}, {c}, X}. It can be verified that (X,G) is a strongly G-space. Every topological space (X, τ) is a strongly G-space if G= τ. More over if G ∈ { SO(X, τ), αO(X, τ), PO(X, τ), βO(X, τ)} then (X,G) is a strongly G-space.

Example 3.15 Let X = {a,b,c}, G={{a}, {a,b}, X}, G′ = {{b,c}, {c}, ϕ}. It can be verified that (X,G) is not a strongly G-space. Further if G $\in$ { gO(X, τ), rgO(X, τ), pgO(X, τ), sgO(X, τ) , gprO(X, τ), pgprO(X, τ)} then (X,G) is not a strongly G-space.

Proposition 3.16 (X, G) is a strongly G-space if and only if *Gcl*(A) is G-closed for every subset A of X.

Proof. Follows from Definition 3.13 and Proposition 3.5.

If G = { ϕ, X} then the structure G is called the indiscrete structure and if G = the power set of X then G is called a discrete structure. The structure G is called locally indiscrete if A$\cap$B = ϕ whenever A$\in$G and B$\in$G. Every partition topology is a locally indiscrete structure but the converse is not true. If G = { {x}: x $\in$X} then G is called the point structure on X. A structure G on X is of rank r if every member of G has exactly r elements.

Proposition 3.17 Let (X, G) be a G-space. Then the following are equivalent.

(i) G is closed under arbitrary union.

(ii) G′ is closed under arbitrary intersection.

(iii) *Gint*(*A*) is G-open in (X,G) for every subset A of X.

(iv) *Gcl*(A) is G-closed for every subset A of X.

Proof. Straight forward.

Definition 3.18 Suppose (X,G) is a G-space and A is a subset of X. If U$\in$ G and if x$\in$U then U is called a G-neighbourhood of x in (X,G). The G-derived set of A is defined as Gd(A) = {x$\in$X : for every G-neighbourhood U of x, U$\cap$ (A-x) $\neq \phi$}.

Proposition 3.19 Suppose (X,G) is a G-space and A is a subset of X. Then

(i) *Gcl*(A) = {x$\in$X : for every G-neighbourhood U of x, U$\cap$ A $\neq \phi$}.

(ii) *Gcl*(A) = A$\cup$Gd(A).

Proof. Straight forward.

Definition 3.20 Suppose (X,G) is a G-space and A is a subset of X. The G-boundary of A is defined as GFr(A) = *Gcl*(A)$\cap$*Gc*l(X-A).

Proposition 3.21 Suppose (X,G) is a G-space and A is a subset of X. Then

(i). The G-boundary of A = the G-boundary of X-A.

(ii). The G-boundary of A need not be a G-closed set in (X,G).

(iii). GFr(A) = *Gcl*(A)-*Gint*(A).

(iv). GFr(A) $\cap$ *Gint*(A) = ϕ.

(v). *Gcl*(A) = *Gint*(A)$\cup$GFr(A).

(vi). X = *Gint*(A)$\cup$GFr(A)$\cup$ *Gint*(X-A).

Proof. Straight forward.

4 (G ,D)-Continuous functions.

In this section the most general version of continuity is defined and its basic properties are studied.

Definition 4.1 Let (*X* , G) and (*Y,D)* be any a G-space and a D-space respectively. and f: $X \to Y$ be a function. .Then f is (G, D) – continuous if for every $V \in D$, $f^{-1}(V) \in G$.

Example 4.2 Let $X = \{a, b, c\}$ and $Y = \{1, 2, 3, 4\}$.Let G. = {{a},{b},{a, b}}, D = {{1, 2},{3}}. Let $f: X \to Y$ be defined by f(a) = 1, f(b) = 3, f(c) = 4. Then f is (G., D)-continuous from *X* to *Y*. Define $g: X \to Y$ by g(a) = 1, g(b) = 4, g(c) =3. Then g is not (G., D)-continuous from *X* to *Y*.

Proposition 4.3 Let (X,τ) and (Y,σ) be topological spaces. Then f: (X, τ) → (Y, σ) is

(i) continuous if and only if it is (C(X,τ),C(Y,σ)) – continuous.

(ii) semi-continuous if and only if it is (SO(X,τ), σ) – continuous.

(iii) semi-continuous if and only if it is (SC(X,τ),C(Y,σ))– continuous.

(iv) α-continuous if and only if it is (αO(X,τ), σ) – continuous

(v) α-continuous if and only if it is (αC(X,τ), C(Y,σ))– continuous.

(vi) β-continuous if and only if it is (βO(X,τ), σ) – continuous

(vii) β- continuous if and only if it is (βC(X,τ), C(Y,σ))– continuous.

Proof. f: (X, τ) → (Y, σ) is continuous if and only if the inverse image of every closed set in (Y,σ) is closed in (X,τ) that proves (i). f: (X, τ) → (Y, σ) is semi-continuous if and only if the inverse image of a set in σ is semi-open in (X,τ) which proves (ii). f: (X,τ) → (Y,σ) is semi-continuous if and only if the inverse image of every closed set in (Y,σ) is semi-closed in (X,τ), proving (iii). f: (X,τ) → (Y,σ) is α-continuous if and only if the inverse image of a set in σ is α-open in (X,τ) that proves (iv).. f: (X,τ) → (Y,σ) is α-continuous if and only if the inverse image of every closed set in (Y,σ) is α-closed in (X,τ) that implies (v). This completes the proof.

Proposition 4.4 Let (X,G) and (Y, D) be a G-space and a D-space respectively. Then f: X → Y is (G,D)-continuous if and only if it is (G′, D′)-continuous.

Proof. Follows from Definition 4.1 and from the fact that $A \in G$ if and only if $X \setminus A \in G'$ and $B \in D$ if and only if $Y \setminus B \in D'$ where $A \subseteq X$ and $B \subseteq Y$.

Proposition 4.5 Let (X,G) and (Y, D) be a G-space and a D-space respectively. Suppose f: X → Y is (G,D)-continuous . Then $f^{-1}(Dint(B)) \subseteq Gint(f^{-1}(B))$ for every subset B of Y.

Proof. Suppose LOW(B) is empty. Then *Dint*(B) = ϕ so that the proof is clear in this case. Now suppose LOW(B) ≠ϕ. Then for every $B \subseteq Y$ we have $f^{-1}(Dint(B)) =$

$f^{-1}(\cup\{ O : O\in LOW(B) \}) = \cup\{ f^{-1}(O) : O\in LOW(B) \})$

$= \cup\{ f^{-1}(O) : O\in LOW(B)$ with $f^{-1}(O)\in LOW(f^{-1}(B)) \}$

$\subseteq \cup\{ A : A\in LOW(f^{-1}(B)) \}$

$= Gint(f^{-1}(B))$. This proves the proposition.

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