Golden Graphs-I

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***Abstract*—.In this paper ,we have introduced new definition of golden graphs and hence pure golden graphs and characterized for acyclic graphs as pure golden graphs .Also generated a infinite class of P_n ,a path on n nodes as golden graphs.**

***Keywords*—Trees, Paths, adjacency matrix, characteristic polynomial of a graph G and Golden ratio.**

I. Introduction

The historic evidence of the appearance of Golden ratio (GR) in graph theory, first time GR appeared in graph theory in connection with chromatic polynomials. W.TTutte(1970), Michel O Alberston(1973),Saeid Alikhani and Yee-hock(2009) all dealt with GR in connection with chromatic polynomials. Pavel Chebotarev(2008) ,deal with GR in connection with spanning forest . We are giving an account of GR in graph which we came across while studying the spectral properties of graphs. While studying the spectra of P_4,path on 4 vertices ,we find that its Eigen values are $\frac{1+\sqrt{5}}{2}$, $\frac{1-\sqrt{5}}{2}$, $\frac{-1-\sqrt{5}}{2}$ and $\frac{-1+\sqrt{5}}{2}$ which are nothing but Golden ratio (Divine ratio).Interestingly ,we asked the question which graphs have Eigen values as Golden ratio. In this paper we have proved logically that, there are infinite class P_n, a path on n nodes which have GR as Eigen value and path P_4 is the only pure golden tree.

II. Preliminaries

Let G be a graph without loops or multiple links having n nodes. Then the adjacency matrix of G , $A(G) = A$, is a square matrix, symmetric matrix of order n ,whose elements A_{ij} are ones or zeros if the corresponding nodes are adjacent or not, respectively. This matrix has (not necessarily distinct) real-valued Eigen values ,which are denoted by $\lambda_{1,}\lambda_2,......,\lambda_{n.}$ The set of Eigen values of A together with their multiplicities form the spectrum of G ,which will be represented here as Spec $(G)=\{[\lambda_1]^{m_1},[\lambda_2]^{m_2}.........[\lambda_n]^{m_n}\}$,where λ_i is the i^{th} Eigen value with m_i multiplicity. Here the Eigen values are assumed to be labelled in a non-increasing manner .

$$\lambda_1 \geq \lambda_2 \geq ...... \geq \lambda_{n.}$$

Let $P_{n,}C_n, K_n$ be the path graph .the cycle graph and the complete graph on n nodes ,respectively .The path P_n is a tree with two nodes of degree 1and the other two nodes with degree 2.A tree is a acyclic graph(with out cycles) .A cycle C_n is a graph on n nodes containing a single cycle through all nodes .

Theorem2.1 [1]: Let $\lambda_1 \geq \lambda_2 \geq ...... \geq \lambda_{n.}$ be the Eigen values of the adjacency matrix of a simple graph G .Then the following are equivalent .

a) G is a bipartite graph .

b) For all $1 \leq i \leq n, \lambda_{n+1-i} = -\lambda_i$.

Theorem2.2 [1]: Let $\phi(G,x) = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + ...C_n$ be the characteristic polynomial of a graph G ,then the co-efficient of $\phi(G,x)$ satisfy:

a) $C_0 = 1$

b) $C_1 = o$

c) $-C_2$ is the number of edges in G .

d) $-C_3$ is twice the triangles in G.

Theorem2.3[1]: Let G be a graph of order n,size m and k be the number of components of G,then $m \geq n-k$ holds. Further ,the equality holds if and only if each component of G is a tree.

III. MAIN RESULTS.

We defined pure golden graph and Golden graph as follows,

Definition.3.1: A graph G is said to be Pure golden graph, if all the Eigen values of G are Golden ratios (i.e. $\frac{1+\sqrt{5}}{2}$, $\frac{1-\sqrt{5}}{2}$, $\frac{-1-\sqrt{5}}{2}$ and $\frac{-1+\sqrt{5}}{2}$)

Definition3.2: A graph G is said to be golden graph, if at least one of the Eigen values of G are Golden ratios.

Lemma.3.3: A graph G is a pure golden tree if and only if $G = P_4$,a path on four nodes.

Proof: Let G be a pure golden tree of order n,size m and only Eigen values of G are

$$\lambda_1 = \frac{1+\sqrt{5}}{2},\ \lambda_2 = \frac{-1+\sqrt{5}}{2},\ \lambda_3 = \frac{1-\sqrt{5}}{2},\ \lambda_4 = \frac{-1-\sqrt{5}}{2}$$

Cleary $\lambda_1 = -\lambda_4$ & $\lambda_2 = -\lambda_3$.

By the property of a bipartite graph[Theorm:2.1] and hence for a tree, the multiplicities of λ_1, λ_4 be l and of λ_2, λ_3 be k.Thus , the characteristic polynomial of G can be expressed as

$$\phi(G,x) = (x^2-\lambda_1^2)^l (x^2-\lambda_2^2)^k \ldots\ldots\ldots\ldots\ldots\ldots \text{(1)}$$

By expanding the equation (1) ,we have

$$\phi(G,x) = \left[(x^2)^l - {}^lC_1(x^2)^{l-1}\lambda_1^2 + {}^lC_2(x^2)^{l-2}\lambda_1^4 - \ldots\right] \times \left[(x^2)^k - {}^kC_1(x^2)^{k-1}\lambda_2^2 + {}^kC_2(x^2)^{k-2}\lambda_2^4 - \ldots\right]$$

$$= x^{2l+2k} - \left[{}^kC_1\lambda_2^2 + {}^lC_1\lambda_1^2\right]x^{2l+2k-2} + \ldots..$$

By the properties of characteristic polynomial of graph [Theorm:2.2], we have $n = 2l + 2k$, $m = {}^kC_1\lambda_2^2 + {}^lC_1\lambda_1^2$

But $\lambda_1^2 = \left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{3+\sqrt{5}}{2}$ & $\lambda_2^2 = \left(\frac{-1+\sqrt{5}}{2}\right)^2 = \frac{3-\sqrt{5}}{2}$

And thus , $m = l\left(\frac{3+\sqrt{5}}{2}\right) + k\left(\frac{3-\sqrt{5}}{2}\right)$

$$= \frac{3l+3k}{2} + \frac{\sqrt{5}(l-k)}{2}$$

The equation , $2m = (3l+3k) + \sqrt{5}(l-k)$ holds ,only when $l-k=0$ .Since l , k and n are non-negative integers. Thus , $l = k$ must hold .This proves that all roots are of same multiplicities and hence the equation(1) becomes

$$\phi(G,x) = \left[(x^2-\lambda_1^2)(x^2-\lambda_2^2)\right]^l \ldots\ldots\ldots\ldots\ldots\ldots\text{(2)}$$

Thus, we have $n = 4l$ & $m = 3l$ .As G is a tree ,and therefore $3l = m = n-1 = 4l-1 \Rightarrow l = 1$.Thus , G is tree of order 4.

But only trees of order 4 are $K_{1,3}$ & P_4 .Hence, G must be P_4 ,as none of the Eigen values of $K_{1,3}$ are golden ratio.

Converse is obvious.

Theorem.3.4: An acyclic graph G is pure golden if and only if every component of G is P_4.

Proof: Suppose G is a pure golden graph of order n and size m .Now we assert that, each component of G is P_4.

On the similar argument arguments as in the first half of the proof of lemma 3.3, we have the characteristic polynomial of G as

$$\phi(G,x) = \left[(x^2-\lambda_1{}^2)(x^2-\lambda_2{}^2)\right]^l$$

By the properties of the degree and co-efficient of $\phi(G,x)$,we have $n = 4l$ & $m = 3l$.

Thus , $m = n - l$ holds . G contains exactly l components and each of them is a tree[Theorem2.3] .But by the lemma.3.3 ,each component of G is P_4 and hence the assertion holds.

Converse is obvious.

Theorem. 3.5: P_n is golden graph if and only if $n = 5k-1$.

Proof: Let P_n be a golden graph .We know that spectrum of P_n is $2\cos\left(\frac{\pi k}{n+1}\right)$, where $k = 1,2.......n$.

Therefore $2\cos\left(\frac{\pi k}{n+1}\right) = \frac{1+\sqrt{5}}{2}$, *for* some k

$$\Rightarrow \quad \cos\left(\frac{\pi k}{n+1}\right) = \frac{1+\sqrt{5}}{4}$$

We know that , $\cos 36^\circ = \frac{1+\sqrt{5}}{4}$

$\therefore \quad \frac{\pi k}{n+1} = 2l\pi \pm \alpha$,where $\alpha \in I$

$$\Rightarrow \quad \frac{\pi k}{n+1} = 2l\pi \pm 36^\circ$$

$$\Rightarrow \quad \frac{k}{n+1} = 2l \pm \frac{1}{5}$$

$$\Rightarrow \quad 5k = 10l(n+1) \pm (n+1)$$

$$\Rightarrow \quad 5k = (n+1)(10l \pm 1)$$

$\therefore \quad l = 0$, because l being the number of full rotation and π representing only half of the rotation must be zero, the above equation becomes

$$\Rightarrow \quad 5k = \pm(n+1)$$

$$\Rightarrow \quad 5k = (n+1)$$

$$\Rightarrow \quad n = 5k-1.$$

$\therefore$ If P_n is golden graph, then $n = 5k-1$.

For the converse, it is enough to claim that $\phi(P_{5k-1})$ is divisible by $x^2 + x - 1$.

We prove this by induction on k .If $k = 1$, we get $\phi(P_{5k-1}) = \phi(P_4) = (x^2 + x - 1)(x^2 + x - 1)$.

Hence true for $k = 1$.

Next , we assume that the result is true for $k - 1$ .This implies that $x^2 + x - 1 \big/ \phi(P_{5k-6})$.

Now , $\phi(P_{5k-1)} = \phi(P_4) \times \phi(P_{5k-5}) - \phi(P_3) \times \phi(P_{5k-6})$.

Since by induction hypothesis, $x^2 + x - 1 \big/ \phi(P_{5k-6})$.

We see that $x^2 + x - 1 \big/ \phi(P_{5k-6})$.

Thus by induction $\phi(P_{5k-1})$ is divisible by $x^2 + x - 1$.

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