Regular pre semi I totally continuous functions in Ideal Topological Spaces

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Abstract

The authors introduced $rpsI$-closed sets and $rpsI$-open sets in ideal topological spaces and established their relationships with some generalized sets in ideal topological spaces. The aim of this paper is to introduce $rpsI$-totally continuous, totally $rpsI$-continuous, strongly $rpsI$-continuous functions and characterize their basic properties.

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1 Introduction

In 1980, Jain [2] introduced totally continuous functions. In 1995, Nour [4] introduced the concept of totally semi continuous functions as a generalization of totally continuous functions and several properties of totally semi-continuous functions were obtained. We introduced the $rpsI$-closed sets and $rpsI$-open sets in ideal topological spaces. Also we introduced $rpsI$-continuous functions. In this paper, we introduce the concept of $rpsI$-totally continuous, strongly $rpsI$-continuous, $rpsI$-homeomorphisms and rps^*I-homeomorphisms in ideal topological spaces. Furthermore, basic properties of these functions and preservation theorems of $rpsI$-totally continuous functions.

2 Preliminaries

For a subset A of an ideal topological space(X, τ, I), $cl^*(A)$ and $int^*(A)$ denote the closure of A and interior of A respectively. A^c denotes the complement of A in X. Now we recall the following definitions.

Definition 2.1. *A subset A of an ideal topological space (X, τ, I) is called*

i) semi-I-open [1] if $A \subseteq cl^*(int(A))$.

ii) semi pre I-open [1] if $A \subseteq cl(int(cl^*(A)))$.

iii) αI- open [1] if $A \subseteq int(cl^*(int(A)))$.

iv) regular I-open [3] if $A = int(cl^*(A))$.

Definition 2.2. *A subset A of an ideal topological space (X, τ, I) is called*

i) regular generalized I-closed (rgI-closed) [5] if $cl^*(A) \subseteq U$ whenever $A \subseteq U$ and U is regular I-open.

ii) regular pre semi I-closed [5] ($rpsI$-closed) if $spIcl(A) \subseteq U$ whenever $A \subseteq U$ and U is regular generalized I-open.

The complement of $rpsI$-closed set is $rpsI$-open set. $rpsIcl(A)$ is the smallest $rpsI$-closed set containing A.

Definition 2.3. *A function $f : (X, \tau) \to (Y, \sigma)$ is called totally continuous [2] if the inverse image of every open subset of Y is clopen in X.*

Definition 2.4. *A function $f : (X, \tau, I) \to (Y, \sigma)$ is called*

i) $rpsI$-irresolute [6] if $f^{-1}(A)$ is $rpsI$-closed in X, for every $rpsI$-closed subset A of Y.

ii) A function $f : (X, \tau, I) \to (Y, \sigma)$ is called $rpsI$-continuous [6] if the inverse image of every closed subset of Y is $rpsI$-closed in X.

3 $rpsI$-totally continuous functions, totally $rpsI$-continuous functions and strongly $rpsI$-continuous

In this section, the notion of totally $rpsI$-continuous, $rpsI$-totally continuous functions and strongly $rpsI$-continuous are introduced. Characterizations and some relationships between $rpsI$-totally continuous functions and other similar functions are obtained. Also some basic properties of $rpsI$-totally continuous functions are investigated.

Definition 3.1. *A function $f : (X, \tau, I) \to (Y, \sigma, J)$ is called rpsI-totally continuous function if the inverse image of every rpsI-open subset of Y is clopen in X.*

Theorem 3.2. *A function $f : (X, \tau, I) \to (Y, \sigma, J)$ is rpsI-totally continuous if and only if the inverse image of every rpsI-closed subset of Y is clopen in X.*

Proof. Let F be any $rpsI$-closed set in Y. Then F^c is $rpsI$-open in Y. By definition, $f^{-1}(F^c)$ is clopen in X. But $f^{-1}(F^c) = (f^{-1}(F))^c$ which is clopen in X. This implies $f^{-1}(F)$ is clopen in X. Conversely suppose V is $rpsI$-open Y, then V^c is $rpsI$-closed in Y. By hypothesis $f^{-1}(V^c)$ is clopen in X. But $f^{-1}(V^c) = (f^{-1}(V))^c$ which is clopen in X, which implies $f^{-1}(V)$ is clopen in X. Thus, inverse image of every $rpsI$-open set in Y is clopen in X. Therefore f is $rpsI$-totally continuous.

Theorem 3.3. *Every rpsI-totally continuous function is totally continuous.*

Proof. Suppose $f : (X, \tau, I) \to (Y, \sigma, J)$ is $rpsI$-totally continuous. Let U be any open subset of Y. Since every open set is $rpsI$-open, U is rpsI-open in Y and f is $rpsI$-totally continuous, it follows $f^{-1}(U)$ is clopen in X. This proves the theorem. The converse of the above theorem need not be true as seen from the following example.

Example 3.4. *Let $X = Y = Z = \{a, b, c\}, \tau = \{\phi, X, \{a\}, \{b, c\},\ I = \{\phi, \{a\}\}$, $\sigma = \{\phi, Y, \{a\}\}$. Define a functions $f : (X, \tau, I) \to (Y, \sigma, J)$ by $f(a) = a,\ f(b) = b$, $f(c) = c$. Clearly the inverse image of every open set is clopen. Therefore f is totally continuous. But f is not rpsI-totally continuous, because for the rpsI-open set $\{a, b\}$, $f^{-1}(\{a, b\}) = \{a, b\}$ is not clopen in X.*

Theorem 3.5. *Let $f : X \to Y$ be a function, where X and Y are ideal topological spaces. Then the following are equivalent.*

i. f is rpsI-totally continuous

ii. for each $x \in X$ and each rpsI-open set V in Y with $f(x) \in V$, there is a clopen set U in X such that $x \in U$ and $f(U) \subseteq V$.

Proof. $(i) \Rightarrow (ii)$. Suppose f is $rpsI$-totally continuous and V be any $rpsI$-open set in Y containing $f(x)$ so that $x \in f^{-1}(V)$. Since f is $rpsI$-totally continuous, $f^{-1}(V)$ is clopen in X. Let $U = f^{-1}(V)$, then U is clopen in X and $x \in U$. Also $f(U) = f(f^{-1}(V)) \subseteq V$. This implies $f(U) \subseteq V$.
$(ii) \Rightarrow (i)$. Let V be $rpsI$-open set in Y. Let $x \in f^{-1}(V)$ be any arbitrary point. This implies $f(x) \in V$. By (ii) there is a clopen set $f(G) \subseteq X$ containing x such that $f(G) \subseteq V$, which implies $G \subseteq f^{-1}(V)$. We have $x \in G \subseteq f^{-1}(V)$. This implies $f^{-1}(V)$ is clopen neighbourhood of each of its points. Hence it is clopen set in X. Therefore f is $rpsI$-totally continuous.

Theorem 3.6. *If a function $f : (X, \tau, I) \rightarrow (Y, \sigma, J)$ is rpsI-totally continuous then f is continuous but not conversely.*

Proof. Let V be an open set in Y. Then V is rpsI-open in Y. Since f is $rpsI$-totally continuous, $f^{-1}(V)$ is both open and closed in X. Thus f is continuous. Converse of the above theorem need not be true as seen from the following example

Example 3.7. *Let $X = Y = \{a, b, c, d\}, \tau = \{\phi, X, \{a, c\}, \{d\}, \{a, c, d\}\}$, $I = \{\phi, \{a\}\}$, $\sigma = \{\phi, Y, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}$. Define a functions $f : (X, \tau, I) \rightarrow (Y, \sigma, J)$ by $f(a) = d$, $f(b) = c$, $f(c) = d$, $f(d) = b$. Then f is continuous but not rpsI-totally continuous. Because the subset $\{b\}$ is rpsI-open in Y but $f^{-1}(\{b\}) = \{d\}$ is not closed in X.*

Example 3.8. *Let $X = Y = \{a, b, c, d\}, \tau = \{\phi, X, \{a\}, \{b, c, d\}, \{a, c, d\}\}$, $I = \{\phi, \{a\}\}$, $\sigma = \{\phi, Y, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}$. Define a functions $f : (X, \tau, I) \rightarrow (Y, \sigma, J)$ by $f(a) = b$, $f(b) = d$, $f(c) = c$, $f(d) = d$. Then f is perfectly continuous but not rpsI-totally continuous. Because the subset $\{c\}$ is rpsI-open in Y but $f^{-1}(\{c\}) = \{c\}$ is not clopen in X.*

Theorem 3.9. *The composition of two rpsI-totally continuous functions is rpsI-totally continuous.*

Proof. Let $f : (X, \tau, I) \rightarrow (Y, \sigma, J)$ and $g : (Y, \sigma, J) \rightarrow (Z, \eta, K)$ be any two $rpsI$-totally continuous functions. Let V be $rpsI$-open set in Z. Since g is $rpsI$-totally continuous, $g^{-1}(V)$ is clopen and hence open in Y. Since every open set is $rpsI$-open, $g^{-1}(V)$ is $rpsI$-open in Y. Further, since f is $rpsI$-totally continuous, $f^{-1}(g^{-1}(V)) = (gof)^{-1}(V)$ is clopen in X. Hence gof is $rpsI$-totally continuous function.

Definition 3.10. *A function $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma, J)$ is called strongly rpsI-continuous if the inverse image of every rpsI-open set of Y is open in X.*

Theorem 3.11. *If a function $f : (X, \tau, I) \rightarrow (Y, \sigma, J)$ is rpsI-totally continuous then f is strongly rpsI-continuous but not conversely.*

Proof. Proof follows from Definition.

Example 3.12. *Let* $X = Y = \{a, b, c, d\}, \tau = \{\phi, X, \{a, c\}, \{d\}, \{a, c, d\}\}$*,* $I = \{\phi, \{a\}\}$*,* $\sigma = \{\phi, Y, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}$*. Define a functions* $f : (X, \tau, I) \rightarrow (Y, \sigma, J)$ *by* $f(a) = f(b) = f(c) = d$*,* $f(d) = a$*. Then* f *is strongly rpsI-continuous but not rpsI-totally continuous. Because the subset a is rpsI-open in* Y *but* $f^{-1}(a) = d$ *is not closed in* X*.*

Theorem 3.13. *Let* X *be a discrete topological space and* Y *be any ideal space and* $f : (X, \tau, I) \rightarrow (Y, \sigma, J)$ *be a function. If* f *is strongly rpsI-continuous then* f *is rpsI-totally continuous.*

Proof. Let V be *rpsI*-open in Y. Since f is strongly *rpsI*-continuous, $f^{-1}(V)$ is open in X. Also since X is a discrete space, we have $f^{-1}(V)$ is closed in X and so f is *rpsI*-totally continuous.

Definition 3.14. *A function* $f : (X, \tau, I) \rightarrow (Y, \sigma)$ *is called totally rpsI-continuous if* $f^{-1}(V)$ *is rpsI-clopen in* (X, τ, I) *for each open set* V *in* (Y, σ)*.*

Theorem 3.15. *Every totally rpsI-continuous function is rpsI-continuous.*

Proof. proof follows from definition The converse of the above statements need not be true as seen from the following example.

Example 3.16. *Let* $X = Y = \{a, b, c\}, \tau = \{\phi, X, \{a\}, \{a, b\}$*,* $I = \{\phi, \{b\}\}$*,* $\sigma = \{\phi, Y, \{b\}, \{a, b\}\}$*. . Define a function* $f : (X, \tau, I) \rightarrow (Y, \sigma, J)$ *by* $f(a) = b$*,*$f(b) = a$*,* $f(c) = c$*. Then* f *is rpsI-continuous but not totally rpsI-continuous because the set* $\{b\}$ *is open in* Y *but* $f^{-1}(b) = \{a\}$ *which is rpsI-open and not rpsI-closed in* X*.*

Theorem 3.17. *Every rpsI-totally continuous function is totally rpsI-continuous.*

Proof. Suppose $f : (X, \tau, I) \rightarrow (Y, \sigma, J)$ is *rpsI*-totally continuous. Let U be any open subset of Y. Since every open set is *rpsI*-open, U is rpsI-open in Y and f is *rpsI*-totally continuous, it follows $f^{-1}(U)$ is clopen in X. Then $f^{-1}(U)$ is *rpsI*-clopen in X. This proves the theorem.

The converse of the above theorem need not be true as seen from the following example.

Example 3.18. *Let* $X = Y = Z = \{a, b, c\}, \tau = \{\phi, X, \{a\}, \{b, c\}$*,* $I = \{\phi, \{a\}\}$*,* $\sigma = \{\phi, Y, \{a\}\}$*. Define a functions* $f : (X, \tau, I) \rightarrow (Y, \sigma, J)$ *by* $f(a) = a$*,* $f(b) = b$*,* $f(c) = c$*. Clearly the inverse image of every open set is rpsI-clopen in* X*. Therefore* f *is totally rpsI-continuous. But* f *is not rpsI-totally continuous, because for the rpsI-open set* $\{a, b\}$*,* $f^{-1}(\{a, b\}) = \{a, b\}$ *is not clopen in* X*.*

4 $rpsI$-homeomorphisms

In this section, we introduce the concept of rpsI-homeomorphisms and study its relationship with homeomorphisms. We also introduce a new class of functions rps^*I-homeomorphisms which form a subclass of rpsI-homeomorphisms. We prove that the set of all rps^*I-homeomorphisms from (X, τ, I) onto itself is a group under the composition of functions.

Definition 4.1. *A bijection $f : (X, \tau, I) \to (Y, \sigma)$ is called rpsI-homeomorphism if both f and f^{-1} are rpsI-continuous.*
We say that the spaces (X, τ, I) and (Y, σ) are rpsI-homeomorphic if there exists an rpsI-homeomorphism from (X, τ, I) onto (Y, σ).

Theorem 4.2. *Every homeomorphism is an rpsI-homeomorphism.*

Proof. Let $f : (X, \tau, I) \to (Y, \sigma)$ be a homeomorphism. Then f and f^{-1} are continuous and f is a bijection. Since every continuous function is $rpsI$-continuous, it follows that f is $rpsI$-homeomorphism.
The converse of the above theorem need not be true as seen from the following example.

Example 4.3. *Let $X = Y = \{a, b, c, d\}, \tau = \{\phi, X, \{a\}, \{b, d\}, \{a, b, d\}\}$, $I = \{\phi, \{a\}\}$, $\sigma = \{\phi, Y, \{b, c, d\}\}$. Define a function $f : (X, \tau, I) \to (Y, \sigma, J)$ by $f(a) = a$, $f(b) = b$, $f(c) = c$, $f(d) = d$. Then f is a rpsI homeomorphism but not homeomorphism. Because the subset $(f^{-1})^(\{a\}) = \{a\}$ is closed in Y but not closed in X.*

Remark 4.4. *The composition of two rpsI-homeomorphism need not be rpsI-homeomorphism.*

Example 4.5. *Let $X = Y = Z = \{a, b, c, d\}, \tau = \{\phi, X, \{c\}, \{a, b, d\}\}$, $I = \{\phi, \{a\}\}$, $\sigma = \{\phi, Y, \{b\}, \{a, c, d\}\}$, $J = \{\phi, \{b\}\}$ and $\eta = \{\phi, Z, \{a\}, \{b, c, d\}\}$, $\mathcal{K} = \{\phi, \{b\}\}$. Define a function $f : (X, \tau, I) \to (Y, \sigma, J)$ by $f(a) = a$, $f(b) = b$, $f(c) = c$, $f(d) = d$ and $g : (Y, , J) \to (Z, \eta, K)$ by $g(a) = c$, $g(b) = d$, $g(c) = b$, $g(d) = a$. Then f and g are both rpsI-homeomorphisms but $g \circ f$ is not rpsI-homeomorphism. Because the subset $b, d\}$ is closed in X, $((g \circ f)^{-1})^{-1}(\{b, d\}) = \{a, c, d\}$ is not rpsI-closed in Z.*

Definition 4.6. *A bijection $f : (X, \tau, I) \to (Y, \sigma)$ is called rps^*I-homeomorphism if both f and f^{-1} are rpsI-irresolute.*
We say that the spaces (X, τ, I) and (Y, σ) are rps^*I-homeomorphic if there exists an rps^*I-homeomorphism from (X, τ, I) onto (Y, σ).
We denote the family of all rps^*I-homeomorphism of an ideal topological space (X, τ, I) onto itself by $rps^*I - h(X, \tau)$.

Proof. Let $f : (X, \tau, I) \to (Y, \sigma)$ be a rps^*I-homeomorphism. Then f and f^{-1} are $rpsI$-irresolute and f is bijection. Therefore f and f^{-1} are $rpsI$-continuous. Therefore f is $rpsI$-homeomorphism.
The converse is not true as seen from the following example.

Theorem 4.7. *Let $f : (X, \tau, I) \to (Y, \sigma, J)$ and $g : (Y, \sigma, J) \to (Z, \eta, K)$ be rps^*I-homeomorphisms. Then their composition $g \circ f : (X, \tau, I) \to (Z, \eta, K)$ is rps^*I-homeomorphism.*

Proof. Suppose f and g are rps^*I-homeomorphisms. Then f and g are $rpsI$-irresolute. Let U be $rpsI$-open in Z. Since g is $rpsI$-irresolute, $g^{-1}(U)$ is $rpsI$-open in Y. Since f is $rpsI$-irresolute, $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is $rpsI$ open in X. Hence gof is $rpsI$-irresolute. Also for an $rpsI$-open set G in X, we have $(g \circ f)(G) = g(f(G)) = g(U)$ where $U = f(G)$. By hypothesis, $f(G)$ is $rpsI$-open in Y and so again by hypothesis, $g(f(G))$ is an $rpsI$-open set in Z. That is $(gof)(G)$ is an $rpsI$-open set in Z and therefore $(gof)^{-1}$ is $rpsI$-irresolute. Also $g \circ f$ is a bijection. This proves $g \circ f$ is rps^*I-homeomorphism.

Theorem 4.8. *The set $rps^*I - h(X, \tau)$ from (X, τ, I) onto itself is a group under the composition of functions.*

Proof. Let $f, g \in rps^*I - h(X, \tau)$. Then $g \circ f \in rps^*I - h(X, \tau)$. We know that the composition of functions is associative and the identity element $I : (X, \tau, I) \to (X, \tau, I)$ belonging to $rps^*I - h(X, \tau)$ serves as the identity element. If $f \in rps^*I - h(X, \tau)$ then $f^{-1} \in rps^*I - h(X, \tau)$. This proves $rps^*I - h(X, \tau)$ is a group under the operation of composition of functions.

Theorem 4.9. *Let $f : (X, \tau, I) \to (Y, \sigma)$ be an rps^*I-homeomorphism. Then f induces an isomorphism from the group $rps^*I - h(X, \tau)$ onto the group $rps^*I - h(Y, \sigma)$.*

Proof. Let $f \in rps^*I - h(X, \tau)$. We define a function $\psi f : rps^*I - h(X, \tau) \to rps^*I - h(Y, \sigma)$ by $\psi(f(h) = f \circ h \circ f^{-1}$ for every $h \in rps^*I - h(X, \tau)$. Then f is a bijection. Further for all $g, h \in rps^*I - h(X, \tau), \psi f(g \circ h) = f \circ (g \circ h) \circ f^{-1}) = (f \circ g \circ f^{-1}) \circ (f \circ h \circ f^{-1}) = \psi f(g) \circ \psi f(h)$.

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