Solving Fuzzy Two-Stage Programming Problem with Discrete Fuzzy Vector

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***Abstract*—The fuzzy two-stage programming problem with discrete fuzzy vector is hard to solve. In this paper, in order to solve this class of model, we design a algorithm by which we solve its deterministic equivalent programming to obtain optimal solution. Finally, two numerical examples are provided for showing the effectiveness of this algorithm.**

***Keywords*—Fuzzy programming, fuzzy variable; two-stage, expected value.**

I. Introduction

Since real-world situations are often not deterministic, conventional mathematical programming models are incapable to tackle all practical decision problems. Since Zadeh's pioneering work\cite{Zadeh}, there are much research on possibility theory\cite{Dubois, Klir,Yazenin}. Fuzzy decision making models have

provided an important aspect in dealing with practical decision making problems\cite{Buckley, Inuiguchi, Tanaka, Liu1, Liu2,Liu6,Liu7}.

Based on the credibility theory\cite{BLiuYKLiu}, some new fuzzy programming were presented\cite{Gao, Xiaoxiahuang, YKLiu}. In \cite{YKLiu}, the author gave a class of fuzzy programming called fuzzy programming with recourse or fuzzy two-stage programming. The fuzzy two-stage programming is used to solve the problem in which first a decision, called the first-stage decision, is given, after the uncertain information formulated by fuzzy variables is realized, the second-stage decision is taken. Therefore, this fuzzy two-stage programming is a dynamic one. For solving the model presented, the author designed a hybrid intelligent algorithm combining fuzzy simulation, neural network and genetic algorithm (GA). However, the algorithm designed only obtains the approximate optimal solution of the model.

To the best of the author's knowledge, there is no research on fuzzy programming problem with discrete fuzzy variables. The author tries to do something in this area. Under this consideration, to our purpose, in this paper we consider the fuzzy two-stage programming with discrete fuzzy variables. Since the computational complexity of the expected value of the fuzzy variable, this class of fuzzy two-stage programming is very hard to solve. In order to solve it, we design a algorithm that can obtain its optimal solution by finding a so-called {\textit{optimal realized value array.}}

This paper is organized as follows. In Section 2, we recall some necessary preliminaries on the fuzzy variable and the fuzzy expected value operator. Then, we give a theorem that is a basis on which the

algorithm is designed in Section 3. Finally, two numerical examples are given to show the effectiveness of the algorithm.is available.

II. PRELIMINARIES

Let ξ be a normalized discrete fuzzy variable with following possibility distribution function

$$\mu(r)=\begin{cases}\mu_1, & \text{if } r=a_1\\ \mu_2, & \text{if } r=a_2\\ \cdots & \\ \mu_n, & \text{if } r=a_n,\end{cases}$$

then the expected value of ξ is computed by the following formula\cite{BLiuYKLiu}

$$E[\xi]=\sum_{i=1}^{n}a_ip_i,$$

where $a_i's$ are assumed to satisfy $a_1\le a_2\le\cdots\le a_n$, and the weights are given by

$$p_i=\frac{1}{2}(\max_{j=1}^{i}\mu_j-\max_{j=0}^{i-1}\mu_j)+\frac{1}{2}(\max_{j=i}^{n}\mu_j-\max_{j=i+1}^{n+1}\mu_j)$$

($\mu_0=0,\mu_{n+1}=0$) for $i=1,2,\cdots,n$.

The fuzzy variables $\xi_1,\xi_2,\cdots,\xi_n$ are said to be independent if and only if

$$\mathrm{Pos}\{\xi_i\in B_i, i=1,2,\cdots,n\}=\min_{1\le i\le n}\mathrm{Pos}\{\xi\in B_i\}$$

for any sets $B_1,B_2,\cdots,B_n$ of $\Re$ \cite{Liu5}. The vector $(\xi_1,\cdots,\xi_n)$ is called a fuzzy vector if and only if $\xi_1,\xi_2,\cdots,\xi_n$ are fuzzy variables\cite{Liu5}. If all of the fuzzy variables $\xi_1,\xi_2,\cdots,\xi_n$ are the discrete ones, then fuzzy vector $(\xi_1,\cdots,\xi_n)$ is called a discrete one.

III. FUZZY TWO-STAGE MODEL

A. Fuzzy two-stage model with discrete fuzzy vector

In \cite{YKLiu}, a new class of fuzzy programming called fuzzy two-stage programming was presented as follows

$$\begin{cases}\min\limits_{\mathbf{x}} & c^T\mathbf{x}+\mathbf{E}_\xi[\min\limits_{\mathbf{y}}\ \mathbf{q}(\xi)^{\mathbf{T}}\mathbf{y}]\\ s.t. & Ax=b\\ & W(\xi)\mathbf{y}+\mathbf{T}(\xi)\mathbf{x}=\mathbf{h}(\xi)\\ & \mathbf{x}\ge\mathbf{0},\mathbf{y}\ge\mathbf{0}\end{cases}\qquad(1)$$

where $\mathbf{x}\in\Re^{n_1}$ is the first-stage decision, $\mathbf{y}\in\Re^{n_2}$ is the second-stage decision. Corresponding to $\mathbf{x}$ are the first-stage vector and matrices c, b and A, of sizes $n_1\times 1$, $m_1\times 1$ and $m_1\times n_1$, respectively. For a given realization $\hat{\xi}$ of fuzzy vector ξ, the second-stage problem data $q(\hat{\xi})$, $h(\hat{\xi})$, $T(\hat{\xi})$ and $W(\hat{\xi})$ become known, where $q(\hat{\xi})$ is $n_2\times 1$, $h(\hat{\xi})$ is $m_2\times 1$, $T(\hat{\xi})$ is $m_2\times n_1$, and $W(\hat{\xi})$ is $m_2\times n_2$.

For each fixed first-stage decision $\mathbf{x}$ and realization $\hat{\xi}$, the second-stage decision $\mathbf{y}^*$ is the optimal solution to the following model

$$\begin{cases}\min\limits_{\mathbf{y}} & q(\hat{\xi})^T\mathbf{y}\\ s.t. & W(\hat{\xi})\mathbf{y}+\mathbf{T}(\hat{\xi})\mathbf{x}=\mathbf{h}(\hat{\xi})\\ & y\ge 0.\end{cases}$$

Let $\mathcal{Q}(\mathbf{x})=\mathbf{E}_\xi[\mathbf{Q}(\mathbf{x},\xi)]$ and $Q(\mathbf{x},\xi)=\min\limits_{\mathbf{y}}\{\mathbf{q}(\xi)^{\mathbf{T}}\mathbf{y}|\mathbf{W}(\xi)\mathbf{y}+\mathbf{T}(\xi)\mathbf{x}=\mathbf{h}(\xi),\mathbf{y}\ge\mathbf{0}\}$, then the model (1) is equivalent to the following programming

$$\begin{cases}\min\limits_{\mathbf{x}} & c^T\mathbf{x}+\mathcal{Q}(\mathbf{x})\\ s.t. & A\mathbf{x}=\mathbf{b}\\ & \mathbf{x}\ge\mathbf{0}.\end{cases}$$

On the fuzzy two-stage programming problem, the interested readers may refer to \cite{YKLiu}.

For the sake of the demand of the research, in the rest of this paper, we assume that $\xi = (\xi_1, \cdots, \xi_m)$ in the model (1) is a discrete fuzzy vector with the following possibility distribution

$$\mu(\mathbf{r}) = \begin{cases} \mu_1, & \text{if } \mathbf{r} = \hat{\xi}_1 \\ \mu_2, & \text{if } \mathbf{r} = \hat{\xi}_2 \\ \cdots & \\ \mu_s, & \text{if } \mathbf{r} = \hat{\xi}_s \end{cases}$$

where ξ_i are independent discrete variables for $i = 1, \cdots, m$.

Note that the model (1) with the above discrete fuzzy vector is not equivalent to the following model:

$$\begin{cases} \min\limits_{\mathbf{x},\mathbf{y}} & c^T\mathbf{x} + \sum\limits_{j=1}^{s} \mathbf{p_j q}(\hat{\xi}_j)^T \mathbf{y_j} \\ s.t. & A\mathbf{x} = \mathbf{b} \\ & W(\hat{\xi}_j)\mathbf{y_j} + \mathbf{T}(\hat{\xi}_j)\mathbf{x} = \mathbf{h}(\hat{\xi}_j), \\ & j = 1, \cdots, s \\ & \mathbf{x}, \mathbf{y_1}, \cdots, \mathbf{y_j} \geq \mathbf{0} \end{cases}$$

where

$$p_k = \frac{1}{2}(\max_{i=1}^{k} \mu_i - \max_{i=0}^{k-1} \mu_i) + \frac{1}{2}(\max_{i=k}^{s} \mu_i - \max_{i=k+1}^{s+1} \mu_i)$$

$(\mu_0 = \mu_{s+1} = 0)$ for $k = 1, \cdots, s$.

Next we give a theorem by which a algorithm in the next subsection is designed to solve this class of fuzzy two-stage programming problem with the discrete fuzzy vector.

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Theorem 1. Let x^* and $\mathbf{y}_1^*, \cdots, \mathbf{y}_s^*$ be the optimal solutions of the following model

$$\begin{cases} \min\limits_{\mathbf{x},\mathbf{y}} & c^T\mathbf{x} + \mathbf{p_1 q}(\hat{\xi}_1)^T \mathbf{y_1} + \cdots + \mathbf{p_s q}(\hat{\xi}_s)^T \mathbf{y_s} \\ s.t. & A\mathbf{x} = \mathbf{b} \\ & W(\hat{\xi}_j)\mathbf{y_j} + \mathbf{T}(\hat{\xi}_j)\mathbf{y_j} = \mathbf{h}(\hat{\xi}_j), \\ & j = 1, \cdots, s \\ & \mathbf{x}, \mathbf{y_1}, \cdots, \mathbf{y_s} \geq \mathbf{0} \end{cases} \tag{2}$$

where

$$p_k = \frac{1}{2}(\max_{i=1}^{k} \mu_i - \max_{i=0}^{k-1} \mu_i) + \frac{1}{2}(\max_{i=k}^{s} \mu_i - \max_{i=k+1}^{s+1} \mu_i) \tag{3}$$

$(\mu_0 = \mu_{s+1} = 0)$ for $k = 1, \cdots, s$.

If the inequality

$$q(\hat{\xi}_1)^T \mathbf{y}_1^* \leq \cdots \leq \mathbf{q}(\hat{\xi}_s)^T \mathbf{y}_s^*, \tag{4}$$

holds, then $\mathbf{x}^*$ is the optimal solution of the model (1).

Proof. Obviously, $\mathbf{y}_j^*$ is the optimal solution to the following model

$$\begin{cases} \min & q(\hat{\xi}_j)^T \mathbf{y_j} \\ s.t. & W(\hat{\xi}_j)\mathbf{y_j} + \mathbf{T}(\hat{\xi})\mathbf{x}^* = \mathbf{h}(\hat{\xi}_j) \\ & \mathbf{y_j} \geq \mathbf{0}, \end{cases}$$

otherwise, $\mathbf{y}_j^*$ can not be the optimal solution to the model (2) for $j = 1, \cdots, s$.

Since $\mathbf{y_1^*}, \cdots, \mathbf{y_s^*}$ satisfy the condition (4), and $\mathbf{x}^*$ is the feasible solution to the model (1), we have

$$\mathcal{Q}(\mathbf{x}^*) = \mathbf{p_1 q}(\hat{\xi}_1)^{\mathbf{T}}\mathbf{y_1^*} + \cdots + \mathbf{p_s q}(\hat{\xi}_s)^{\mathbf{T}}\mathbf{y_s^*}$$

where p_j is computed by the formula (3) for $j = 1, \cdots, s$. Noting that $\mathbf{x}^*$ is the optimal solution of the model (2), we have that $\mathbf{x}^*$ is also the optimal solution of the model (1). The proof is complete.

B. Solving of the model

According to Theorem 1, in order to solve the model (1), we need to look for a array $(\mathbf{r_1^*}, \cdots, \mathbf{r_s^*})$ where $\mathbf{r_i^*} \in \{\hat{\xi}_1, \cdots, \hat{\xi}_s\}$ for $i = 1, \cdots, s$, called the *optimal realized value array*, such that $\mathbf{y_1^*}, \cdots, \mathbf{y_s^*}$ satisfy the condition

$$q(\mathbf{r_1^*})^{\mathbf{T}}\mathbf{y_1^*} \leq \cdots \leq \mathbf{q}(\mathbf{r_s^*})^{\mathbf{T}}\mathbf{y_s^*}, \tag{5}$$

where $\mathbf{y_1^*}, \cdots, \mathbf{y_s^*}$ are the optimal solution of the following model

$$\begin{cases} \min\limits_{\mathbf{x},\mathbf{y}} & c^T\mathbf{x} + \mathbf{p_1 q}(\mathbf{r_1^*})^{\mathbf{T}}\mathbf{y_1} + \cdots + \mathbf{p_s q}(\mathbf{r_s^*})^{\mathbf{T}}\mathbf{y_s} \\ s.t. & A\mathbf{x} = \mathbf{b} \\ & W(\mathbf{r_j^*})\mathbf{y_j} + \mathbf{T}(\mathbf{r_j^*})\mathbf{y_j} = \mathbf{h}(\mathbf{r_j^*}), \\ & j = 1, \cdots, s \\ & \mathbf{x}, \mathbf{y_1}, \cdots, \mathbf{y_s} \geq \mathbf{0} \end{cases} \tag{6}$$

and p_k are computed by the formula

$$p_k = \frac{1}{2}(\max_{i=1}^{k} \mu_i - \max_{i=0}^{k-1} \mu_i) + \frac{1}{2}(\max_{i=k}^{s} \mu_i - \max_{i=k+1}^{s+1} \mu_i) \tag{7}$$

$(\mu_0 = \mu_{s+1} = 0)$ where μ_k is the possibility of realized value $\mathbf{r_k^*}$ of the fuzzy vector ξ for $k = 1, \cdots, s$.

For finding the *optimal realized value array* $(\mathbf{r_1^*}, \cdots, \mathbf{r_s^*})$ to obtain the optimal solution of the model (1), we give the algorithm as follows.

Algorithm

Step 0. Set $t = 0$. Let $(\mathbf{r_1^t}, \cdots, \mathbf{r_s^t})$ be the current array. Randomly initialize the array $(\mathbf{r_1}, \cdots, \mathbf{r_s})$ where $\mathbf{r_k} \in \{\hat{\xi}_1, \cdots, \hat{\xi}_s\}$ for $k = 1, \cdots, s$. Let $\mathbf{r} \in \Re^{\mathbf{m}}$.

Step 1. Let$(\mathbf{r_1^t}, \cdots, \mathbf{r_s^t}) = (\mathbf{r_1}, \cdots, \mathbf{r_s})$.

Step 2. In the model (6), let $\mathbf{r_k^*} = \mathbf{r_i^t}$ for $k = 1, \cdots, s$. After computing p_k according to the formula (7) for $k = 1, \cdots, s$, solve the model (6) to obtain the optimal solution $\mathbf{x}^*, \mathbf{y_1^*}, \cdots, \mathbf{y_s^*}$. If $\mathbf{y_1^*}, \cdots, \mathbf{y_s^*}$ satisfy the condition (5), then $(\mathbf{r_1^*}, \cdots, \mathbf{r_s^*})$ is the *optimal realized value array* and $\mathbf{x}^*$ is the optimal solution of the model (1), the algorithm stops. Otherwise, go to the next step.

Step 3. Randomly take two number i and j from the set $\{1, \cdots, s\}$ such that $i < j$. Let $\mathbf{r} = \mathbf{r_i^t}$, $\mathbf{r_i^t} = \mathbf{r_j^t}$, and $\mathbf{r_j^t} = \mathbf{r}$. Set $(\mathbf{r_1}, \cdots, \mathbf{r_s}) = (\mathbf{r_1^t}, \cdots, \mathbf{r_s^t})$ and $t = t + 1$. Return step 1.

IV. NUMERICAL EXAMPLES

Example 1. Consider Example 13 in [13] as follows

$$\begin{cases} \min\limits_{\mathbf{x}} & 2x_1 + 3x_2 + E_\xi[\min 2y_1 + y_2] \\ s.t. & x_1 \le 1 \\ & x_2 \le 1 \\ & y_1 + y_2 \ge 1 - x_1 \\ & y_1 \ge \xi - x_1 - x_2 \\ & x_1, x_2, y_1, y_2 \ge 0 \end{cases}$$

where ξ is a fuzzy variable takeing the values 0, 1, and 2 with possibility $\frac{1}{3}$, 1 and $\frac{1}{2}$, respectively.

In [13], the above problem is converted to its deterministic equivalent programming

$$\begin{cases} \min\limits_{\mathbf{x}} & 2x_1 + 3x_2 + Q(x_1, x_2) \\ s.t. & x_1 \le 1 \\ & x_2 \le 1 \\ & x_1, x_2 \ge 0 \end{cases}$$

where

$$Q(x_1, x_2) = \begin{cases} -\frac{4}{3}x_1 - \frac{2}{3}x_2 + 2 & \text{if } x_1 + x_2 > 1 \\ -\frac{3}{2}x_1 - \frac{5}{6}x_2 + \frac{15}{6} & \text{if } x_1 + x_2 \le 1. \end{cases}$$

By calculating the above deterministic programming, we obtain that the optimal solution of the original problem is $(x_1^*, x_2^*) = (0, 0)$, and the optimal value is $\frac{15}{6}$.

For showing the effectiveness of the algorithm designed in subsection B, we apply it to solve Example 1.

Consider the following model

$$\begin{cases} \min\limits_{\mathbf{x},\ \mathbf{y}} & 2x_1 + 3x_2 + \sum_{j=1}^{3} p_j(2y_{j1} + y_{j2}) \\ s.t. & x_1 \le 1 \\ & x_2 \le 1 \\ & y_{j1} + y_{j2} \ge 1 - x_2,\ \ j = 1, 2, 3 \\ & y_{j1} \ge \hat{\xi}_j - x_1 - x_2,\ \ j = 1, 2, 3 \\ & x_1, x_2, y_{j1}, y_{j2} \ge 0, j = 1, 2, 3 \end{cases} \tag{8}$$

where $\hat{\xi}_j$ is the realized value of the fuzzy variable ξ with the possibility μ_j for $j = 1, 2, 3$, and p_j is computed by the formula

$$p_j = \frac{1}{2}(\max_{i=1}^{j} \mu_i - \max_{i=0}^{j-1} \mu_i) + \frac{1}{2}(\max_{i=j}^{3} \mu_i - \max_{i=j+1}^{4} \mu_i) \tag{9}$$

$(\mu_0 = \mu_4 = 0)$ for $j = 1, 2, 3$.

Next we apply the algorithm to solve the example 1. In the first iteration we assume that $\hat{\xi}_1 = 2$, $\hat{\xi}_2 = 0$ and $\hat{\xi}_3 = 1$ in the model (8), compute $p_1 = \frac{1}{3}$, $p_2 = 0$ and $p_3 = \frac{2}{3}$ according to the formula (9), then by solving the model (8) obtain the optimal solution

$$\begin{aligned} & (x_1^*, x_2^*, y_{11}^*, y_{12}^*, y_{21}^*, y_{22}^*, y_{31}^*, y_{32}^*) \\ = \ & (0.1812, 0, 1.8188, 0, 110.7978, 94.9196, 0.8188, 0). \end{aligned}$$

Since the inequality $2y_{11}^* + y_{12}^* \le 2y_{12}^* + y_{22}^* \le 2y_{13}^* + y_{13}^*$ does not hold, $(x_1^*, x_2^*) = (0.1812, 0)$ is not the optimal solution of Example 1.

In the second iteration, letting $\hat{\xi}_1 = 0$, $\hat{\xi}_2 = 1$ and $\hat{\xi}_3 = 2$ in the model (8) and computing the weights $p_1 = \frac{1}{6}$, $p_2 = \frac{1}{2}$ and $p_3 = \frac{1}{3}$ according to the formula (9), then we solve the model (8) and obtain the optimal solution

$$\begin{aligned}&(x_1^*, x_2^*, y_{11}^*, y_{12}^*, y_{21}^*, y_{22}^*, y_{31}^*, y_{32}^*)\\ =\ &(0,0,0,1,1,0,2,0).\end{aligned}$$

Since $2y_{11}^* + y_{12}^* = 1,\ 2y_{12}^* + y_{22}^* = 2$ and $2y_{13}^* + y_{13}^* = 4$, by Theorem 1, we know that $(x_1^*, x_2^*) = (0,0)$ is the optimal solution of Example 1, and the optimal value is

$$\begin{aligned}&2x_1^* + 3x_2^* + \tfrac{1}{6}(2y_{11}^* + y_{12}^*) + \tfrac{1}{2}(2y_{21}^* + y_{22}^*) + \tfrac{1}{3}(2y_{31}^* + y_{32}^*)\\ &= \tfrac{15}{6}.\end{aligned}$$

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Example 2. Consider the following two-stage programming problem

$$\begin{cases} \min\limits_{\mathbf{x}} & x_3 + E_\xi[\min\ 2\xi_1 y_1 - y_2] \\ s.t. & x_1 + x_2 + x_3 \le 10 \\ & x_1 - 2x_2 + x_3 \le 15 \\ & -x_1 + x_2 + 2\xi_2 y_1 + y_2 \le 7 \\ & x_1 + x_3 + y_1 + y_2 \le 25 \\ & x_1, x_2, x_3, y_1, y_2 \ge 0 \end{cases}$$

where ξ_1 and ξ_2 are two independent discrete fuzzy variables taking values 1, 2 and 2, 3 with possibility 0.4 ,1 and 1, 0.2, respectively. By solving the following model using the algorithm in subsection 3.2

$$\begin{cases} \min\limits_{\mathbf{x},\ \mathbf{y}} & x_3 + \sum\limits_{j=1}^{4} p_j(2\hat{\xi}_{1j} y_{1j} - y_{2j}) \\ s.t. & x_1 + x_2 + x_3 \le 10 \\ & x_1 - 2x_2 + x_3 \le 15 \\ & -x_1 + x_2 + 2\hat{\xi}_{2j} y_{1j} + y_{2j} \le 7,\ \ j = 1,2,3,4 \\ & x_1 + x_3 + y_{1j} + y_{2j} \le 25,\ \ j = 1,2,3,4 \\ & x_1, x_2, x_3, y_{1j}, y_{2j} \ge 0,\ \ j = 1,2,3,4 \end{cases}$$

where $(\hat{\xi}_{11}, \hat{\xi}_{21}) = (1,3),\ (\hat{\xi}_{12}, \hat{\xi}_{22}) = (1,1), (\hat{\xi}_{13}, \hat{\xi}_{23}) = (2,1)$ and $(\hat{\xi}_{14}, \hat{\xi}_{24}) = (2,3)$ with possibility $\mu_1 = 0.2,\ \mu_2 = 0.4,\ \mu_3 = 1$ and $\mu_4 = 0.2$, respectively, and $p_1 = 0.1, p_2 = 0.1, p_3 = 0.7, p_4 = 0.1$ by the formula

$$p_j = \frac{1}{2}(\max_{i=1}^{j} \mu_i - \max_{i=0}^{j-1} \mu_i) + \frac{1}{2}(\max_{i=j}^{4} \mu_i - \max_{i=j+1}^{5} \mu_i)$$

$(\mu_0 = \mu_5 = 0)$ for $j = 1,2,3,4$, we obtain the optimal solution

$$\begin{aligned}&(x_1^*, x_2^*, x_3^*, y_{11}^*, y_{21}^*, y_{12}^*, y_{22}^*, y_{13}^*, y_{23}^*, y_{14}^*, y_{24}^*)\\ =\ &(9,0,0,0,16,0,16,0,16,0,16).\end{aligned}$$

Since $2\hat{\xi}_{11} y_{11}^* - y_{21}^* \le 2\hat{\xi}_{12} y_{12}^* - y_{22}^* \le 2\hat{\xi}_{13} y_{13}^* - y_{23}^* \le 2\hat{\xi}_{14} y_{14}^* - y_{24}^*$, we know that $(x_1^*, x_2^*, x_3^*) = (9,0,0)$ is the optimal solution of Example 2, and (y_{1j}^*, y_{2j}^*) is the second-stage optimal solution for the realized value $(\hat{\xi}_{1j}, \hat{\xi}_{2j})$ of the fuzzy vector (ξ_1, ξ_2) and the first-stage optimal solution (x_1^*, x_2^*, x_3^*) for $j = 1,2,3,4$, respectively.

References

[1] J.J. Buckley, "Solving possibilistic linear programming problems", Fuzzy Sets and Systems, Vol 31, pp. 329-341, 1989.

[2] D. Dubois, H. Prade, Possibility Theory, Plenum, New York, 1988.

[3] J. Gao and B. Liu, "Fuzzy multilevel programming with a hybrid intelligent algorithm", Computer & Mathematics with Applications, Vol 49, pp. 1539–1548, 2005.

[4] X.X. Huang, "Fuzzy chance-constrained portfolio selection", Applied Mathematics and Computation, Vol 177, 500-507, 20006.

[5] M. Inuiguchi, H. Ichihashi, Y. Kume, "Modality constrained programming problems: a unified aproach to fuzzy mathematical programming problems in the setting of possibility theory", Information Sciences, Vol 67, pp. 93-126, 1993.

[6] G.J. Klir, "On fuzzy-set interpretation of possibility theory", Fuzzy Sets and Systems, Vol 108, pp. 263-373, 1999.

[7] B. Liu, H. Iwamura, "Chance constrained programming with fuzzy arameters", Fuzzy Sets and Systems, Vol 94, pp. 227-237, 1998.

[8] B. Liu, H. Iwamura, "Fuzzy programming with fuzzy decisions and fuzzy simulation-based genetic algorithm", Fuzzy Sets and Systems, Vol 122, pp. 253-262.

[9] B. Liu, Y.-K. Liu, "Expected value of fuzzy variable and fuzzy expeced value models", IEEE Tranctions on Fuzzy Systems, Vol 10, pp. 445-450, 2002.

[10] B. Liu, Uncertainty Theory: An Introducetion to its Axiomatic Foundations, Apringer-Verlag, Berlin, 2004.

[11] B. Liu, "Dependent-chance programming with fuzzy decisions", IEEE Transactions on Fuzzy Syestems, Vol 7, pp. 354-360, 1999.

[12] B. Liu, "Dependent-chance programming in fuzzy environments", Fuzzy Sets and Systems, Vol 109, pp. 97-106, 2000.

[13] Y.K. Liu, "Fuzzy programming with recourse", International Journal of Uncertainty, Fuzziness & Knowledge-Based Systems, Vol 13, pp. 381-413, 2005.

[14] H. Tanaka, P.Guo, H.-J. Zimmermann, "Possibility distribution of fuzzy decision variables obtained form possibilistic linear programming problems", Fuzzy Sets and Systems, Vol 113, pp. 323-332, 2000.

[15] A.V. Yazenin, "On the problem of possibilitic optimization", Fuzzy Sets and Systems, Vol 81, pp. 133-140, 1996.

[16] L. Zadeh, "Fuzzy sets", Information and Control, Vol 8, pp. 338-363, 1965.