The Banach Algebra Valued Functions of Bounded $k\varphi -$ Variation of Two Variables in the Sense of Riesz–Korenblum

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Abstract **- In this study we present the functions of bounded $\mathrm{k}\varphi -$ variation of two variables in the sense of Riesz-Korenblum. Moreover we proved that the class of two variables functions of bounded$\mathrm{k}\varphi^* -$ variation from σ into $\mathbb{B}$ is a Banach space and defined an extension E_f on $(\sigma, \mathbb{B})$, where $\sigma = \sigma_1 \times \sigma_2, \sigma_1, \sigma_2$ are non-empty compact subsets of R.**

Keywords **- Banach Algebra Valued Functions, $\mathrm{k}\varphi -$ variation, Compact subsets of R, functions of two variables.**

1. INTRODUCTION

When working with numbers such as real numbers $x \in \boldsymbol{R}$or complex numbers $z \in \boldsymbol{C}$, there are unambiguous notion of a *magnitude* $|x|$ *or* $|z|$ of a number, with which to measure which numbers are larger and which are small. One can also use this notion of magnitude to define a distance $|x - y|$ *or* $|z - w|$ between two real numbers *x, y* $\in \boldsymbol{R}$, or between two complex numbers *z, w* $\in \boldsymbol{C}$, thus giving a quantitative measure of which pairs of numbers are close and which ones are far apart. This situation becomes more complicated however when dealing with objects with more degrees of freedom. Consider for instance the problem of determining the *magnitude* of a three-dimensional rectangular box. There are several measures for such a magnitude: length, width, height, volume, surface area, diameter (*i.e* length of the diagonal), eccentricity, and so forth. Unfortunately, these magnitudes do not give equivalent comparisons: box A may be longer and have more volume than box B, but box B may be wider and have more surface area, and so forth. Because of this one abandon the idea that there should only be one notion of *magnitude* for boxes, and instead accept that there are instead multiplicities of such notions, all of which have some utility. Thus for some applications one may wish to distinguish the large volume boxes from the small volume boxes, while in others one may want to distinguish the eccentric boxes from the round boxes. Of course, there are several relationships between the different notions of magnitude (e.g. the isoperimetric inequality allows one to obtain the upper bound for the volume in terms of the surface area), so the situation is not as disorganized as it may first appear.

Now we turn to *functions* with a fixed domain and range (e.g. functions *f: [1,1]*→***R*** from the interval [1,1] to the real line **R**). These objects have infinitely many degrees of freedom, and so it should not be surprising that there are now infinitely many distinct notions of *magnitude*, all of which provide a different answer to the question "*how large is a given function f*?", or to the closely related question *"how close together are two functions f,g?"*, in some cases, certain functions may have infinite magnitude by one such measure, and finite magnitude by the another; similarly, a pair of functions may be very close by one measure and very far apart by another. Again this situation may seem chaotic, but it simply reflects the facts that functions have many distinct characteristics – some are tall, some are broad, some are smooth, some are oscillatory, and so forth – depending on the application at hand. One may want to give more weight to one of these characteristics than to others. In analysis, this is embodied in the variety of standard *function spaces*, and their associated *norms*, which are available to describe functions both qualitatively and quantitatively. While these spaces and norms are mostly distinct from each other, they are certainly interrelated, for instance, through such basic facts of analysis such as approximability by test functions (or in some cases by polynomials), by embedding such as sobolev embedding, and by interpolation theorems.

More formally, a function space is a class X of functions together with a *norm* which assigns a non-negative number $\|f\|_X$ to every function f in X; this function is the function space's way of measuring how large a function is. It is common (though not universal) for the class X of functions to consist precisely of those functions for which the definition of the norm $\|f\|_X$ makes sense and is finite; thus the mere fact that a function *f* has membership in a function space X conveys some qualitative information about that function (it may imply some regularity, some decay, some boundedness, or some integrability on the function f), while the norm $\|f\|_X$ supplements this qualitative information with a more quantitative measurement of the function (e.g. how regular is *f?* how much decay does *f* have? by which constant is *f* bounded? what is the integral of *f*?). Typically, we assumed that the function space X and its associated norm $\|\cdot\|_X$ obey a certain number of axioms; for instance, a rather standard set of axioms is that X is a real or complex vector space, that the norm is non-degenerate ($\|f\|_X > 0$, for non-zero *f*), homogeneous of degree 1, and obeys the triangle inequality$\|f + g\|_X \leq \|f\|_X + \|g\|_X$; furthermore, the space X when viewed using the metric *d(f,g)* = $\|f - g\|_X$ is a complete metric space [1]. Spaces satisfying all of these axioms are known as *Banach spaces*. A majority (but certainly not all) of the standard function spaces are Banach spaces. More also, a Banach space which is also closed under multiplication is refers to as a Banach algebra.

The concept of functions of bounded variation has been well-known since Jordan gave the complete characterization of functions of bounded variation as a difference of two increasing functions in 1881. This class of functions immediately proved to be important in connection with the rectification of curves and with the Dirichlet's theorem on convergence of Fourier series. Functions of bounded variation exhibit so many interesting properties that make them a suitable class of functions in a variety of contexts with wide applications in pure and applied mathematics [9].

Riesz in 1910 generalized the notion of Jordan and introduced the notion of bounded $p - variation(1 \leq p < \infty)$ and showed that, for $1 < p < \infty$, this class coincides with the class of functions absolutely continuous with the derivative in spaceL_p. On the other hand, this notion of bounded $p - variation$ was generalized by Medvedev in 1953 who introduce the concept of bounded $\varphi - variation$ in the sense of Riesz and also showed a Riesz's Lemma for this class of functions [7].

Korenblum in 1975 introduce the nation of bounded $k - variation$ variation. This concept differs from others due to the fact that it introduces a distribution function k that measures intervals in the domain of the function and not in the range [5]. In 1985, Cyphert and Kelingos showed that a function u is of bounded $k - variation$ if it can be written as the difference of two k-decreasing functions. In 2010, Park J., introduced the notion of functions of $bounded\ k\phi - variation$ on a compact interval $[a, b] \subset R$ which is a combination of concept of bounded $k - variation$ and bounded $\varphi - variation$ in the sense of Schramm [8], and in 2010 Aziz et al. showed that the space of bounded $k - variation$ satisfies Matkowski's weak condition [9].

Castillo et al. introduce the notion of bounded $k - variation$ in the sense of Riesz-Korenblum, which is a combination of the notions of bounded $p - variation$ in the sense of Riesz and bounded $k - variation$ in sense of Korenblum. More also, Castillo et al. in 2013 introduce the concept of bounded $k\phi - variation$ in the sense of Riesz-Korenblum, which is a combination of the notions of bounded $\varphi - variation$ in the sense of Riesz and bounded $k - variation$ in the sense of the functions of bounded $k - variation$ and bounded $\varphi - variation$ in the sense of Riesz. The same study proved that the space generated by this class of functions is a Banach space with a given norm and that the uniformly bounded composition operator satisfies Minkowski's weak condition in this space [2]. Recently, Choudhaly et al [1] had shown that the space of functions of bounded $k\varphi - variation$ is a Banach algebra. In this paper we present the functions of bounded$k\varphi - \mathrm{variation}$ of two variables in the sense of Riesz-Korenblum. Moreover we proved that the class of two variables functions of bounded$k\varphi^* - \mathrm{variation}$ from σ into $\mathbb{B}$ is a Banach space and defined an extension E_f on$(\sigma, \mathbb{B})$, *where* $\sigma = \sigma_1 \times \sigma_2, \sigma_1, \sigma_2$ are non-empty compact subsets of ***R***.

2. PRELIMINARY

In this section we present some definitions and preliminary results relating to the notion of functions of bounded $k\varphi - variation$ in the sense of Riesz – Korenblum.

Definition 1. *Partition of an interval [a, b]*: A partition say π of an interval $[a, b]$ is a set of points $a = t_0 < t_1 < t_2 < \cdots < t_n = b$ such that $\pi: a = t_0 < t_1 < t_2 < \cdots < t_n = b$ [6].

Definition 2. *Bounded Variation*: Let $f: [a,b] \to \mathbb{R}$ be a function. For each partition $\pi: a = t_0 < t_1 < t_2 < \cdots < t_n = b$ of the interval $[a, b]$, we define

$$V(f;[a,b]) = \sup_{\pi} \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|,$$

Where the supremum is taken over all partitions π of the interval $[a,b]$. If $V(f;[a,b]) < \infty$, we say that f has bounded variation. Denoted by $BV[a,b]$ the collection of all functions of bounded variation on $[a,b]$ [7].

Definition 3. A function $\varphi: [0,\infty) \to [0,\infty)$ is said to be a φ - function if it satisfies the following properties.

(a) φ is continuous on $[0,\infty)$.

(b) $\varphi(t) = 0$ if and only if $t = 0$.

(c) φ is strictly increasing.

(d) $\lim_{t\to\infty} \varphi(t) = \infty$ [2].

Definition 4. (Conditions ∞_1 and Δ_2). Let φ be a convex φ – function, then

(a) φ Satisfies the condition ∞_1 if $\lim_{t\to\infty}(\varphi(t)/t) = \infty$.

(b) φ Satisfies the condition $\Delta_2(\infty)$ if there is $C > 0$, $x_0 > 0$ such that $\varphi(2t) \le C\varphi(t)$, $t \ge x_0$ [8].

Definition 5. *Bounded φ – variation*: Let φ be a φ – function and $f: [a,b] \to \mathbb{R}$ be a function. For each partition $\pi: a = t_0 < t_1 < t_2 < \cdots < t_n = b$ of the interval $[a, b]$, we define

$$V_\varphi^R(f) = V_\varphi^R(f;[a,b]) = \sup_{\pi} \sum_{i=1}^{n} \varphi\left(\frac{|f(t_i)-f(t_{i-1})|}{|t_i - t_{i-1}|}\right)|t_i - t_{i-1}|,$$

Where the supremum is taken over all partitions π of the interval $[a,b]$. If $V_\varphi^R(f;[a,b]) < \infty$, we say that f has bounded φ – variation in the sense of Riesz. Denoted by $V_\varphi^R[a,b]$ the collection of all functions of bounded φ – variation in the sense of Riesz on $[a,b]$. This set of functions share similar properties with $BV[a,b]$. In fact if φ is convex then, $V_\varphi^R[a,b] \subset BV[a,b]$ and if $\lim_{t\to\infty}(\varphi(t)/t) = \gamma < \infty$, then $V_\varphi^R[a,b] = BV[a,b]$ [3].

Definition 6. Let $x_2 \in [a_2,b_2]$. Consider the function $f(\cdot,x_2): [a_1,b_1] \times \{x_2\} \to \mathbb{R}$ the $\varphi - variation$ in the sense of Riesz of the function $f(\cdot,x_2)$ of one variable defined by $f(\cdot,x_2)(t) = f(t,x_2), t \in [a_1,b_1]$, on the interval $[x_1,y_1]$, is the quantity

$$V^R_{\varphi,[x_1,y_1]} f(\cdot,x_2) \coloneqq \sup_{\prod_1} \sum_{i=1}^{m} \varphi\left[\frac{|\Delta_{10} f(t_i,x_2)|}{|\Delta t_i|}\right]|\Delta t_i|,$$

Where the supremum is taken over all partitions $\prod_1 = \{t_i\}_{i=0}^m (m \in \mathbb{N})$ of the interval $[x_1,y_1]$ [8].

Definition 7. Similarly, let $x_1 \in [a_1,b_1]$ and $f(x_1,\cdot): \{x_1\} \times [a_2,b_2] \to \mathbb{R}$, the $\varphi - variation$ in the sense of Riesz of the function $f(x_1,\cdot)$ of one variable defined by $f(x_1,\cdot)(t) = f(x_1,t), t \in [a_2,b_2]$, on the interval $[x_2,y_2]$, is the quantity

$$V^R_{\varphi,[x_2,y_2]} f(x_1,\cdot) \coloneqq \sup_{\prod_2} \sum_{j=1}^{n} \varphi\left[\frac{|\Delta_{01} f(x_1,s_j)|}{|\Delta s_j|}\right]|\Delta s_j|,$$

Where the supremum is taken over all partitions $\prod_2 = \{t_j\}_{j=0}^n (n \in \mathbb{N})$ of the interval $[x_2,y_2]$.

Definition 8. The $\varphi - bidimentional$ variation in the sense of Riesz is define by the formula

$$V_\varphi^R(f) \coloneqq \sup_{\prod_1,\prod_2} \sum_{i=1}^{m}\sum_{j=1}^{n} \varphi\left[\frac{|\Delta_{11} f(t_i,s_j)|}{|\Delta t_i||\Delta s_j|}\right] \cdot |\Delta t_i||\Delta s_j|,$$

Where the supremum is taken over the set of all partitions $(\prod_1,\prod_2)$ of the rectangle $I_a^b \subset R^2$ [8].

Definition 9. A function $k: [0,1] \to [0,1]$ is said to be a $k - function$ if it satisfies the following properties:

(a) k is continuous with $k(0) = 0$ and $k(1) = 1$.
(b) k is concave (down), increasing, and
(c) $\lim_{t\to 0}(k(t)/t) = \infty$.
The set of all $k - function$ will be denoted by $\mathcal{K}$. Note that every $k - function$ is sub-additive; that is,

$$k(t_1 + t_2) \le k(t_1) + k(t_2), \qquad t_1, t_2 \in [0,1]$$

Then, for all partitions $\pi: a = t_0 < t_1 < \cdots < t_n = b$ of $[a, b]$, we have

$$1 = k(1) = k\left(\sum_{i=1}^{n} \frac{t_i - t_{i-1}}{b-a}\right) \le \sum_{i=1}^{n} k\left(\frac{t_i - t_{i-1}}{b-a}\right).$$

Korenblum introduces the definition of bounded $k - variation$ as follows:

Definition 10. A real value function f on $[a, b]$ is said to be of bounded k – variation, if

$$kV(f) = kV(f;[a,b]) = \sup_{\pi} \frac{\sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|}{\sum_{i=1}^{n} k((t_i - t_{i-1})/(b-a))} < \infty,$$

Where the supremum is taken over all partitions π of the interval $[a, b]$. We denote by $kV[a, b]$ the collection of all functions of bounded k – variation on $[a, b]$ [2].

Definition 11. *Bounded* $k\varphi$ – *variation*:Let φbe a φ – function, $k \in \mathcal{K}$, and $f : [a, b] \to \mathbb{R}$ be a function. For each partition $\pi : a = t_0 < t_1 < t_2 < \cdots < t_n = b$ of the interval $[a, b]$, we define $kV_\varphi^R(f) = kV_\varphi^R(f;[a,b])$

$$= \sup_{\pi} \frac{\sum_{i=1}^{n} \varphi(|f(t_i) - f(t_{i-1})|/|t_i - t_{i-1}|)|t_i - t_{i-1}|}{\sum_{i=1}^{n} k((t_i - t_{i-1})/(b-a))},$$

Where the supremum is taken over all partitions π of the interval $[a, b]$. If$kV_\varphi^R(f;[a,b]) < \infty$, we say that f has bounded $k\varphi$ – variation in the sense of Riesz – Korenblum. Denoted by $V_\varphi^R[a, b]$ the collection of all functions of bounded $k\varphi$ – variation in the sense of Riesz - Korenblumon $[a, b]$.

Remark. Note that the class$V_\varphi^R[a, b]$ is not empty since for an affine function $f: [a, b] \to \mathbb{R}$ is defined by $d(t) := dt + e, t \in [a, b], where\ d\ and\ e$ are fixed real numbers. For a given partition $\pi : a = t_0 < t_1 < t_2 < \cdots < t_n = b$ of $[a, b]$ we have.

$$\frac{\sum_{i=1}^{n} \varphi(|f(t_i) - f(t_{i-1})|/|t_i - t_{i-1}|)|t_i - t_{i-1}|}{\sum_{i=1}^{n} k((t_i - t_{i-1})/(b-a))} = \frac{\varphi(|d|)(b-a)}{\sum_{i=1}^{n} k((t_i - t_{i-1})/(b-a))}$$

Taking the supremum over all partitions π of the interval $[a, b]$, the greater value of the right side of the above expression is obtain for the partition $\pi : a = t_0 < t_1 = b$ and in this case we get

$$\frac{\varphi(|d|)(b-a)}{k((t_1 - t_0)/(b-a))} = \frac{\varphi(|d|)(b-a)}{k(1)} = \varphi(|d|)(b-a)$$

Therefore$kV_\varphi^R(f;[a,b]) = \varphi(|d|)(b-a)$.

Now let σ be any non-empty compact subset of $\mathbb{R}$ and $I = [a, b]$ be the smallest closed interval containing σ. Let$\coprod(\sigma)$ be the class of all partitions of σ. That is$\coprod(\sigma) = \{t : t = \{t_i\}_{i=0}^{n}\ is\ an\ increasing\ finite\ sequence\ in\ \sigma\}$.

Definition 12.*Extension of a function f:* for a given function $f: \sigma \to \mathbb{B}$ where σ is a non-empty compact subset of **R**. defined the function$E_f: I \to \mathbb{B}\ by\ E_f(x) = f(\alpha(x))$, where

$$\alpha(x) = \begin{cases} x, & if\ x \in \sigma \\ \sup\{t: [x,t] \subset I \setminus \sigma\}, & otherwise \end{cases}$$

Obviously, if E_f is an extension of f and is constant on the gap in σ [10].

3 GENERALIZATION IN TWO VARIABLES

We now demonstrate the results for functions of two variables. Let $\sigma_1\ and\ \sigma_2$ be two non-empty compact subsets of **R** and let **R** $=[a_1, b_1] \times [a_2, b_2] \subset \boldsymbol{R}^2$ be the smallest closed rectangle containing $\sigma = \sigma_1 \times \sigma_2$.

Definition 1. For a given Banach algebra $\mathbb{B}$ and σ_1, σ_2 are non-empty compact subsets of***R*,a** function $f: \sigma \to \mathbb{B}$ is said to be of bounded $k\varphi - variation$ (that is $f \in kV_\varphi^{\mathbb{B}}[a, b]$, if

$$kV_\varphi^{\mathbb{B}}(f;[a,b]) = kV_\varphi^{\mathbb{B}}(f, \sigma, D) < \infty$$

Where $kV_\varphi^{\mathbb{B}}(f, \sigma, D)$

$$= \sup_{\pi, D} \frac{\sum_{i=1}^{n}\sum_{j=1}^{m} \varphi(|\Delta f(s_i, t_j)|/|\Delta s_i \Delta t_j|)|\Delta s_i \Delta t_j|}{\sum\sum k((\Delta s_i \Delta t_j)/(b-a))}$$

In which $\Delta s_i = s_i - s_{i-1}$, $\Delta f(s_i, t_j) = f(s_i, t_j) - f(s_i, t_{j-1}) - f(s_{i-1}, t_j) + f(s_{i-1}, t_{j-1})$ and $D = s \times t$ is a rectangular grid on σ obtained from any two partitions $s = \{s_i\}_{i=0}^{n} \in \coprod(\sigma_1)\ and\ t = \{t_j\}_{j=0}^{m} \in \coprod(\sigma_2)$.

Definition 2. For a given function $f: \sigma \to \mathbb{B}$, $where\ \sigma = \sigma_1 \times \sigma_2$. We defined the function$E_f: \boldsymbol{R} \to \mathbb{B}\ by\ E_f(x_1, x_2) = f(\alpha(x_1), \alpha(x_2))$ where

$$\alpha(x_i) = \begin{cases} x_i, & if\ x_i \in \sigma_i \\ \sup\{t: [x_i, t] \subset [a_i, b_i] \setminus \sigma_i\}, & otherwise \end{cases}$$

for $i = 1,2,3, \ldots$

Inview of the above definition,suppose $f \in kV_\varphi^R(\sigma, \mathbb{B})$ is such that the marginal functions $f(a_1, \cdot) \in kV_{\varphi_2}^R(\sigma_2, \mathbb{B})$ and $f(\cdot, a_2) \in kV_{\varphi_1}^R(\sigma_1, \mathbb{B})$ then f is said to be of bounded$k\varphi^* - variation\ over\ \sigma(that\ is\ f \in kV_{\varphi^*}^R(\sigma, \mathbb{B}))$. If $f \in kV_{\varphi^*}^R(\sigma, \mathbb{B})$ then each of the marginal functions $f(\cdot, t) \in kV_{\varphi_1}^R(\sigma_1, \mathbb{B})$ and $f(s, \cdot) \in kV_{\varphi_2}^R(\sigma_2, \mathbb{B})$, where $t \in \sigma_2$ *and* $s \in \sigma_1$ are fixed. For $\mathbb{B} = \boldsymbol{C}$, we omit writing $\boldsymbol{C}$, the class $kV_{\varphi^*}^R(\sigma, \mathbb{B})$ *reduces to* $kV_{\varphi^*}^R(\sigma)$. For $\sigma = [a, b]$ we simply write $kV_{\varphi^*}^R[a, b]$ or more generally $kV_{\varphi^*}^R(\boldsymbol{R})$ *when* $\sigma = \boldsymbol{R}$.

Theorem 3. Let σ_1 *and* σ_2 be non-empty compact subsets **R** such that $\sigma = \sigma_1 \times \sigma_2$ *and* $\mathbb{B}$ be a commutative unital Banach algebra. If $f \in kV_{\varphi^*}^R(\sigma, \mathbb{B})$ then $kV_\varphi^R(f, \sigma, \mathbb{B}) = kV_\varphi^R(E_f, \boldsymbol{R}, \mathbb{B})$, where **R** is the smallest closed rectangle containing σ.

Proof: **R** is the smallest closed rectangle containing $\sigma = \sigma_1 \times \sigma_2$ and $E_f|\sigma = f$ implies $kV_\varphi^R(f, \sigma, \mathbb{B}) \le kV_\varphi^R(E_f, \boldsymbol{R}, \mathbb{B})$.

Let $R = s \times t$ be any rectangular grid of $[a_1, b_1] \times [a_2, b_2]$, *where* $s = \{s_i\}_{i=1}^m \in \coprod([a_1, b_1])$ *and* $t = \{t_j\}_{j=1}^m \in \coprod([a_2, b_2])$. Consider $u = s \cap \sigma_1 \in \coprod(\sigma_1)$ and $v = t \cap \sigma_2 \in \coprod(\sigma_2)$. Then $D = u \times v$ is a rectangular grid of $\sigma = \sigma_1 \times \sigma_2$.for the simplicity of the proof, suppose there is only one $(s_k, t_l) \in R$ such that $(s_k, t_l) \in R \setminus D$. Where $(s_{k-1}, s_k) \cap \sigma_1 = (s_k, s_{k+1}) \cap \sigma_1 = \emptyset$ and $(t_{l-1}, t_l) \cap \sigma_2 = (t_l, t_{l+1}) \cap \sigma_2 = \emptyset$. Then

$$sup_{\pi,D} \frac{\sum_{s\times t} \varphi\left(\frac{\left|\Delta E_f(s_i t_j)\right|}{|\Delta s_i \Delta t_j|}\right)|\Delta s_i \Delta t_j|}{\sum k\left(\frac{\Delta s_i \Delta t_j}{b-a}\right)} =$$

$$\sup_{\pi,D} \frac{\sum_{i=1}^{k-2}\sum_{j=1}^{l-2} \varphi(\frac{\left|\Delta E_f(s_i t_j)\right|}{|\Delta s_i \Delta t_j|})|\Delta s_i \Delta t_j|}{\sum k((\Delta s_i \Delta t_j)/(b-a))} + \frac{\sum_{i=1}^{k-2}\sum_{j=1}^{l-2} \varphi(\frac{\left|\Delta E_f(s_{i-1} t_{j-1})\right|}{|\Delta s_{i-1} \Delta t_{j-1}|})|\Delta s_{i-1} \Delta t_{j-1}|}{\sum k((\Delta s_{i-1} \Delta t_{j-1})/(b-a))} + \frac{\sum_{i=1}^{k-2}\sum_{j=1}^{l-2} \varphi(\frac{\left|\Delta E_f(s_{i-1} t_j)\right|}{|\Delta s_{i-1} \Delta t_j|})|\Delta s_{i-1} \Delta t_j|}{\sum k((\Delta s_{i-1} \Delta t_j)/(b-a))} + \frac{\sum_{i=1}^{k-2}\sum_{j=1}^{l-2} \varphi(\frac{\left|\Delta E_f(s_i t_{j-1})\right|}{|\Delta s_i \Delta t_{j-1}|})|\Delta s_i \Delta t_{j-1}|}{\sum k((\Delta s_i \Delta t_{j-1})/(b-a))} + \frac{\sum_{i=1}^{k-2}\sum_{j=1}^{l-2} \varphi(\frac{\left|\Delta E_f(s_k t_l)\right|}{|\Delta s_k \Delta t_l|})|\Delta s_k \Delta t_l|}{\sum k((\Delta s_k \Delta t_l)/(b-a))} + \frac{\sum_{i\ge k+1}^{m}\sum_{j\ge l+1}^{n} \varphi(\frac{\left|\Delta E_f(s_i t_j)\right|}{|\Delta s_i \Delta t_j|})|\Delta s_i \Delta t_j|}{\sum k((\Delta s_i \Delta t_j)/(b-a))}.$$

Since $E_f(s_k, t_l) = f(\alpha(s_k, t_l))$, for $(u \times v)^* = (u \times v) \cup \{\alpha(s_k, t_l)\}$, we get

$$\sup_{\pi,D} \frac{\sum_{s\times t} \varphi\left(\frac{\left|\Delta E_f(s_i t_j)\right|}{|\Delta s_i \Delta t_j|}\right)|\Delta s_i \Delta t_j|}{\sum k\left(\frac{\Delta s_i \Delta t_j}{b-a}\right)} \le \sup_{\pi,D} \frac{\sum_{u\times v} \varphi\left(\frac{\left|\Delta f(u_i v_j)\right|}{|\Delta s_i \Delta t_j|}\right)|\Delta u_i \Delta v_j|}{\sum k\left(\frac{\Delta u_i \Delta v_j}{b-a}\right)}$$

Hence, the result follows from $kV_\varphi^R(E_f, \sigma, \mathbb{B}) \le kV_\varphi^R(f, \sigma, \mathbb{B})$.

Remark 4. From the theorem it is obvious that for a given function $f: \sigma \to \mathbb{B}$, $f \in kV_{\varphi^*}^R(\sigma, \mathbb{B})$ if and only if $E_f \in kV_{\varphi^*}^R(\sigma, \mathbb{B})$.

Remark 5. The map $F : kV_{\varphi^*}^R(\sigma, \mathbb{B}) \to kV_{\varphi^*}^R(R, \mathbb{B})$ defined by $F(f) = E_f$ for all $f \in kV_{\varphi^*}^R(\sigma, \mathbb{B})$ is a linear isometry.

Theorem 6. $(kV_{\varphi^*}^R(\sigma, \mathbb{B}), \|\cdot\|_{kV_{\varphi^*}^R(\sigma,\mathbb{B})})$ is a Banach space with respect to the point wise operations, where $\mathbb{B}$ is a commutative unital Banach algebra and $\sigma = \sigma_1 \times \sigma_2$ in which σ_1 *and* σ_2 are non-empty compact subsets of **R.**

Proof: let $\{f_k\}_{k=1}^\infty$ be a Cauchy sequence in $kV_{\varphi^*}^R(\sigma, \mathbb{B})$. Therefore it converges uniformly to some function say f *on* σ.From theorem 2.6 [8] we get

$$\lim_{k\to\infty} kV_{\varphi^2}^R((f_k(a_1,\cdot) - f(a_1,\cdot)), \sigma_2, \mathbb{B}) = 0 \quad (1)$$

$$\lim_{k\to\infty} kV_{\varphi^1}^R((f_k(\cdot, a_2) - f(\cdot, a_2)), \sigma_1, \mathbb{B}) = 0 \quad (2)$$

Now for any rectangular grid D of σ

$$\begin{aligned} kV_\varphi^R(f_k, \sigma, \mathbb{B}, D) &\le kV_\varphi^R(f_k - f_l, \sigma, \mathbb{B}, D) \\ &\quad + kV_\varphi^R(f_l, \sigma, \mathbb{B}, D) \\ &\le kV_\varphi^R(f_k - f_l, \sigma, \mathbb{B}) + kV_\varphi^R(f_l, \sigma, \mathbb{B}) \end{aligned}$$

this implies $kV_\varphi^R(f_k, \sigma, \mathbb{B}) \le kV_\varphi^R(f_k - f_l, \sigma, \mathbb{B}) + kV_\varphi^R(f_l, \sigma, \mathbb{B})$

and $\left|kV_\varphi^R(f_k, \sigma, \mathbb{B}) - kV_\varphi^R(f_l, \sigma, \mathbb{B})\right| \le kV_\varphi^R(f_k - f_l, \sigma, \mathbb{B}) \to 0$ *as* $k, l \to \infty$

hence,

$\left\{kV_\varphi^R(f_k, \sigma, \mathbb{B})\right\}_{k=1}^\infty$ *is a Cauchy sequence in* $\boldsymbol{R}$ *and is bounded by* $M > 0$. Therefore

It is obvious that $kV_\varphi^R(f, \sigma, \mathbb{B}, D) \le M < \infty$ and thus, with (1) and (2) implies that $f \in kV_{\varphi^*}^R(\sigma, \mathbb{B})$. Moreover

$$kV_\varphi^R(f_k - f, \sigma, \mathbb{B}, D) = \frac{\sum_{i=1}^m\sum_{j=1}^n \varphi(\frac{\left|\Delta(f_k - f)(s_i t_j)\right|}{|\Delta s_i \Delta t_j|})|\Delta s_i \Delta t_j|}{\sum k((\Delta s_i \Delta t_j)/(b-a))}$$

$$= \lim_{l\to\infty} \frac{\sum_{i=1}^m\sum_{j=1}^n \varphi(\frac{\left|\Delta(f_k - f_l)(s_i t_j)\right|}{|\Delta s_i \Delta t_j|})|\Delta s_i \Delta t_j|}{\sum k((\Delta s_i \Delta t_j)/(b-a))}$$

$$\le \lim_{l\to\infty} kV_\varphi^R(f_k - f_l, \sigma, \mathbb{B}) \to 0 \text{ as } k \to \infty$$

Thus the result follows from (1) and (2).

Corollary 7. If we replace $(\sigma, \mathbb{B})$ *by* $([a_1, b_1] \times [a_2, b_2])$ in the above theorem we also have Banach space with respect to the point wise operations.

4 Recommendation for Further Studies:

The study initially has the desire to expand the concept to several variables, hence that can be undertaken by any potential researcher.

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