

Bayesian Analysis of Extended Lomax Distribution

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Abstract: In this paper, the Markov chain Monte Carlo (MCMC) method is used to estimate the parameters of Extended Lomax distribution, based on a complete sample. The proposed methodology can be used to obtain the Bayes estimates the parametric function such as reliability function, hazard function, etc under different loss functions and is also suitable for empirical modeling. A real data set is considered for illustration under non-informative and informative set of priors.

Keyword: Extended Lomax distribution, Markov chain Monte Carlo (MCMC), Bayesian estimation, OpenBUGS.

Introduction

The Lomax (1954) or Pareto II distribution has been quite widely applied in a variety of contexts. Although introduced originally for modeling business failure data, the Lomax distribution has been used for reliability modeling and life testing (e.g. Hassan and Al-Ghamdi, 2009) and applied to income and wealth distribution data (Harris, 1968; Atkinson and Harrison, 1978), firm size (Corbellini et al., 2007) and queuing problems. It has also found application in the biological sciences and even for modeling the distribution of the sizes of computer files on servers (Holland et al. 2006). Some authors such as Bryson (1974), have suggested the use of this distribution as an alternative to the exponential distribution when the data are heavy tailed.

The Lomax distribution can be motivated in a number of ways. For example, Balkema and de Haan (1974) show that it arises as the limit distribution of residual lifetime at great age; Dubey (1970) shows that it can be derived as a special case of a particular compound gamma distribution; and Tadikamalla (1980) relates the Lomax distribution to the Burr family distributions. On the other hand, the Lomax distribution has itself been used as the basis for several generalizations.

Adding parameters to a well established family of distributions is a useful method for obtaining new families of distributions, Marshall and Olkin(1997) introduced an interesting approach of adding additional parameter to an existing model.

The two parameter extension of Lomax distribution is a special case of generalized Pareto distribution and also a special case of the q-exponential distribution. The q-exponential extends this distribution to support on a bounded interval. The Lomax parameters are given by

$$\alpha = \frac{2-q}{q-1}; \lambda = \frac{1}{\lambda_q(q-1)}$$

The Lomax distribution is special case of the 4 – parameters generalized beta II distribution.

In this article the Bayes estimates of two-parameter Extended Lomax distribution model is considered when both parameters are unknown. It is observed that the Bayes estimates cannot be computed explicitly. In this paper, it is assumed that the parameters have uniform and gamma priors, and they are independently distributed under the priors. We use Gibbs sampling technique in OpenBUGS (Thomas, A. (2010)) to generate the Markov chain Monte Carlo (MCMC) samples from the posterior density function. Based on the generated samples, we can compute the Bayes estimates of the unknown parameters and also construct highest posterior density credible intervals. We have also estimated the reliability function. To demonstrate proposed method a real data set has been taken.

The Extended Lomax Model

The two parameters extended Lomax distribution has a shape and a scale parameter. The random variable X follows extended Lomax distribution with the shape and scale parameters as $\lambda > 0$ and $\alpha > 0$, respectively, if it has the following probability density function (pdf)

$$f(x; \alpha, \lambda) = \frac{\alpha \lambda (1+x)^{\lambda-1}}{\{(1+x)^\lambda - (1-\alpha)\}^2} \quad (x > 0, \alpha > 0, \lambda > 0) \quad (1.1) \quad \text{And it will be denoted by MOEL2}(\alpha, \lambda)$$

The corresponding cumulative distribution function (cdf) is

$$F(x) = \frac{(1+x)^\lambda - 1}{(1+x)^\lambda - (1-\alpha)}; (\alpha, \lambda) > 0; x \geq 0 \quad (1.2)$$

The hazard function is $h(x) = \frac{\lambda (1+x)^{\lambda-1}}{(1+x)^\lambda - (1-\alpha)}; x \geq 0, \alpha > 0, \lambda > 0 \quad (1.3)$

The reliability/survival function is given by

$$R(x) = \frac{\alpha}{(1+x)^\lambda - (1-\alpha)}; x \geq 0, \alpha > 0, \lambda > 0 \quad (1.4)$$

The quantile function is

$$x_p = \left\{ 1 + \frac{\alpha p}{1-p} \right\}^{1/\lambda} - 1; 0 < p < 1 \quad (1.5)$$

The median is given by

$$\text{median}(x) = \left\{ 1 + \alpha \right\}^{1/\lambda} - 1$$

The mode of the extended Lomax distribution is

$$x = \left\{ \frac{(1-\lambda)(1-\alpha)}{(1+\lambda)} \right\}^{1/\lambda} - 1 \quad (1.6)$$

The random deviate generate is given by

$$x = \left\{ 1 + \frac{\alpha u}{1-u} \right\}^{1/\lambda} - 1; 0 < u < 1 \quad (1.7)$$

Where, u has the U(0,1) distribution

Density and Hazard Function Shape

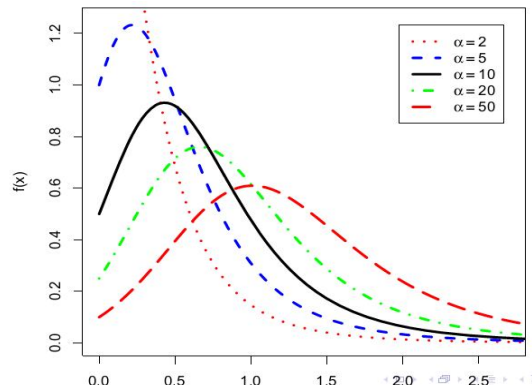


Figure 1: Shape of Density Function for $\lambda=1$ and different values of α

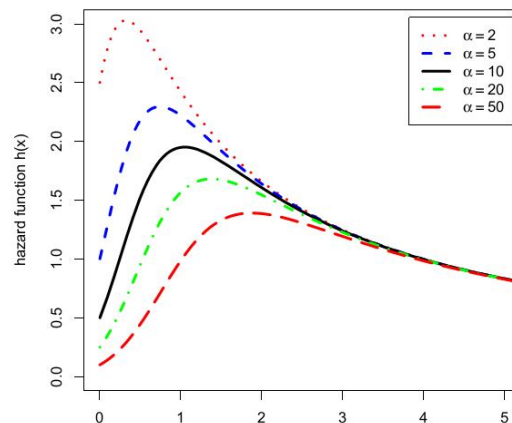


Figure 2: Shape of hazard function for $\lambda=1$ and different values of α

Maximum Likelihood Estimation

Let $\underline{X} = (X_1, \dots, X_n)$ be sample from a distribution with density function (1.1). The log likelihood function of the parameter $l(\alpha, \lambda)$ is given by

$$l(\alpha, \lambda) = n \log \alpha + n \log \lambda + (\lambda - 1) \sum_{i=1}^n (1 + x_i) - 2 \sum_{i=1}^n \log \left\{ (1 + x_i)^\lambda - (1 - \alpha) \right\} \quad (1.8)$$

Therefore, to obtain the MLE's of α and λ we can maximize equation (1.8) directly with respect to α and λ or we can solve the following two non-linear equations using iterative procedure

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} - 2 \sum_{i=1}^n \frac{1}{\left\{ (1 + x_i)^\lambda - (1 - \alpha) \right\}} \quad (1.9)$$

$$\frac{\partial l}{\partial \lambda} = \frac{n}{\lambda} + \sum_{i=1}^n \log(1 + x_i) - 2 \sum_{i=1}^n \frac{(1 + x_i)^\lambda \log(1 + x_i)}{\left\{ (1 + x_i)^\lambda - (1 - \alpha) \right\}} \quad (1.10)$$

We have the asymptotic normality results as follows:

$$\sqrt{n}(\hat{\theta} - \theta) \rightarrow N_2(0, I^{-1}(\theta)); \theta = (\alpha, \lambda)$$

Where $I(\theta)$ is the Fisher information matrix. This asymptotic behavior remains valid if $I(\theta)$ replaced by consistent estimator or more simply by the sample information matrix given by

$$I(\hat{\theta}) = \left(\frac{-\partial^2 l}{\partial \theta_i \partial \theta_j} \right)_{\theta=\hat{\theta}} \quad \theta = (\alpha, \lambda) \quad (1.11)$$

The observed information matrix $I(\theta)$ provides an asymptotic estimate of the covariance matrix or the expected information matrix. The asymptotic distribution is generally used to construct approximate confidence interval. The $(1 - 2Y)100\%$ confidence intervals of the parameters $\theta = (\alpha, \lambda)$ can be constructed as

$$\hat{\alpha} \pm z_{\gamma} \sqrt{\text{var}(\hat{\alpha})} \text{ and } \hat{\lambda} \pm z_{\gamma} \sqrt{\text{var}(\hat{\lambda})} \quad (1.12) \text{ where } z_{\gamma} \text{ is the upper } \gamma^{\text{th}} \text{ percentile of the standard normal distribution.}$$

Bayesian estimation using Markov Chain Monte Carlo

Markov chain Monte Carlo (MCMC) method is the Monte Carlo integration using Markov chains and this technique is used by the Gibbs sample, which provides a way for extracting samples from the posteriors often specified up to proportionality, as a product of prior and likelihood function. The idea of Gibbs sampler is to sample the component of target vector of the variable in step by step and each step is based on the assumption that all other are known to stay at most recent values. This type of sampling procedure was first introduced by Geman and Geman(1984) but applicability for the statistical modeling with the Bayesian principle was proposed by Gelfand and Smith(1990).

By assuming some arbitrarily chosen initial values the algorithm is started for the related variables and generating the values of the variate from the various full conditionals in a cyclic order. That means, in every time value of the variable is generated from the full conditional, which is influenced by the at most last value of all other conditioning variables and after each cycle of iteration it is updated by sampling a new value from its full conditional. The process is repeated until and unless the generating chains achieve a systematic pattern of convergence. Hence after that the generated variate values can be considered as the random samples from the corresponding posteriors. For this readers are referred to Gammerrman(1997) and Robert and Casella(2004) for details of the procedures and the related convergence test.

To generate the posterior sample the most widely used piece of software for applying Bayesian inference is the OpenBUGS (Thomas, 2010). This OpenBUGS software offer a user, interface based on the dialogue boxes and the menu commands, through the model may then be analyses using Markov chain Monte Carlo techniques. This is a fully extensible modular framework for constructing and analyzing Bayesian probability models Gilks et al.(1996) and Lunn et al., 2000) for existing probability models. Since, the extended Lomax distribution is not available in OpenBUGS. Thus, it requires incorporation of a module in ReliaBUGS, (Kumar et al. 2010) subsystem of OpenBUGS to estimate parameters of this distribution.

A model is written in component Pascal for extended Lomax distribution to perform full Bayesian analysis in OpenBUGS using the method described in (Thomas et al.(2006), Thomas(2010), Kumar et al.(2010) and Lunn et al., (2013)). The model code can be obtained from the author.

Prior Distributions

(i) **Uniform prior:** It frequently happens that the experimenter or researcher knows in advance that the probable values of parameter θ lies over a finite range $[a, b]$ but has no strong opinion about any subset of values over this range. In that case a uniform distribution over $[a, b]$ may be a good approximation for the parameter as the prior distribution, the probability density function of the uniform prior is given as

$$g(\theta) = \frac{1}{b-a}; \quad 0 < a < \theta < b$$

(ii) **Gamma prior:** The most widely used prior distribution of the parameter θ is the gamma distribution with parameters 'a' (>0) and 'b' (>0) denoted as $\theta \sim G(a, b)$ with pdf given by

$$g(\theta) = \frac{b^a}{\Gamma a} \theta^{a-1} e^{-b\theta}; \quad \theta > 0, (a, b) > 0$$

Data Analysis

To demonstrate the given methodology, we have to present the one real data set. The data represent 46 repair times (in hours) for an airborne communication transceiver (Chhikara and Folks, 1977). The data are presented below:

Data set : 0.2, 0.3, 0.5, 0.5, 0.5, 0.5, 0.6, 0.6, 0.7, 0.7, 0.7, 0.8, 0.8, 1.0, 1.0, 1.0, 1.0, 1.1, 1.3, 1.5, 1.5, 1.5, 1.5, 2.0, 2.0, 2.2, 2.5, 2.7, 3.0, 3.0, 3.3, 3.3, 4.0, 4.0, 4.5, 4.7, 5.0, 5.4, 5.4, 7.0, 7.5, 8.8, 9.0, 10.3, 22.0, 24.5

MLE and asymptotic confidence intervals

An extended Lomax model is used to fit this data set. We have started the iterative procedure by maximizing the log likelihood function given in equation (1.8) directly with an initial guess for $\alpha=1.0$ and $\lambda=0.5$, far away from the solution. We have used *optim()* function in R software (Venables et al., 2010), with option, Newton-Raphson method. The iterative process stopped only after 9 iterations. We obtained $\hat{\alpha} = 7.0338$ and $\hat{\lambda} = 1.9132$ and the corresponding log-likelihood value = -102.4137. An estimate of variance-covariance matrix, using equation (1.11) is given by

$$\begin{pmatrix} \text{var}(\hat{\alpha}) & \text{cov}(\hat{\alpha}, \hat{\lambda}) \\ \text{cov}(\hat{\alpha}, \hat{\lambda}) & \text{var}(\hat{\lambda}) \end{pmatrix} = \begin{bmatrix} 0.01961901 & -0.03665022 \\ -0.03665022 & 0.08170092 \end{bmatrix}$$

Thus, equation (1.12) we can construct the approximate 95% confidence interval for the parameters of extended Lomax model based on MLE's. Table 1 shows the MLE's with their standard errors and approximate 95% confidence intervals for α and λ .

Table 1. Maximum likelihood estimate, standard error and 95% confidence interval

Parameter	MLE	Std. Error	95% confidence interval
Alpha	7.0338	0.1400	(6.7594, 7.3082)
Lambda	1.9132	0.2858	(1.3530, 2.4733)

To check the validity of the model, we compute the Kolmogorov-Smirnov (KS) distance between the empirical distribution function and the fitted distribution function when the parameters are obtained by the method of maximum likelihood.

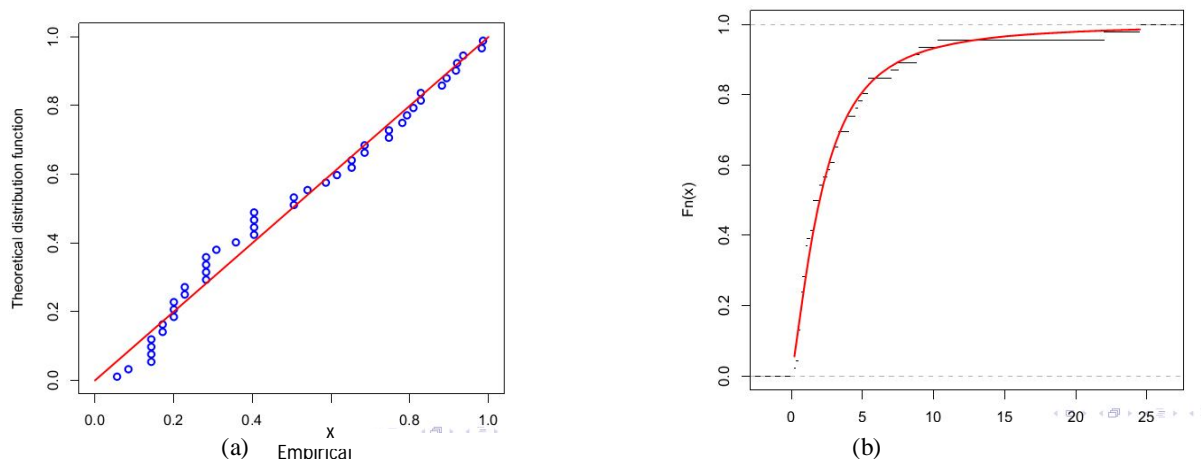


Figure 3: The graph of (a) Empirical distribution function and fitted distribution function. (b) Probability-probability (p-p) plot using MLE's as estimate

The result of K-S test is $D=0.0994$ with the corresponding p -value = 0.754. Therefore, the high p -value clearly indicates that extended Lomax model can be used to analyze this data set and also plot the empirical distribution function in Figure 3(a) which shows that the estimated extended Lomax model provides excellent fit to the given data.

The other graphical method widely used for checking whether a fitted model is in agreement with the data is Probability-Probability (P-P) plots in model validation. As can be seen from the straight line pattern in Figure 3(b) the extended Lomax model fits the data very well.

Bayesian estimation under uniform prior for scale parameter and gamma prior for shape parameter

In this section I provide the Bayes estimates of the shape and scale parameters based on the uniform priors. The Bayesian model is constructed by specifying a prior distribution for α and λ and then multiplying with the likelihood function to obtain the posterior distribution function. For the current model, since there are no more constraints for the parameters other than the non-negative conditions and we have no reason to prefer one value over another for each of the parameters, it is convenient to assume independent uniform distribution on R^+ for the parameters. It is assumed that α follows $U(a,b)$ and λ follows $G(c,d)$ and they are independent. Therefore the joint posterior density is proportional to the likelihood function. We choose $a=0, b=30$ and $c=d=0.001$ to run MCMC in OpenBUGS.

For getting a random sample of specified size from the posterior one can either consider a long Gibbs sequence, ignore an initial transient phase and then take suitably spaced outcomes or, alternatively, proceed by generating independent parallel chains for long run and the picking up the observation, after convergence from each chain at the same relative position. The gaps in the single chain are chosen to make serial correlation negligibly small.

We run the model to generate two Markov chains at the length of 40,000 with different starting points of the parameters. We have chosen initial values $\alpha=1.0$ and $\lambda=0.5$ for the first chain and $\alpha=10.0$ and $\lambda=5$ for the second chain. The convergence is monitored using trace and BGR plots. We find that the Markov Chains converge together after approximately 3000 observations. Therefore, the burn in period of 5000 samples is more than enough to erase the effect of starting point (initial values). Finally, samples of size 7000 are formed from the posterior by picking up equally spaced every fifth outcome i.e. $thin=5$, starting from 5001. This is done to minimize the auto correlation among the generated deviates. Therefore, we have the posterior sample $(\alpha_{1i}, \lambda_{1i}), i = 1, 2, \dots, 7000$ from chain 1 and $(\alpha_{2i}, \lambda_{2i}), i = 1, \dots, 7000$ from chain 2.

The chain 1 is considered for convergence diagnostics plots. The visual summary is based on posterior sample obtained from chain1, where as the numerical summary is presented for both the chains.

Convergence diagnostics

The first step in making an inference from MCMC analysis is to ensure that an equilibrium has indeed been reached by the Markov chain, i.e. that the chain has converged. There are several ways to check for convergence. Most methods use at least two parallel chains. Visual checks are routinely used to confirm convergence.

(i) History Plot:

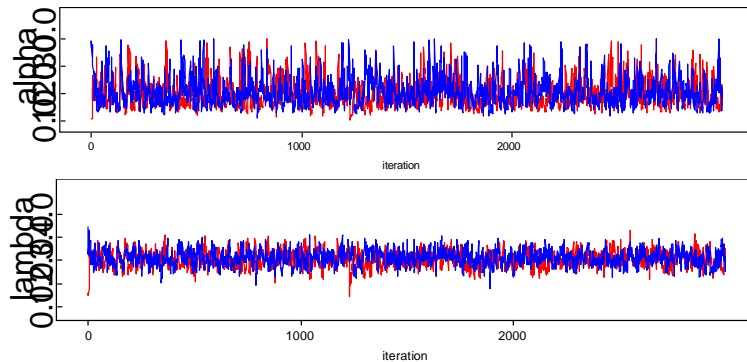


Figure 4: Sequential realization of the parameters α and λ .

Sequential realization of the parameters α and λ for the iteration number 3001 to 5000 can be observed in Figure (4), the dotted lines represents the mean of all 2000 sample values. Visual inspection of the time series plot produced by “history” suggests that the Markov chains have converged.

It looks like nice oscillograms around a horizontal line without any trend. The Markov chain is most likely to be sampling from the stationary distribution and is mixing well.

(ii) Brook-Gelman-Rubin(BGR) Diagnostics

The Brooks-Gelman-Rubin (Gelman and Rubin, 1992 and Brooks and Gelman, 1998) convergence diagnostics is based on running multiple parallel chains from over dispersed starting values.

OpenBUGS provides the Brooks-Gelman-Rubin statistics for assessing convergence. For a given parameter, this statistics assesses the variability within parallel chains as compared to variability between parallel chains. The model is judge to have converged if the ratio of between to within variability is close to 1. Plots of this statistics can be obtained in OpenBUGS by

clicking the “bgr diag”. The green line represents the between variability, the blue line represents the within variability, and the red line represents the ratio.

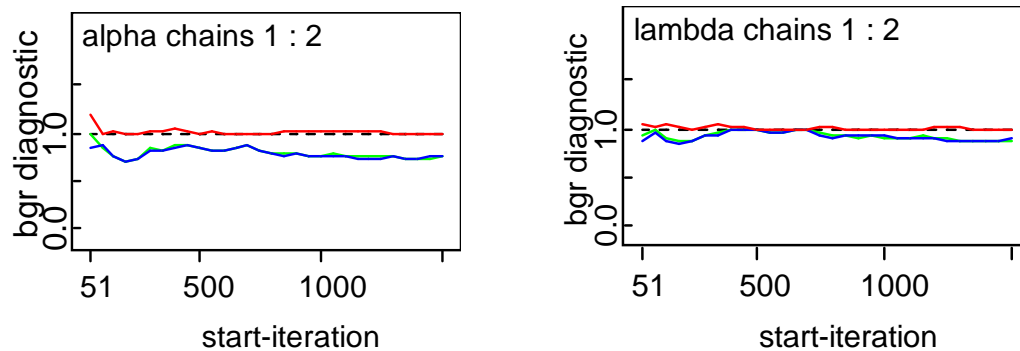


Figure 5: The BGR plots for α and λ .

Here, for this model, evidence for convergence comes from the red line being close to 1 on the y-axis and from the blue and green lines being stable (horizontal) across the width of the plot. From the figure (5) it is clear that convergence is achieved. Thus, the posterior summary statistics can be obtained.

Posterior Summary

Once the convergence has been achieved, it will need to run the simulation for a further number of iterations to obtain samples that can be used for posterior inference. By saving the more samples, the more accurate will be obtained in posterior estimation. One way to assess the accuracy of the posterior estimates is by calculating the Monte Carlo error (MC error) for each parameters. As a rule of thumb, the simulation should be run until the Monte Carlo error (MC error) for each parameter of interest is less than about 5% of the sample standard deviation (SD). The Monte Carlo error (MC error) and sample standard deviation (SD) are reported in the numerical summary table.

In Table 2, various quantities of interest and their numerical values based on MCMC sample of posterior characteristics under uniform priors are considered. The numerical summary is based on final posterior sample (MCMC output) of 7000 samples for α and λ .

Numerical Summary

Table 2: Numerical summaries based on MCMC samples of posterior characteristics under uniform priors:

Characteristics	Chain 1		Chain 2	
	Alpha	Lambda	Alpha	Lambda
Mean	9.6876	2.0269	9.5562	2.0186
Standard deviation	4.7912	0.3352	4.7905	0.3343
MC error	0.0746	0.0048	0.0688	0.0049
Minimum	1.3990	0.9384	1.3190	0.7907
2.5 th percentile	3.2458	1.3930	3.1696	1.3730
First quartile	6.1998	1.7920	6.1453	1.7898
Median	8.6685	2.0155	8.5210	2.0150
Third quartile	12.1725	2.2530	11.8900	2.2410
97.5 th percentile	21.6400	2.7030	21.9813	2.6880
Maximum	30.0000	3.2400	30.0000	2.2760
Mode	6.6452	1.9743	6.9456	2.0252

We know that the posterior mean, median and mode are Bayes estimates under square error, absolute error and zero error loss functions respectively. In Table 2, the computed values of posterior quantities mean, median, mode, 2.5th and 97.5th percentiles are given. The five point summary (minimum, first quartile(Q_1), median, third quartile(Q_3) and maximum) is also computed. Note that once we have the MCMC samples, it is possible to compute the Bayes estimates with respect to any other loss function also.

Credible and Highest Probability Density (HPD) intervals

Symmetric credible intervals: To compute the credible intervals of α and λ , order $\alpha_1, \dots, \alpha_M$ as $\alpha_{(1)}, \dots, \alpha_{(M)}$ and $\lambda_1, \dots, \lambda_M$ as $\lambda_{(1)}, \dots, \lambda_{(M)}$ respectively. Then $100(1-2\gamma)$ symmetric credible intervals of α and λ respectively become $[\alpha_{(M\gamma)}, \dots, \alpha_{(M(1-\gamma))}]$ and $[\lambda_{(M\gamma)}, \dots, \lambda_{(M(1-\gamma))}]$.

Highest Probability Density (HPD) interval

The HPD intervals are of the shortest length of any of the Bayesian credible intervals. Let $\alpha_{(1)} < \dots < \alpha_{(M)}$ be the ordered $\alpha_{(i)}$. From the following $100(1-2\gamma)\%$ credible intervals of $[\alpha_{(1)}, \dots, \alpha_{(1-\gamma)M}], \dots, [\alpha_{(\gamma M)}, \dots, \alpha_{(M)}]$; choose that interval which has the shortest length. Similarly, we can construct a $100(1-\gamma)\%$ HPD credible interval of λ . The highest

probability density (HPD) intervals are computed using the algorithm described by Chen and Shao (1999) under the assumption of unimodal marginal posterior distribution. Table 3, shows the 95% symmetric credible and HPD intervals under uniform prior for α and gamma prior for λ based on the 7000 posterior samples.

Table 3: Credible Interval and HPD Credible Interval (95%) under uniform and gamma priors

Parameter	Sample	Credible Interval	HPD Credible zInterval
Alpha	Chain 1	(3.2458,21.6400)	(2.6060,19.8200)
	Chain 2	(3.1698,21.9813)	(2.3290,19.7400)
Lambda	Chain 1	(1.3930,2.7030)	(1.4020,2.7100)
	Chain 2	(1.3730,2.6880)	(1.4110,2.7180)

Once a random sample from the posterior is obtained, desired (posterior) characteristics e.g. reliability function is studied. Also any function $G(\alpha)$ (or $g(\lambda)$) of the parameters of interest α or λ can be estimated by simply considering $\{G(\alpha_{1i}), G(\lambda_{1i})\}$, $i = 1, \dots, 7000$ from chain 1 and $\{G(\alpha_{2i}), G(\lambda_{2i})\}$, $i = 1, \dots, 7000$ from chain 2 as a MCMC sample of the desired parametric function $G(\alpha)$ (or $G(\lambda)$).

Visual Summary

The probability histograms of α and λ generated by MCMC method along with their kernel density estimates is plotted. The kernel density is plotted using R software, (Venables and Ripley, 2002 and Venables et al., 2010) under Gaussian in the kernel selecting suitable bandwidth. Figure (6) provides the kernel density estimates of α and λ . The MCMC samples (actual realizations) are plotted along the x-axis.

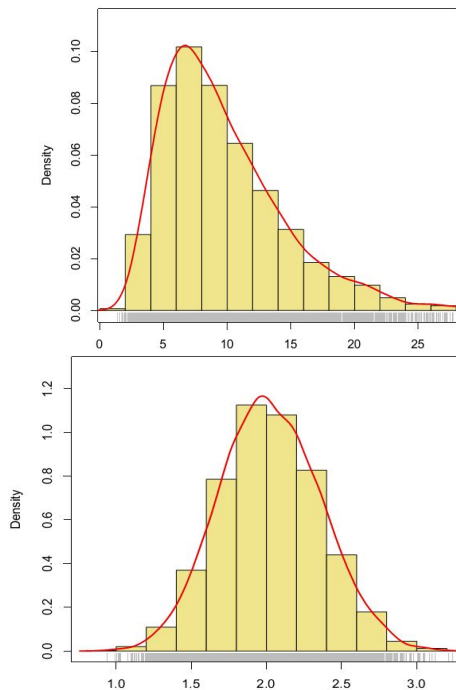


Figure 6: Kernel density estimate and histogram of α (a) and λ (b) based on MCMC samples.

Estimation of reliability function

In this section, the main aim was to demonstrate the effectiveness of proposed methodology. For this we have estimated the reliability function using MCMC samples under uniform prior for α and gamma prior for λ . Since due to an effective MCMC technique therefore, any function of the parameters can be estimated. The empirical reliability function has been used to make the comparison more meaningful. The figure 7(a), exhibits the estimated reliability function using Bayes estimate based on MCMC output under independent uniform prior for α and gamma prior for λ and the empirical reliability function.

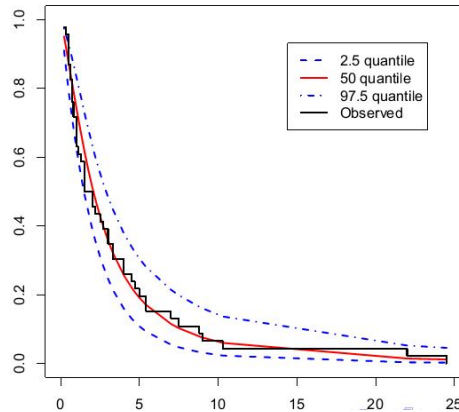


Figure 7: The estimated reliability function (dashed line) based on MCMC samples and the empirical reliability function (solid line) based on uniform priors.

It can be observed that the MCMC method works quite well for uniform and gamma priors for the parameters α and λ respectively.

Application and Uses

The acceptance and applicability of Bayesian methods have increased in recent years. Today, with advances in computation and methodology, researchers are using Bayesian methods to solve an increasing variety of complex problems. In many applications, Bayesian methods provide important computational and methodological advantages over classical techniques.

Reliability engineering covers all the processes of conception, design, testing, estimation, optimization, maintainability and availability of the product. The reliability of the products can be enhanced by applying life data analysis techniques and other reliability engineering methodologies.

One of the most important parts of reliability engineering is reliability modeling for life data analysis, which deals with the specific issues to study and predict the lifetime of the products using statistical parametric distributions or nonparametric methods, and subsequently these methodologies and results can be applied to product testing and prediction, optimization of warranty policy and quality and reliability enhancement. It may also be applicable for the survival or medical data analysis.

Conclusion

In this paper, the extended Lomax distribution with scale parameter α , and shape parameter λ has been discussed and estimate the value of these parameters based on one real data set in Bayesian framework by using the Markov chain Monte Carlo (MCMC) method. The MCMC method is more effective method as compare to the usual method of estimation.

Bayesian analyses is carried out by considering uniform prior for scale parameter α and gamma prior for shape parameter λ . The posterior samples are drawn in to OpenBUGS. The numerical summary for the extended Lomax distribution has been worked out under different set of priors based on the MCMC samples. We constructed the credible and highest probability density (HPD) interval, so we can easily obtain the probability intervals for the parameters through the MCMC samples from the empirical distribution. The stated methodology can be used to estimate any function of the parameter such as reliability and hazard function.

The calculation of probability interval for the parameters of this distribution is more easier and accurate than that constructed based on MLE which needs the local estimate of the variance covariance matrix of MLE or the Fisher information matrix.

The paper successfully described the scope of Markov Chain Monte Carlo (MCMC) technique in the extended Lomax model. Such procedures can be developed for other similar distributions as well. The only requirement is to develop a proper and efficient algorithm in OpenBUGS.

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