Fekete-Szegö Problems Involving Certain Integral Operator

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Abstract. In this paper, we obtain Fekete-Szegö inequality for certain normalized analytic function $f(z)$ defined on the open unit disc for which $\frac{z(J_{\eta,\lambda}f(z))'}{J_{\eta,\lambda}f(z)}, (0 \leqslant \eta - 1 < \lambda < \eta < 2)$ lies in a region starlike with respect to 1 and is symmetric with respect to the real axis. Also certain applications of the main result for a class of functions defined by Hadamard product are given.

Keywords: Caputo's differentiation operator;; Hadamard product;Analytic functions; Starlike functions; Fekete-Szegö inequality.

1 Introduction

Let A denote the class of all analytic functions $f(z)$ defined on the unit disk $U = \{z : |z| < 1\}$ and A_o be the family of function $f(z) \in A$ normalized by the conditions $f(0) = 0, f'(0) = 1$.

$f \in A$ has the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, (z \in U). \tag{1.1}$$

Let S be the family of functions $f \in A_o$ which are univalent. Let $\phi(z)$ be an analytic function with positive real part on A with $\phi(0) = 1$,$\phi'(0) > 0$ which maps the unit disk U onto a region starlike with respect to 1 which is symmetric with respect to the real axis. Let $S^*(\phi)$ be the class of functions in $f \in S$ for which $\frac{zf'(z)}{f(z)} \prec \phi(z)$,$(z \in U)$ and $C(\phi)$ be the class of functions in $f \in S$ for which $1 + \frac{zf''(z)}{f'(z)} \prec \phi(z)$,$(z \in U)$. Where $\prec$ denotes the subordination between analytic functions. These classes where introduced and studied by Ma and Minda [4]. They obtained the Fekete-Szegö inequality for the functions in the class $C(\phi)$. Since $f \in C(\phi)$ if and only if $zf'(z) \in S^*(\phi)$,we get the Fekete-Szegö inequality for functions in the class $S^*(\phi)$. For a brief history of Fekete-Szegö problem for class of starlike, convex, and close-to convex functions, see [3]. Many authors went through the same investigation by involving certain integral operators,see for example [1, 2].

Salah and Darus have involved Caputo's definition of the fractional-order derivative

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau$$

$n-1 < Re\,(\alpha) \leq n, n \in N$, and the parameter α is allowed to be real or even complex, a is the initial value of the function f .Together with the generalization operator of Salagean derivative operator and Libera integral operator

$$\Omega^{\lambda} f\,(z) = \Gamma\,(2-\lambda)\, z^{\lambda} D_z^{\lambda} f\,(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma\,(2-\lambda)\,\Gamma\,(n+1)}{\Gamma\,(n-\lambda+1)} a_n z^n$$

for any real λ.They have introduced the integral operator

$$J_{\eta,\lambda} f\,(z) = \frac{\Gamma\,(2+\eta-\lambda)}{\Gamma\,(\eta-\lambda)} z^{\lambda-\eta} \int_0^z \frac{\Omega^{\eta} f\,(\xi)}{(z-\xi)^{\lambda+1-\eta}} d\xi$$

η(real number) and $(\eta - 1 < \lambda \leq \eta < 2)$. That has the following taylor series expansion

$$J_{\eta,\lambda} f\,(z) = z + \sum_{n=2}^{\infty} \frac{(\Gamma\,(n+1))^2\,\Gamma\,(2+\eta-\lambda)\,\Gamma\,(2-\eta)}{\Gamma\,(n+\eta-\lambda+1)\,\Gamma\,(n-\eta+1)} a_n z^n.$$

And satisfies $J_{0,0} = f(z)$, $J_{1,1} = zf^{'}(z)$. (For further details you may refer to [5]).

Making use of the previous integral operator we can introduce a generalization of the class $S^*(\phi)$ by the following definition.

Definition 1.1 Let $\phi(z)$ be a univalent function with respect to 1 which maps the unit disk U onto a region in the right half plane which is symmetric with respect to the real axis, $\phi(0) = 1$ and $\phi^{'}(0) > 0$. A function $f \in A$ is in the class $J_{\lambda}^{\eta}(\phi)$ if

$$\frac{z(J_{\eta}, \lambda f(z))^{'}}{J_{\eta}, \lambda f(z)} \prec \phi(z) \tag{1.2}$$

it's obvious that if we put $\eta = \lambda = 0$ the class $J_{\lambda}^{\eta}(\phi)$ is reduced to the class $S^*(\phi)$.

Next we will follow the same technics of many authors (see for example [1,2,3] in order to solve Fekete-Szegö problem with respect to the class $J_{\lambda}^{\eta}(\phi)$.

To prove our main result, we need the following lemma

Lemma 1.1 See [4]: If $p_1(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is analytic function with positive real part in U , then

$|c_2 - \nu c_1^2| \leq \begin{cases} -4\nu + 2, & \text{if } \nu \leq 0 \\ 2\ , & \text{if } 0 \leq \nu \leq 1 \\ 4\nu - 2, & \text{if } \nu \geq 1 \end{cases}$ When $\nu < 0$ or $\nu > 1$, the equality holds if and only if $p_1(z)$ is $(1+z)/(1-z)$ or one of its rotations. If $0 < \nu < 1$, then equality holds if and only if $p_1(z)$ is $(1+z^2)/(1-z^2)$ or one of its rotations. If $\nu = 0$,the equality holds if and only if $p_1(z) = (\frac{1}{2} + \frac{1}{2}\gamma)\frac{1+z}{1-z} + (\frac{1}{2} - \frac{1}{2}\gamma)\frac{1-z}{1+z}$, $(0 \leq \gamma \leq 1)$ or one of its rotations. If $\nu = 1$, the equality holds if and only if $p_1(z)$ is the reciprocal of one of the functions such that the equality holds in the case of $\nu = 0$. Also the above upper bound is sharp,it can be improved as follows when $0 < \nu < 1$: $|c_2 - \nu c_1^2| + \nu |c_1|^2 \leq 2$, $(0 < \nu \leq \frac{1}{2})$ and

$|c_2 - \nu c_1^2| + (1-\nu)|c_1|^2 \leq 2$, $(\frac{1}{2} < \nu \leq 1)$

2 Fekete-Szegö Problem

Our main result is the following:

Theorem 2.1. Let $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$. If $f(z)$ given by (1.1) belongs to $J_\lambda^\eta(\phi)$, then

$$|a_3 - \mu a_2^2| \leq$$

$$\begin{cases} \frac{(2-\eta)(3-\eta)(2+\eta-\lambda)(3+\eta-\lambda)}{72} B_2 - \frac{(2-\eta)^2(2+\eta-\lambda)^2\mu}{16} B_1^2 + \frac{(2-\eta)(3-\eta)(2+\eta-\lambda)(3+\eta-\lambda)}{72} B_1^2, & \text{if } \mu \leq \sigma_1; \\ \frac{(2-\eta)(3-\eta)(2+\eta-\lambda)(3+\eta-\lambda)}{72} B_1, & \text{if} \sigma_1 \leq \mu \leq \sigma_2; \\ -\frac{(2-\eta)(3-\eta)(2+\eta-\lambda)(3+\eta-\lambda)}{72} B_2 + \frac{(2-\eta)^2(2+\eta-\lambda)^2\mu}{16} B_1^2 - \frac{(2-\eta)(3-\eta)(2+\eta-\lambda)(3+\eta-\lambda)}{72} B_1^2, & \text{if } \mu \geq \sigma_2, \end{cases}$$

where

$$\sigma_1 := \frac{2(3-\eta)(3+\eta-\lambda)\left\{(B_2 - B_1) + B_1^2\right\}}{9(2-\eta)(2+\eta-\lambda)B_1^2}$$

$$\sigma_2 := \frac{2(3-\eta)(3+\eta-\lambda)\left\{(B_2 + B_1) + B_1^2\right\}}{9(2-\eta)(2+\eta-\lambda)B_1^2}$$

the result is sharp.

Proof. For $f(z) \in J_\lambda^\eta(\phi)$, let

$$p(z) = \frac{z(J_{\eta,\lambda} f(z))'}{J_{\eta,\lambda} f(z)} = 1 + b_1 z + b_2 z^2 + \cdots. \tag{2.1}$$

From (2.1), we obtain

$$a_2 = \frac{(2-\eta)(2+\eta-\lambda)}{4} b_1$$

and

$$a_3 = \frac{(2-\eta)(3-\eta)(2+\eta-\lambda)(3+\eta-\lambda)}{72}(b_2 + b_1^2)$$

since $\phi(z)$ is univalent and $p \prec \phi$, the function

$$p_1(z) = \frac{1 + \phi^{-1}(p(z))}{1 - \phi^{-1}(p(z))} = 1 + c_1 z + c_2 z^2 + \cdots$$

is analytic and has a positive real part in U. Also we have

$$p(z) = \phi\left(\frac{p_1(z) - 1}{p_1(z) + 1}\right). \tag{2.2}$$

And from (2.2), we obtain

$$b_1 = \frac{1}{2} B_1 c_1$$

and

$$b_2 = \frac{1}{2}B_1(c_2 - \frac{1}{2}c_1^2) + \frac{1}{4}B_2c_1^2$$

.

Therefore we have

$$a_3 - \mu a_2^2 = \frac{(2-\eta)(3-\eta)(2+\eta-\lambda)(3+\eta-\lambda)}{144}B_1\{c_2 - \upsilon c_1^2\} \tag{2.3}$$

where

$$\upsilon = \frac{1}{2}\left(1 - \frac{B_2}{B_1} + \frac{9(2-\eta)(2+\eta-\lambda)\mu - 4(3-\eta)(3+\eta-\lambda)}{4(3-\eta)(3+\eta-\lambda)}B_1\right).$$

Our result now follows by an application of Lemma 1.1. To show that the bounds are sharp, we define the functions $K_n^\phi (n = 2, 3, ...)$ by

$$\frac{z(J_{\eta,\lambda}K_n^\phi(z))'}{J_{\eta,\lambda}K_n^\phi(z)} = \phi(z^{n-1})$$

,

$$K_n^\phi(0) = 0 = (K_n^\phi(0))' - 1$$

and the function F_γ and G_γ, $(0 \leq \gamma \leq 1)$ by

$$\frac{z(J_{\eta,\lambda}F_\gamma(z))'}{J_{\eta,\lambda}F_\gamma(z)} = \phi\left(\frac{z(z+\gamma)}{1+\gamma z}\right)$$

,

$$F_\gamma(0) = 0 = (F_\gamma(0))' - 1$$

and

$$\frac{z(J_{\eta,\lambda}G_\gamma(z))'}{J_{\eta,\lambda}G_\gamma(z)} = \phi\left(-\frac{z(z+\gamma)}{1+\gamma z}\right)$$

,

$$G_\gamma(0) = 0 = (G_\gamma(0))' - 1.$$

Clearly the functions $K_n^\phi, F_\gamma, G_\gamma \in J_\lambda^\eta(\phi)$. Also we write $K^\phi := K_2^\phi$. If $\mu < \sigma_1$ or $\mu > \sigma_2$, then the equality holds if and only if f is K^ϕ or one of its rotations. When $\sigma_1 < \mu < \sigma_2$, the equality holds if and only if f is K_3^ϕ or one of its rotations. If $\mu = \sigma_1$ then the equality holds if and only if f is F_γ or one of its rotations. If $\mu = \sigma_2$ then the equality holds if and only if f is G_γ or one of its rotations.

Remark 2.2. If $\sigma_1 \leq \mu \leq \sigma_3$, then, in view of Lemma 1.1, Theorem 2.1.can be improved. Let σ_3 be given by

$$\sigma_3 := \frac{2(3-\eta)(3+\eta-\lambda)(B_1^2 + B_2)}{9(2-\eta)(2+\eta-\lambda)B_1^2}.$$

If $\sigma_1 \leq \mu \leq \sigma_3$, then

$$|a_3-\mu a_2^2|+\frac{2(3-\eta)(3+\eta-\lambda)}{9(2-\eta)(2+\eta-\lambda)}\left[B_1-B_2+\frac{9(2-\eta)(2+\eta-\lambda)\mu-2(3-\eta)(3+\eta-\lambda)}{2(3-\eta)(3+\eta-\lambda)}B_1^2\right]|a_2^2|$$

$$\leq \frac{(2-\eta)(3-\eta)(2+\eta-\lambda)(3+\eta-\lambda)}{72}B_1.$$

If $\sigma_3 \leq \mu \leq \sigma_2$, then

$$|a_3-\mu a_2^2|+\frac{2(3-\eta)(3+\eta-\lambda)}{9(2-\eta)(2+\eta-\lambda)}\left[B_1+B_2-\frac{9(2-\eta)(2+\eta-\lambda)\mu-2(3-\eta)(3+\eta-\lambda)}{2(3-\eta)(3+\eta-\lambda)}B_1^2\right]|a_2^2|$$

$$\leq \frac{(2-\eta)(3-\eta)(2+\eta-\lambda)(3+\eta-\lambda)}{72}B_1$$

3 Applications to functions defined by fractional derivatives.

For two analytic functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$,their convolution is defined to be the function $(f * g)(z)$ given by $(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n$. For fixed $g \in A_o$, let $J_\lambda^{\eta,g}(\phi)$ be the class of functions $f \in A_o$ for which $(f * g) \in J_\lambda^{\eta}(\phi)$.

Definition 3.1.Let $f(z)$ be analytic function in a simply connected region of the $z-plane$ containing the origin. The fractional derivative of order γ is defined by

$$D_z^\gamma f(z) = \frac{1}{\Gamma(1-\gamma)}\frac{d}{dz}\int_0^z \frac{f(\xi)}{(z-\xi)^\gamma}d\xi, (0 \leq \gamma < 1).$$

Where the multiplicity of $(z-\xi)^\gamma$ is removed by requiring that $\log(z-\xi)$ is real for $(z-\xi) > 0$. Using the above definition and its known extensions involving fractional derivative and fractional integrals, Owa and Srivastava [6] introduced the operator $\Omega^\gamma : A_o \to A_o$ defined by

$$\Omega^\gamma f(z) = \Gamma(2-\gamma)z^\gamma D_z^\gamma f(z), (\gamma \neq 2, 3, 4, ...)$$

The class $J_\delta^{\eta,\gamma}(\phi)$ consists of functions $f \in A_o$ for which $\Omega^\gamma f \in J_\delta^{\eta}(\phi)$. Note that $J_\delta^{\eta,\gamma}(\phi)$ is the special case of the class $J_\lambda^{\eta,g}(\phi)$when

$$g(z) = z + \sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2-\gamma)}{\Gamma(n+1-\gamma)}z^n$$

.

Let

$$g(z) = z + \sum_{n=2}^{\infty} g_n z^n, (g_n > 0).$$

Since $J_{\eta,\lambda} f(z) \in J_{\lambda}^{\eta,g}(\phi)$ if and only if $J_{\eta,\lambda} f(z) * g(z) \in J_{\lambda}^{\eta}(\phi)$, we obtain the coefficient estimate for functions in the class $J_{\lambda}^{\eta,g}(\phi)$, from the corresponding estimate for functions in the class $J_{\lambda}^{\eta}(\phi)$.

Applying Theorem 2.1. for the function $J_{\eta,\lambda} f(z) * g(z)$ we get the following Theorem 3.2 after obvious change of the parameter μ

Theorem 3.2. Let $g(z) = z + \sum_{n=2}^{\infty} g_n z^n, (g_n > 0)$, and let the function $\phi(z)$ be given by $\phi(z) = 1 + \sum_{k=1}^{\infty} B_k z^k$. If $J_{\eta,\lambda} f(z)$ given by (1.3) belongs to $J_{\lambda}^{\eta,g}(\phi)$, then

$$|a_3 - \mu a_2^2| \leq$$

$$\begin{cases} \frac{1}{g_3}\left[\frac{(2-\eta)(3-\eta)(2+\eta-\lambda)(3+\eta-\lambda)}{72} B_2 - \frac{(2-\eta)^2(2+\eta-\lambda)^2 \mu g_3}{16 g_2^2} B_1^2 + \frac{(2-\eta)(3-\eta)(2+\eta-\lambda)(3+\eta-\lambda)}{72} B_1^2\right], & \text{if } \mu \leq \sigma_1; \\ \frac{1}{g_3}\left[\frac{(2-\eta)(3-\eta)(2+\eta-\lambda)(3+\eta-\lambda)}{72} B_1\right], & \text{if } \sigma_1 \leq \mu \leq \sigma_2; \\ \frac{1}{g_3}\left[-\frac{(2-\eta)(3-\eta)(2+\eta-\lambda)(3+\eta-\lambda)}{72} B_2 + \frac{(2-\eta)^2(2+\eta-\lambda)^2 \mu g_3}{16 g_2^2} B_1^2 - \frac{(2-\eta)(3-\eta)(2+\eta-\lambda)(3+\eta-\lambda)}{72} B_1^2\right], & \text{if } \mu \geq \sigma_2, \end{cases}$$

Where

$$\sigma_1 := \frac{2g_2^2(3-\eta)(3+\eta-\lambda)\left\{(B_2 - B_1) + B_1^2\right\}}{9g_3(2-\eta)(2+\eta-\lambda)B_1^2}$$

$$\sigma_1 := \frac{2g_2^2(3-\eta)(3+\eta-\lambda)\left\{(B_2 + B_1) + B_1^2\right\}}{9g_3(2-\eta)(2+\eta-\lambda)B_1^2}.$$

The result is sharp.

Since

$$(\Omega^{\gamma} J_{\eta,\lambda} f)(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\gamma)}{\Gamma(n+1-\gamma)} \theta(n) a_n z^n$$

,and

$$\left(\theta(n) = \frac{\left(\Gamma(n+1)\right)^2 \Gamma(2+\eta-\lambda)\Gamma(2-\eta)}{\Gamma(n+\eta-\lambda+1)\Gamma(n-\eta+1)}\right).$$

We have

$$g_2 := \frac{\Gamma(3)\Gamma(2-\gamma)}{\Gamma(3-\gamma)} = \frac{2}{2-\gamma} \tag{3.1}$$

$$g_3 := \frac{\Gamma(4)\Gamma(3-\gamma)}{\Gamma(4-\gamma)} = \frac{6}{(2-\gamma)(3-\gamma)} \tag{3.2}$$

Using (3.1) and (3.2), Theorem 3.2. reduces to the following

Theorem 3.3. Let $g(z) = z + \sum_{n=2}^{\infty} g_n z^n, (g_n > 0)$, and let the function $\phi(z)$ be given by $\phi(z) = 1 + \sum_{k=1}^{\infty} B_k z^k$. If $J_{\eta,\lambda} f(z)$ given by (1.3) belongs to $J_{\lambda}^{\eta,g}(\phi)$, then

$$|a_3 - \mu a_2^2| \leq$$

$$\begin{cases} \frac{(2-\gamma)(3-\gamma)}{6}\left[\frac{(2-\eta)(3-\eta)(2+\eta-\lambda)(3+\eta-\lambda)}{72}B_2 - \frac{3(2-\gamma)(2-\eta)^2(2+\eta-\lambda)^2\mu}{32(3-\gamma)}B_1^2 + \frac{(2-\eta)(3-\eta)(2+\eta-\lambda)(3+\eta-\lambda)}{72}B_1^2\right], \text{if } \mu \leq \sigma_1; \\ \frac{(2-\gamma)(3-\gamma)}{6}\left[\frac{(2-\eta)(3-\eta)(2+\eta-\lambda)(3+\eta-\lambda)}{72}B_1\right], \text{ if } \sigma_1 \leq \mu \leq \sigma_2; \\ \frac{(2-\gamma)(3-\gamma)}{6}\left[-\frac{(2-\eta)(3-\eta)(2+\eta-\lambda)(3+\eta-\lambda)}{72}B_2 + \frac{3(2-\gamma)(2-\eta)^2(2+\eta-\lambda)^2\mu}{32(3-\gamma)}B_1^2 - \frac{(2-\eta)(3-\eta)(2+\eta-\lambda)(3+\eta-\lambda)}{72}B_1^2\right], \text{if } \mu \geq \sigma_2, \end{cases}$$

Where

$$\sigma_1 := \frac{4(3-\gamma)(3-\eta)(3+\eta-\lambda)\left\{(B_2 - B_1) + B_1^2\right\}}{27(2-\gamma)(2-\eta)(2+\eta-\lambda)B_1^2}$$

$$\sigma_2 := \frac{4(3-\gamma)(3-\eta)(3+\eta-\lambda)\left\{(B_2 + B_1) + B_1^2\right\}}{27(2-\gamma)(2-\eta)(2+\eta-\lambda)B_1^2}.$$

The result is sharp.

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