A Study on the Queue Length of the State-Dependent of an Unreliable Machine

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Abstract: In this paper, we consider a state-dependent queueing system in which the machine is subject to random breakdowns. Jobs arrive at the system randomly following a Poisson process with state-dependent rates. Service times and repair times are exponentially distributed. The machine may fail to operate with probability depending on the number of jobs completed since the last repair. The main result of this paper is the matrix-geometric solution of the steady-state queue length from which many performance measurements such as mean queue length, machine utilization are obtained.

Key Words : Laplace transform, Markov chain, Matrix-geometric, Phase-type distribution, Steady-state probability.

1. Introduction

We consider in this paper a single-machine queueing system subject to random breakdowns. Jobs arrive to the system in accordance with a Poisson process with state-dependent rates. If the machine is available when a job arrives, it is immediately processed for a random amount of time following an exponential distribution. If the machine is busy or in repair, then the arriving job is put in a buffer, awaiting service. The machine's operational time, that is the sum of service times since the last repair, follows a general distribution. Once a breakdown occurs it takes a random amount of time for repair.

2. Determination of Breakdown probability

We derive the machine breakdown probability. Let $\{X_k, k \geq 1\}$ be a sequence of service times that are independent identically distributed (i.i.d.) random variables following an exponential distribution of rate μ. Without loss of generality, we may assume that the service rate is unit, $\mu = 1$. Denote by Y the machine operational time since the last repair. We assume that Y has a general cumulative distribution function (CDF) F and a probability density function (PDE) $g = F/dt$. the Laplace transform (LT) of a function $g(t)$ is denoted by $g^*(s)$.

$$g^*(s) = L\{g\} = \int_0^\infty e^{-sy} g(y) \mathrm{d}y \qquad (1)$$

The LT of F and its component $\overline{F} = 1 - F$ are given, respectively, by

$$F^*(s) = \frac{1}{s} g^*(s) \tag{2}$$

$$\overline{F}^*(s) = \frac{1}{s}[1 - g^*(s)] \tag{3}$$

We restrict ourselves to the machine operational time Y having a phase-type distribution with parameter α and T, i.e,. $F(t) = 1 - \alpha e^{Tt} U$ and $g(t) = \alpha e^{Tt} T^0$. Where T^0 satisfies $TU + T^0 = 0$ with U being defined as a column vector with elements equal to 1. Many nonnegative, continuous distributions can be represented or approximated by a phase-type distribution [2]. Based on the service times X_k, we define a random variable S_i as the sum of i consecutive service times $S_i = \sum_{k=1}^{i} X_k$ with $S_0 = 0$. It is known that S_i follows a gamma distribution, or Erlang-k distribution with parameters i and μ.

3.Phase-Type Distribution Probability

If the machine operational time Y has a phase-type distribution with parameter α and T, the probability q_i that the machine will serve at least i jobs is given by

$$q_i = \alpha(i - T/\mu)^{-i} U, \quad i \ge 0 \tag{4}$$

Proof it is true for $i = 0$, $q_0 = \infty$, $U = 1$. For $i > 0$, conditioning on the sum of service time S_i, we certainly have

$$\begin{aligned} q_i &= \Pr\{\text{Atleast } i \text{ jobs completed}\} \\ &= \Pr\{y > S_i\} \\ &= \int_0^\infty \overline{F}(t) \frac{\mu(\mu t)^{i-1}}{(i-1)!} e^{-\mu t} \mathrm{d}t \end{aligned}$$

In comparison with Laplace transforms, the above integral can be thought as some LT evaluated at $s = \mu$. That is

$$q_i = L\left\{ \overline{F}(t) \frac{\mu(\mu t)^{i-1}}{(i-1)!} \right\}\Big|_{s=\mu} \tag{5}$$

Recall the Laplace transform property $L\{t^i f(t)\} = (-i)^i \frac{d^i}{ds^i} L\{f(t)\}$.

Equation (5) now becomes

$$q_i = \frac{\mu^i}{(1-i)!} L\{t^{i-1} \overline{F}(t)\}\Big|_{s=\mu} \tag{6}$$

$$= \frac{\mu^i}{(1-i)!}\frac{d^{i-1}}{ds^{i-1}}\bar{F}^*(s)\Big|_{s=\mu}$$

If the LT of the machine operational time exists and has a rational form, the LT of its complimentary distribution function will also have a rational form. For a phase-type distribution $g(t) = \exp(Tt)T^0$, its LT is [14]

$$g^* = \alpha(sI - T)^{-1}T^0$$

After some algebraic manipulation, it is not hard to derive the LT for

$$\bar{F}^*(s) = \alpha(sI - T)^{-1}U \tag{7}$$

Its $(i-1)$ th derivative is then

$$\frac{d^{i-1}}{ds^{i-1}}\bar{F}^*(s) = \alpha(-1)^{i-1}(i-1)!(sI - T)^{-1}U \tag{8}$$

Substituting into equation (6) yields the desired result.

$$q_i = \mu^i\alpha(\alpha I - T)^i U = \alpha(I - T/\mu)^i U$$

Immediate results of the explicit forms of mixture of exponentials (hyper exponential) and Erlang- distribution.

4. Conditional probability

Suppose that the LT $\bar{F}^*(s)$ has m distinct eigenvalues S_j.

$$\bar{F}^*(s) = \sum_{i=1}^{m}\frac{c_j}{s - s_j} \tag{9}$$

Where c_j are some coefficient. The conditional probability b_j that a breakdown will occur during the next operation given that the machine has processed (i - 1) jobs is given as

$$b_j = \frac{\sum_{i=1}^{n}\frac{(-s_j)}{\mu - s_j}\left(\frac{\mu}{\mu - s_j}\right)^{i-l}}{\sum_{i=1}^{n}c_j\left(\frac{\mu}{\mu - s_j}\right)^{i-l}} \tag{10}$$

Proof By definition, we have

$$b_i = \text{Prob}\{Y \le S_i / Y > S_{i-1}\}$$

Using conditional probability, it is true that

$$b_i = \frac{\text{Pro}\{S_{i-1} < Y \le S_j\}}{\text{Pro}\{Y > S_{i-1}\}}$$
$$= \frac{q_{i-1} - q_i}{q_{i-1}} \tag{11}$$

From equations (6) and (9) we obtain the expression for q_i

$$q_i = \sum_{j=1}^{m} c_j \left(\frac{\mu}{\mu - s_j} \right) \tag{12}$$

Substitution it into $\hat{b}_i$ leads to the desired result after simplification.

If we denote by $\hat{b}_i$ the conditional probability of no breakdown in the next operation, $\hat{b}_i = 1 - b_i$, the relation between q_i and $\hat{b}_i$ is

$$q_i = \prod_{k=1}^{i} \hat{b}_k \tag{13}$$

It is known that many distribution functions do have the LT from as described in equation (9). Such functions are hyper exponentials, Coxian distribution functions, generalized Erlang-k functions with different rates, etc. If the operational time follows an exponential distribution of rate θ, the breakdown probability reduces to a constant $b_i = \dfrac{\theta}{\mu + \theta}$.

5. Stochastic Balance Equations

We define the system state by a pair (i, n), where i is the i-th job that the machine is working on after repair, and n is the number of jobs in the system. The states $(0, n)$ represent the machine is under repair. We assume the machine repair time follow an exponential distribution with rate γ. Since the breakdown can occur only when a job is present in the system, both states $(0,0)$ and $(1,0)$ do not exist. That it is impossible that there is no job in the system when the machine is under repair; it is also impossible that the machine serves one job while the system is empty. Now we define the steady-state probability by

$$\text{P}(i, n) = \text{Pro}\{\text{state in}\,(i, n)\}, 0 \le n \le N, \quad i \ge 0$$

The following steady-state equations can be readily formulated

(i) for states machine in repair $i = 0$, we have the following balance equations

$$(\lambda_1 + \gamma)\mathrm{P}(0,1) = \sum_{i=1}^{\infty} b_i \mu \mathrm{P}(i,1) \tag{14}$$

$$(\lambda_n + \gamma)\mathrm{P}(0,1) = \lambda_{n-1}\mathrm{P}(0,n-1) + \sum_{i=1}^{\infty} b_i \mu \mathrm{P}(i,n), \quad 1 < n \le N \tag{15}$$

(ii) For state $i = 1$, we have

$$(\lambda_1 + \mu)\mathrm{P}(1,1) = \gamma \mathrm{P}(0,1) \tag{16}$$

$$(\lambda_n + \mu)\mathrm{P}(1,n) = \gamma \mathrm{P}(0,n) + \lambda_{n-1}\mathrm{P}(1,n-1), \quad 1 < n \le N \tag{17}$$

(iii) Finally the balance equations for state $i > 1$ are

$$\lambda_0 \mathrm{P}(i,0) = \hat{b}_{i-1}\mu \mathrm{P}(i-1,1) \tag{18}$$

$$(\lambda_n + \mu)\mathrm{P}(i,n) = \lambda_{n-1}\mathrm{P}(i,n-1) + \hat{b}_{i-1}\mu \mathrm{P}(i-1,n+1), \quad 0 < n < N \tag{19}$$

$$\mu \mathrm{P}(i,n) = \lambda_{N-1}\mathrm{P}(i,N-1) \tag{20}$$

The normalization condition is

$$\mathrm{P}(i,n) = 1 \tag{21}$$

To solve these equation, we write them in matrix and vector forms. First we define row vectors,

$$X_i = [\mathrm{P}(i,0)\mathrm{P}(i,1),\ldots \mathrm{P}(i,N)], \quad i \ge 0 \tag{22}$$

Note that for the purpose of consistent notation, we create two artificial states $(0,0)$ and $(1,0)$ so that all vectors X_i are of the same order $N+1$.

Then equations (14),(21) can be written in matrix forms

$$X_0 A_0 + \sum_{i=1}^{\infty} \lambda B = 0 \tag{23}$$

$$X_{i-1}C_{i-1} + \lambda A = 0, \quad i > 0 \tag{24}$$

Where these individual matrices are defined by

$$\mathbf{A_0} = \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & \cdots & 0 \\ 0 & -\lambda_1-\gamma & \lambda_1 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -\lambda_{N-1}-\gamma & \lambda_{N-1} \\ 0 & 0 & \cdots & 0 & -\gamma \end{pmatrix} \tag{25}$$

$$\mathbf{A_i} = \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & \cdots & 0 \\ 0 & -\lambda_1-\mu & \lambda_1 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -\lambda_{N-1}-\mu & \lambda_{N-1} \\ 0 & 0 & \cdots & 0 & -\mu \end{pmatrix}, \quad i > 0 \tag{26}$$

$$\mathbf{B_i} = b_i \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 \\ 0 & \mu & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \mu & 0 \\ 0 & 0 & \cdots & 0 & \mu \end{pmatrix}, \quad i > 0 \tag{27}$$

$$\mathbf{C_0} = \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 \\ 0 & \gamma & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \gamma & 0 \\ 0 & 0 & \cdots & 0 & \gamma \end{pmatrix} \tag{28}$$

$$\mathbf{C_i} = \hat{b}_i \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & \cdots \\ 0 & \mu & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & & \\ 0 & 0 & \cdots & \mu & 0 \end{pmatrix}, \quad i > 0 \tag{29}$$

All matrices are square matrices of order $(N+1)$. Form the balance equations we can see the matrices have the properties of $A_0U + C_0U = 0$, and $A_iU + B_iU + C_iU = 0$.

Let $A = A_i$, $i > 0$, $B_i = b_iB$, $C_i = \hat{b}_iC$, Denoted by $X = [X_0X_1,\ldots,X_i,\ldots]$, and by Q the infinitesimal generator, we have

$$XQ = 0 \tag{30}$$

$$\mathrm{XU} = 1$$

where Q is given in term of A_0, A, B, C_0 and C as

$$\mathbf{Q} = \begin{pmatrix} A_0 & C_0 & 0 & 0 & 0 & \dots \\ b_1B & A & \hat{b}_1C & 0 & 0 & \dots \\ b_2B & o & A & \hat{b}_2C & 0 & \dots \\ b_3B & 0 & 0 & A & \hat{b}_3C & \dots \\ \dots & \dots & \dots & \dots 0 & \dots & \dots \end{pmatrix} \qquad (31)$$

The matrix A is nonsingular. Thus its inverse exists. If we let $H_0 = -C_0A^{-1}$ and $H = -CA^{-1}$, it is possible to express X_i in terms of X_0, Repeatedly using equation.

CONCLUTION : In this model we have embedded in a closed queueing network that has been used in modeling flexible manufacturing system. Due to limited space and other resourses, the total number of jobs is bounded to a constant. In the above paper, we have studied the single unreliable machine as a consequences, the arrival process to a machine is often state-dependent.

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