Boolean Algebras of Pre A*-Algebra

Vijayabarathi.S[#1], Srinivasa Rao.K[*2]

[#]*Assistant Professor, Department of Mathematics, SCSVMV University, Kanchipuram, India*

[*]*Associate Professor, Department of Mathematics, SCSVMV University, Kanchipuram,India*

***Abstract*— In this paper, we obtain Boolean algebras $\Im_{P(A)}$, $\Im_{M(A)}$ from Pre A*-algebra, and proved that $\Im_{P(A)}$, $\Im_{M(A)}$ and $B(A)$ are isomorphic to each other.**

Keywords- Pre A*-Algebra, Centre of Pre A*-Algebra, Boolean Algebra, Isomorphism, Partial Ordering

1. INTRODUCTION

In a draft paper [2], E.G.Maines introduced the concept of Ada (Algebra of disjoint alternatives) $(A,\wedge,\vee,(-)',(-)_{\pi},0,1,2)$ which is however differ from the definition of the Ada of his later paper[9]. While the Ada of the earlier draft seems to be based on extending the If-Then-Else concept more on the basis of Boolean algebras and the later concept is based on C-algebras $(A,\wedge,\vee,')$ introduced by Fernando Guzman and Craig C. Squir [1]. In [3], introduced the concept of A*-algebra $(A,\wedge,\vee,*,(-)^{\sim}(-)_{\pi},0,1,2)$ and studied the equivalence with Ada, C-algebra, Ada's connection with 3-Ring, Stone type representation also introduced the concept of A*-clone, the If-Then-Else structure over A*-algebra and Ideals of A*-algebra. In [4], introduced the concept Pre A*-algebra $(A,\wedge,\vee,(-)^{\sim})$ analogous to C-algebra. In [5], defined partial ordering on Pre-A* algebra by $x\le y$ if and only if $x\wedge y=y\wedge x=x$ and studied the properties of this partial ordering. Given necessary and sufficient conditions for Pre A*-algebra to become a lattice. In[6], defined congruence relation on Pre A*-algebra by $\theta_x=\{(p,q)\in A\times A \mid x\wedge p=x\wedge q\}$ and studied the subdirectly irreducible representation of Pre A*-algebra. In [7], defined ternary operation of Pre A*-algebra , and established Cayley's theorem on Centre of Pre A*-algebra. In [8], proved that if A is a Pre A*-algebra, $x\in A$, then $M_x=\{s\in A/s\le x\}$ is a Pre A*-algebra under the induced operations$\wedge,\vee$ where the complementation is defined by $s^*=x\wedge s^{\sim}$ the relation defined on Pre A* algebra A by $s\le x$ if $s\wedge x=x\wedge s=s$ and the mapping $\alpha_x:A\rightarrow M_x$ defined by $\alpha_x(s)=x\wedge s$ for all $s\in A$ is a homomorphism of A onto M_x with kernel θ_x and hence $A/\theta_x\cong M_x$ also studied the decomposition of Pre A*-algebra. In this paper we prove that three Boolean algebras $B(A),\Im_{P(A)}$ and $\Im_{M(A)}$ are isomorphic to each other.

2. PRELIMANARIES

In this section we recall the definition of Pre A*-algebra and some results from [5,6] which will be required later.

2.1. Definition:

An algebra $(A,\wedge,\vee,(\tilde{-}))$ where A is non-empty set with $1,\wedge,\vee$ are binary operations and $(-)^{\sim}$ is a unary operation satisfying

(a) $x^{\sim\sim}=x,\quad \forall x\in A$

(b) $x\wedge x=x,\quad \forall x\in A$

(c) $x\wedge y=y\wedge x,\quad \forall x,y\in A$

(d) $(x\wedge y)^{\sim}=x^{\sim}\vee y^{\sim},\quad \forall x,y\in A$

(e) $x\wedge(y\wedge z)=(x\wedge y)\wedge z,\quad \forall x,y,z\in A$

(f) $x\wedge(y\vee z)=(x\wedge y)\vee(x\wedge z),\quad \forall x,y,z\in A$

(g) $x\wedge y=x\wedge(x^{\sim}\vee y),\quad \forall x,y\in A$. is called a Pre A*-algebra

2.2. Example:

3 = {0, 1, 2} with operations $\wedge, \vee, (-)^{\sim}$ defined below is a Pre A*-algebra.

$\wedge$	0	1	2
0	0	0	2
1	0	1	2
2	2	2	2

$\vee$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

x	$x^{\sim}$
0	1
1	0
2	2

It can be observed that 2.1(a) and 2.1(b) imply that the varieties of Pre A*-algebras satisfies all dual statements of 2.1(b) to 2.1(g).

Lemma2.3: Every Pre A*-algebra satisfies the following laws[5,6]

(a) $x \vee 1 = x \vee x^{\sim}$ and $x \wedge 0 = x \wedge x^{\sim}$

(c) $x \wedge (x^{\sim} \vee x) = x \vee (x^{\sim} \wedge x) = x$

(d) $(x \vee x^{\sim}) \wedge y = (x \wedge y) \vee (x^{\sim} \wedge y)$

(e) $(x \vee y) \wedge z = (x \wedge z) \vee (x^{\sim} \wedge y \wedge z)$

(f) $x \wedge y = 0$, $x \vee y = 1$, then $y = x^{\sim}$

(g) If $x \vee y = 0$, then $x = y = 0$ and If $x \vee y = 1$, then $x \vee x^{\sim} = 1$

Definition 2.4. [5]: Let A be a Pre A*-algebra. An element $x \in A$ is called central element of A if $x \vee x^{\sim} = 1$ and the set $\{x \in A / x \vee x^{\sim} = 1\}$ of all central elements of A is called the centre of A and it is denoted by $B(A)$.

Theorem 2.5. [5]: Let A be a Pre A*-algebra with 1, then $B(A)$ is a Boolean algebra with the induced operations $\wedge, \vee, (-)^{\sim}$

Lemma2.6. [5]: Let A be a Pre A*-algebra with 1 ,

(a) If $y \in B(A)$ then $x \wedge x^{\sim} \wedge y = x \wedge x^{\sim}, \forall x \in A$ (b) If $x, y \in B(A)$ then $x \wedge (x \vee y) = x \vee (x \wedge y) = x$

Lemma 2.7. [6] : Let $(A, \wedge, \vee, (-)^{\sim})$ be a Pre A*-algebra and let $a \in A$. Then the relation $\theta_a = \{(x, y) \in A \times A / \ a \wedge x = a \wedge y\}$ is (a) a congruence relation (b) $\theta_a \cap \theta_{a^{\sim}} = \theta_{a \vee a^{\sim}}$
(c) $\theta_a \cap \theta_b \subseteq \theta_{a \vee b}$ (d) $\theta_a \cap \theta_{a^{\sim}} \subseteq \theta_{a \wedge a^{\sim}}$

Lemma 2.8 [6]: Let A be a Pre A*-algebra and $a, b \in B(A)$, then $\theta_a \cap \theta_b = \theta_{a \vee b}$

Lemma 2.9 [6]: Let A be a Pre A*-algebra, let Δ_A denote the trivial congruence on A: $\Delta_A = \{(x, x) / x \in A\}$ then (a) $\theta_a = \Delta_A$ if and only if $a \wedge x = x$, $\forall x \in A$ (b) $\theta_a = A \times A$ if and only if $a \wedge x = a$, $\forall x \in A$ (c) $(a, b) \in \theta_a, \theta_b$ then $a = b$ (d) If $a \in B(A)$ then $\theta_a \cap \theta_{a^{\sim}} = \Delta_A$

Theorem 2.10. [6] :Let A be a Pre A*-algebra, then $A/\theta = \{\theta_a / a \in A\}$ is a Pre A*-algebra, is called quotient Pre A*-Algebra, whose operations are defined as $\theta_a \wedge \theta_b = \theta_{a \wedge b}$, $\theta_a \vee \theta_b = \theta_{a \vee b}$ an $(\theta_a)^{\sim} = \theta_{a^{\sim}}$

Definition 2.11.[6]:An algebra A is called subdirectly irreducible provided there is a congruence ϕ on A such that $\phi \neq \Delta_A$, and if $\theta \neq \Delta_A$ is a congruence on A, then $\phi \subseteq \theta$

Theorem 2.12. [6] : **2** and **3** are the only sub-directly irreducible Pre A*-algebras.

Corollary 2.13. [6]: Every Pre A*-algebra is a sub algebra of a product of copies of **3**.

Lemma 2.14. [6]: Let A be Pre A*-algebra with 1 and $a,b \in B(A)$ then(a) θ_a o θ_b $=\theta_{a\wedge b}$ (b) θ_a o θ_b = θ_b o θ_a (c) $\theta_a \subseteq \theta_b$ if and only if $b= a\wedge b$ (d) θ_a o $\theta_{a^{\sim}}$ =A$\times$A

Definition 2.15.[6]:A congruence θ on an algebra A is called a factor congruence on A if there is a congruence ϕ on A such that $\theta \cap \phi = \Delta_A$ and θ o $\phi = A\times A$.

Theorem 2.16.[6]: Let A be a Pre A*-algebra with 1 and θ be congruence on A. If $\theta = \theta_x$ for some $x \in B(A)$, then θ_x is a factor congruence on A.

Now we prove some important properties of Pre A*-algebra

Theorem 2.17: Let A be a Pre A*-Algebra and $x, y \in A$, then $x \vee x^{\sim} \vee y = x \vee y \vee y^{\sim}$

Proof: $x \vee x^{\sim} \vee y = x \vee (y \vee x^{\sim})$

$$
\begin{aligned}
&= x \vee \{y \vee (y^{\sim} \wedge x^{\sim})\} \\
&= (x \vee y) \vee \{x \vee (y^{\sim} \wedge x^{\sim})\} \\
&= (x \vee y) \vee \{x \vee (x^{\sim} \wedge y^{\sim})\} \\
&= (x \vee y) \vee (x \vee y^{\sim}) \\
&= x \vee y \vee y^{\sim}
\end{aligned}
$$

Theorem.2.18:Let A be a Pre A*-Algebra and $x, y \in A$, $\{x \vee (x \wedge y)\} \vee \{x^{\sim} \vee (x^{\sim} \wedge y)\} = x \vee x^{\sim} \vee y$

Proof: $\{x \vee (x \wedge y)\} \vee \{x^{\sim} \vee (x^{\sim} \wedge y)\} = x \vee (x \wedge y) \vee x^{\sim} \vee (x^{\sim} \wedge y)$

$$
\begin{aligned}
&= \{x^{\sim} \vee (x \wedge y)\} \vee \{x \vee (x^{\sim} \wedge y)\} \\
&= x^{\sim} \vee y \vee x \vee y \\
&= x^{\sim} \vee x \vee y \\
&= x \vee x^{\sim} \vee y
\end{aligned}
$$

Theorem.2.19: Let A be a Pre A*-algebra with 1 and $x, y \in A$ such that $x \vee y \in B(A)$ then $x \in B(A)$

Proof: Let a Pre A*-algebra with 1 and $x, y \in A$ such that $x \vee y \in B(A)$ then

$$
\begin{aligned}
1 &= (x \vee y) \vee (x \vee y)^{\sim} \\
&= (x \vee y) \vee (x^{\sim} \wedge y^{\sim}) \\
&= (x \vee y \vee x^{\sim}) \wedge (x \vee y \vee y^{\sim}) \\
&= (x \vee x^{\sim} \vee y) \wedge (x \vee y \vee y^{\sim}) \\
&= (x \vee y \vee y^{\sim}) \wedge (x \vee y \vee y^{\sim}) \quad \text{(Theorem 2.17)} \\
&= x \vee y \vee y^{\sim}
\end{aligned}
$$

Therefore 1= $x \vee y \vee y^{\sim}$ -----(a)

Now $x \vee x^{\sim} = (x \vee x^{\sim}) \wedge 1$

$$
\begin{aligned}
&= (x \vee x^{\sim}) \wedge (x \vee y \vee y^{\sim}) \\
&= [x \wedge (x \vee y \vee y^{\sim})] \vee [x^{\sim} \wedge (x \vee y \vee y^{\sim})]
\end{aligned}
$$

$=[x\wedge(x\vee x^{\sim}\vee y)]\vee[x^{\sim}\wedge(x\vee x^{\sim}\vee y)]$ Theorem 2.17

$=[x\wedge\{x^{\sim}\vee(x\vee y)\}]\vee[x^{\sim}\wedge\{x\vee(x^{\sim}\vee y)\}]$

$=[x\wedge(x\vee y)]\vee[x^{\sim}\wedge(x^{\sim}\vee y)]$ Property 2.1(g)

$=[x\vee(x\wedge y)]\vee[x^{\sim}\vee(x^{\sim}\wedge y)]$

$=(x\vee y)\vee(x^{\sim}\vee y)$

$=x\vee x^{\sim}\vee y$

$=x\vee y\vee y^{\sim}$ Theorem 2.17

$=1$

This shows that $x\in B(A)$

The converse of the above theorem need not be true. For example, in the Pre-A* Algebra A, we know $0\in B(A)$ but $0\vee 2=2\notin B(A)$. We have the following consequence of the above theorem.

Lemma.2.20: Let A be a Pre-A* Algebra with 1, $a,b\in A$ and $a\wedge b\in B(A)$, then $a\in B(A)$

Proof: Let $a\wedge b\in B(A)$, then we have $(a\wedge b)^{\sim}\in B(A)\Rightarrow a^{\sim}\vee b^{\sim}\in B(A)$

$\Rightarrow a^{\sim}\in B(A)$

$\Rightarrow a\in B(A)$

3. The Pre A*-Algebra

We prove that, for each $x\in A$, $P_x=\{x\vee t/t\in A\}$ is itself a Pre-A* algebra under induced operations $\wedge,\vee$ and unary operation is defined by $(x\vee t)^{*}=x\wedge t^{\sim}$. We observe that P_x need not be a sub-algebra of A because the unary operation in P_x is not the restriction of the unary operation on A. Also for each $x\in A$, the set $M_x=\{s\in A/s\leq x\}$ is a Pre A*-algebra under the induced operations$\wedge,\vee$ where the complementation is defined by $s^{*}=x\wedge s^{\sim}$ .We prove that $\Im_{P(A)}=\{P_a\mid a\in B(A)\}$ and $\Im_{M(A)}=\{M_a\mid a\in B(A)\}$ is a Boolean algebras under set, also we establish that $B(A),\Im_{P(A)}$ and $\Im_{M(A)}$ are isomorphic to each other

Theorem 3.1: Let $<A,\wedge,\vee,\ ^{\sim}>$ be a Pre -A* algebra, $x\in A$ and $P_x=\{x\vee t/t\in A\}$, then $<P_x,\wedge,\vee,\ *>$ is a Pre-A* algebra with x as the identity for $\vee$, where $\wedge$and $\vee$ are the operations in A restricted to P_x and for any $x\vee t\in P_x$, here $(x\vee t)^{*}=x\vee t^{\sim}$

Proof:

Let $a,b,c\in A$ .Then $(x\vee a)\vee(x\vee b)=x\vee(a\vee b)\in P_x$ and $(x\vee a)\wedge(x\vee b)=x\vee(a\wedge b)\in P_x$. Thus $\vee,\wedge$ are closed in P_x Consider $(x\vee a)^{**}=\{(x\vee a)^{*}\}^{*}=(x\vee a^{\sim})^{*}=(x\vee a^{\sim\sim})=(x\vee a)$. Therefore $(x\vee a)^{**}=(x\vee a)$.

Now $[(x\vee a)\wedge(x\vee b)]^{*}=[x\vee(a\wedge b)]^{*}=x\vee(a\wedge b)^{\sim}=x\vee(a^{\sim}\vee b^{\sim})=(x\vee a)^{*}\wedge(x\vee b)^{*}$

$(x\vee a)\wedge\{(x\vee a)^{*}\vee(x\vee b)\}=(x\vee a)\wedge\{(x\vee a^{\sim})\vee(x\vee b)\}$

$=(x\vee a)\wedge\{(x\vee(a^{\sim}\vee b)\}$

$=x\vee(a\wedge(a^{\sim}\vee b)$

$=x\vee(a\wedge b)$.

The remaining identities of Pre A*-algebra also hold in P_x because they hold in A. Hence P_x is itself a Pre A*-algebra. Here x is the identity for $\vee$ because $x\vee x\vee a=x\vee a\vee x=x\vee a$ and

$x\vee x^{\sim}$ is the identity for $\wedge$ because $(x\vee x^{\sim})\wedge(x\vee a)=x\vee(x^{\sim}\wedge a)=x\vee a$

Theorem 3.2: Let A be a Pre A*-algebra. Then the following holds.

(i) $P_x = P_y$ if and only if $x = y$

(ii) $P_x \cap P_y = P_{x\vee y}$

(iii) $(P_x)_{x\vee y} = P_{x\vee y}$

Proof: Suppose $P_x = P_y$. Since $x = x \vee x \in P_x = P_y$ and $y = y \vee y \in P_y = P_x$. Therefore $x = y \vee a$ and $y = x \vee b$ for some $a, b \in A$. Now, $x = y \vee a = (y \vee a \vee y) \vee (y \vee y \vee a)$ $= (x \vee y) \wedge (y \vee x)$ $= (x \vee x \vee b) \wedge (x \vee b \vee x)$ $= x \vee b$ $= y$. The converse is trivial

(ii) Suppose $a \in P_x \cap P_y$. Then $a = x \vee b = y \vee c$ for some $b, c \in A$.

Now $a = x \vee x \vee b = x \vee a = x \vee y \vee c \in P_{x\vee y}$. Therefore $P_x \cap P_y \subseteq P_{x\vee y}$

Let $a \in P_{x\vee y}$, then $a = x \vee y \vee b$ for some $b \in A$

Now $a = x \vee y \vee b = x \vee t \in P_x$, where $t = y \vee b$

Again $a = x \vee y \vee b = a = y \vee x \vee b = y \vee s \in P_y$, where $s = x \vee b$. So $a \in P_x \cap P_y$.

Therefore $P_{x\vee y} \subseteq P_x \cap P_y$

Hence $P_x \cap P_y = P_{x\vee y}$

(iii) $(P_x)_{x\vee y} = \{x \vee y \vee a \,/\, a \in P_x\} = \{x \vee y \vee x \vee b \mid b \in A\} = \{x \vee y \vee b \mid b \in A\} = P_{x\vee y}$

Theorem 3.3: Let A be a Pre A*-algebra with 1 and $x \in A$, then the mapping $\alpha_x : A \to P_x$ defined by $\alpha_x(t) = x \vee a$ for all $a \in A$ is a homomorphism of A to P_x with kernel $\theta_{x^\sim}$ and hence $A|\theta_{x^\sim} \cong P_x$

Proof : Let $a, b \in A$, then $\alpha_x(a \vee b) = x \vee a \vee b$ = $x \vee a \vee x \vee b$ = $\alpha_x(a) \vee \alpha_x(b)$ and $\alpha_x(a^\sim) = x \vee a^\sim$ = $(x \vee a)^* = (\alpha_x(a))^*$. Clearly $\alpha_x(a \wedge b) = x \vee (a \wedge b)$ $= (x \vee a) \wedge (x \vee b)$ = $\alpha_x(a) \wedge \alpha_x(b)$. Also $\alpha_x(1) = x \vee 1 = x \vee x^\sim$, which is the identity for $\wedge$ in P_x. Therefore α_x is a homomorphism. Hence by the fundamental theorem of homomorphism $A|\,Ker\alpha_x \cong P_x$ and

$$Ker\alpha_x = \{(a,b) \in A \times A \mid \alpha_x(a) = \alpha_x(b)\}$$

$$= \{(a,b) \in A \times A \mid x \vee a = y \vee b\}$$

$$= \{(a,b) \in A \times A \mid x^\sim \wedge a = y^\sim \wedge b\} = \theta_{x^\sim} \text{ and hence } A|\theta_{x^\sim} \cong P_x.$$

Theorem 3.4: Let A be a Pre A*-algebra with 1 and $a \in B(A)$, then $A \cong P_a \times P_{a^\sim}$.

Proof: Define $\alpha : A \to P_a \times P_{a^\sim}$ by $\alpha(x) = (\alpha_a(x), \alpha_{a^\sim}(x))$ for all $x \in A$. Then by Theorem 3.3, α is well-defined and α is a homogenous. Now we prove that α is one-one.

Let $x, y \in A$ and $\alpha(x) = \alpha(y)$

$$\Rightarrow (\alpha_a(x), \alpha_{a^\sim}(x)) = (\alpha_a(y), \alpha_{a^\sim}(y))$$

$$\Rightarrow (a \vee x, a^\sim \vee x) = (a \vee y, a^\sim \vee y) \Rightarrow a \vee x = a \vee y \text{ and } a^\sim \vee x = a^\sim \vee y$$

Now $x = 1 \vee x = (a \wedge a^\sim) \vee x = (a \vee x) \wedge (a^\sim \vee x) = (a \vee y) \wedge (a^\sim \vee y) = y$.

Finally, we prove that α is onto.

Let $(x, y) \in P_a \times P_{a^\sim}$, then $x = a \vee t$ and $y = a^\sim \vee r$ for some $t, r \in A$. Therefore, $a \vee x = a \vee a \vee t = a \vee t = x$, $a \vee y = a \vee a^\sim \vee r = 1 \vee r = 1$ and $a^\sim \vee x = a^\sim \vee a \vee t = 1 \vee t = 1$, $a^\sim \vee y = a^\sim \vee a^\sim \vee r = a^\sim \vee r = y$.

Now $\alpha(x \wedge y) = (\alpha_a(x \wedge y), \alpha_{a^\sim}(x \wedge y))$

$= (a \vee (x \wedge y), a^{\sim} \vee (x \wedge y))$

$= ((a \vee x) \wedge (a \vee y), (a^{\sim} \vee x) \wedge (a^{\sim} \vee y)) \quad = (x \wedge 1, 1 \wedge y) \ = (x, y)$

Therefore α is onto and is an isomorphism. Therefore $A \cong P_a \times P_{a^{\sim}}$

Theorem 3.5: Let $< A, \wedge, \vee, ^{\sim} >$ be a Pre A*-algebra with 1, then the set $\Im_{P(A)} = \{P_a \mid a \in B(A)\}$ is a Boolean algebra under set inclusion.

Proof: Clearly $(\Im_{P(A)}, \subseteq)$ is a partially ordered set under inclusion. First we show that $P_{a\vee b}$ is the infimum of $\{P_a, P_b\}$ and $P_{a\wedge b}$ is the supremum of $\{P_a, P_b\}$.

From the theorem 3.2 of (ii), shows that $P_{a\vee b}$ is the infimum of $\{P_a, P_b\}$.

Let $t \in P_a$, then $t = a \vee x$ for some $x \in A$.

Now $t = a \vee x = (a \wedge (a \vee b)) \vee x = (a \wedge b) \vee a \vee x \in P_{a\wedge b}$. Therefore $P_a \subseteq P_{a\wedge b}$. Similarly we can prove that $P_b \subseteq P_{a\wedge b}$. Therefore $P_{a\wedge b}$ is an upper bound of P_a, P_b. Suppose P_c is the upper bound of P_a, P_b and $t \in P_{a\wedge b}$, then $t = (a \wedge b) \vee x$ for some $x \in A$.

Now $t = (a \wedge b) \vee x = \{a \wedge (a^{\sim} \vee b)\} \vee x = (a \vee x) \wedge (a^{\sim} \vee b \vee x) = (a \vee x) \wedge (b \vee a^{\sim} \vee x) \in S_c$

(since $a \vee x \in S_a \subseteq S_c$, $b \vee a^{\sim} \vee x \in S_b \subseteq S_c$ and S_c is closed under $\wedge$). Therefore $P_{a\wedge b}$ is the supremum of $\{P_a, P_b\}$. Denote supremum of $\{P_a, P_b\}$ by $P_a \vee P_b$ and infimum of $\{P_a, P_b\}$ by $P_a \wedge P_b$. Now $P_1 \wedge P_a = P_{1\vee a} = P_a$ and $P_0 \vee P_a = P_{0\wedge a} = P_0$. So P_1 is the least element and P_0 is the greatest element of $(\Im_{P(A)}, \subseteq)$. Now for any $a, b, c \in B(A)$, $(P_a \vee P_b) \wedge P_c = P_{(a\wedge b)\vee c} = P_{(a\vee c)\wedge(b\vee c)} = P_{a\vee c} \vee P_{b\vee c} = (P_a \wedge P_c) \vee (P_b \wedge P_c)$.

Also $P_a \wedge P_{a^{\sim}} = P_{a\vee a^{\sim}} = P_1$ and $P_a \vee P_{a^{\sim}} = P_{a\wedge a^{\sim}} = P_0$. Therefore $(\Im_{P(A)}, \subseteq)$ is a complimented distributive lattice and hence it is a Boolean algebra.

Theorem 3.6: Let A be a Pre A*-algebra with 1. Define $\varphi : B(A) \to \Im_{P(A)}$ by $\varphi(a) = P_{a^{\sim}}$ for all $a \in B(A)$. Then φ is an isomorphism.

Proof: Let $a, b \in B(A)$, then $\varphi(a \wedge b) = P_{(a\wedge b)^{\sim}} = P_{a^{\sim}} \wedge P_{b^{\sim}} = \varphi(a) \wedge \varphi(b)$; $\varphi(a \vee b) = P_{(a\vee b)^{\sim}} = P_{a^{\sim}} \vee P_{b^{\sim}}$ $= \varphi(a) \vee \varphi(b)$ and $\varphi(a^{\sim}) = P_{a^{\sim}} = (P_a)^{\sim} = (\varphi(a))^{\sim}$. Clearly φ is both one-one and onto. Hence $B(A) \cong \Im_{P(A)}$

In [5], defined partial ordering on Pre-A* algebra by $x \le y$ if and only if $x \wedge y = y \wedge x = x$ and studied the properties of this partial ordering. Given necessary and sufficient conditions for Pre A*-algebra to become a lattice. In[5,8], proved that if A is a Pre A*-algebra and $x \in A$, then $M_x = \{s \in A / s \le x\}$ is a Pre A*-algebra under the induced operations$\wedge, \vee$ where the complementation is defined by $s^* = x \wedge s^{\sim}$ the relation defined on Pre A* algebra A by $s \le x$ if $s \wedge x = x \wedge s = s$ and the mapping $\alpha_x : A \to M_x$ defined by $\alpha_x(s) = x \wedge s$ for all $s \in A$ is a homomorphism of A onto M_x with kernel θ_x and hence $A/\theta_x \cong M_x$, where $\theta_x = \{(p,q) \in A \times A \mid x \wedge p = x \wedge q\}$. We can easily see that the Pre A*-algebras P_x, A_x are different in general where $x \in A$ .Now, we prove that the set of all A_a's where $a \in B(A)$ is a Boolean Algebra under set inclusion. The following theorem can be proved analogous to theorem 3.5.

Theorem 3.7: Let A be Pre A*-algebra with 1. Then $\Im_{M(A)} = \{M_a \mid a \in B(A)\}$ is a Boolean algebra under set inclusion in which the supremum of $\{A_a, A_b\} = A_{a\vee b}$ and the infimum $\{A_a, A_b\} = A_{a\wedge b}$

The Proof of the following theorem is analogous to that of theorem 3.6

Theorem 3.8: Let A be a Pre A*-algebra with 1, define $\psi : B(A) \to \Im_{\mathrm{M(A)}}$ by $\psi(a) = M_a$ for all $a \in B(A)$. Then ψ is an isomorphism

The following lemma can be proved directly from 3.6 and 3.8

Lemma 3.9: Let A be a Pre A*-algebra with 1, then $B(A), \Im_{P(A)}$ and $\Im_{\mathrm{M(A)}}$ are isomorphic to each other.

References:

[1] Fernando Guzman and Craig C.Squir: The Algebra of Conditional logic, Algebra Universalis 27(1990), 88-110
[2] Manes E.G: The Equational Theory of Disjoint Alternatives, personal communication to Prof.N.V.Subrahmanyam(1989)
[3] Koteswara Rao.P, A*-Algebra an If-Then-Else structures(Doctoral thesis) 1994, Nagarjuna University, A.P., India
[4] Venkateswara Rao.J., On A*-Algebras(Doctoral Thesis) 2000, Nagarjuna University, A.P., India
[5] Venkateswara Rao.J and Srinivasa Rao.K, Pre A*-Algebra as a Poset, African Journal of Mathematics and Computer Science Research.Vol.2 (4), pp 073-080, May 2009.
[6] Venkateswara Rao.J and Srinivasa Rao.K, Congruence relation on Pre A*-Algebra, Journal of Mathematical Sciences , Vol.4, Issue 4,2009, page 295-312.
[7] Srinivasa Rao.K, and Venkateswara Rao.J , Cayley's Theorem on Centre of a Pre -Algebra, International Journal of Computational and Applied Mathematics, Vol.5, No.1, 2010 pp 103- 111
[8] Venkateswara Rao.J, Srinivasa Rao.K, D.Kalyani, Decomposition of Pre A*-Algebra, International Journal of Mathematical Sciences and ApplicationsVol. 1 No. 1, January, 2011
[9] Manes E.G: Ada and the Equational Theory of If-Then-Else, Algebra Universalis 30(1993), 373-394.