On Regular Seminear rings

K.V.R. Srinivas
Professor in Mathematics,
Regency Institute of Technology, Adavipolam, Yanam-533464.
Email:- srinivas_kandarpa06@yahoo.co.in

Abstract:- In this paper mainly we have obtained equivalent conditions on skew semi rings, regular semi near rings and Idempotent semi near rings.

Key words:- skew semi rings, regular semi near rings and Idempotent semi near rings.

introduction:- In this paper we obtained certain Identities on regular semi near rings and also we have obtained certain equivalences on semi near rings,regular semi near ring, and Idempotent semi near rings.

We also obtained results such as $x + x \approx x$, $x*y + y*x + x*y \approx x*y$, $y + y*x + y \approx y$, $x*y*x + x*y + x*y*x \approx x*y*x$, $y*x*y + x*y*x + y*x*y \approx y*x*y$, $x + x*y*x + x \approx x$ using the identities $x*x \approx x$, $x + x*y + x \approx x$, $x + y*x + x \approx x$.

Also by using Greens is relation (i. e, L^+ , R^+ , D^+, L^*, R^* D^*). We have

obtained results for $a + a*b + a = a$, $a + b*a + a = a$, $b + a*b + b = b$, $b*a + b + b*a = b*a$.
Besides this we have also proved the equivalent conditions for idempotent semi near ring like, $x*x \approx x$; $x + x*y + x \approx x \Leftrightarrow x + x \approx x$;
$y + y*x + y \approx y$,
$x*y*x + x*y + x*y*x \approx x*y*x \Leftrightarrow x*y + y*x + x*y \approx x*y$,
$x*y + x + x*y \approx x*y$; $y + x*y + y \approx y \Leftrightarrow (x + y)*(y + x)*(x + y) \approx x*y$.

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First we start with the following Preliminaries

Def 1: <u>Semi near ring</u> :- A semi near ring $(N,+, *)$ is an algebra with two binary operations + and $*$ such that both the additive reduct $(N,+)$ and the reduct $(N, *)$ are semi rings and such that the following distributive law holds,

$$x* (y + z) \approx x*y + x*z \; , \; (x + y) *z \approx x*z + y*z.$$

Def 2: **R**<u>**egular element in a semi near ring in N**</u> :- Two elements a, b are said to be regular if $a*b+b*a+a*b=a*b$, if any two elements in N are regular then N is called regular semi near ring.

Def 3: <u>**Invertible element in a semi near ring**</u> :- two elements a , b are said to be inver tiable semi near ring N

If $a*b + b*a + a*b = a*b$ and $b*a + a*b + b*a = b*a$.

(a) $a*x*a = a$; $x*a*x = x$. (b) $a+x+a = a$; $x+a+x = x$. In this 'a' is called invertiable element in N.

Theorem 1:- A semi near ring which satisfies the identities

$$x*x \approx x\ ,\ x + x*y + x \approx x\ ,\ x + y*x + x \approx x\ ,$$

then it satisfies $x + x \approx x$, $x*y+ y*x + x*y \approx x*y$, $y*x + x*y + y*x \approx y*x$.

Proof :- Let S be a semi near ring which satisfies the above identities.

Then for $a,b \in S$,

$$(a+ a) = (a+ a)*(a+ a) = a*(a+ a) + a*(a+ a) = a*a+a *a+a*a+a*a = a+a+a+a = a+a = a$$

and

$$\begin{aligned}(a*b + b*a + a*b) &= (a*b + b*a + a*b)*(a*b + b*a + a*b)\\ &= a*b *(a*b + b*a + a*b) + b*a *(a*b + b*a + a*b) + a*b *(a*b + b*a + a*b)\\ &= (a*b + a*b*a + a*b) + (b*a*b + (b*a*b)*a + b*a*b) + (a*b + (a*b)a) + a*b)\\ &= a*b + b*a*b+ a*b = a*b.\end{aligned}$$

Also

$$\begin{aligned}(b*a + a*b + b*a) &= (b*a + a*b + b*a)*(b*a + a*b + b*a)\\ &= b*a *(b*a + a*b + b*a) + a*b* (b*a + a*b + b*a) + b*a *(b*a + a*b + b*a)\\ &= b*a*b*a + b*a*a*b + b*a*b*a + a*b*b*a + a*b*a*b + a*b*b*a + b*a*b*a + b*a*a*b + b*a*b*a\\ &= (b*a + b*a*b + b*a) + (a*b*a + a*b + a*b*a) + (b*a + b*a*b + b*a)\\ &= b*a + a*b*a + b*a = b*a.\end{aligned}$$

Remark:- From the above theorem it is observed that 'a*b' and 'b*a' are invertiable elements in N.

Lemma 2:- If N is a semi near ring which satisfies (i) $a*a = a$; (ii) $a+ a*b +a = a$, $a+ b*a +a = a$, then ‘a∗b’ is invertible in N.

Proof:- By using theorem1, we have $a*b + b*a + a*b = a*b$ and $b*a + a*b + b*a = b*a$.

Now, $a*b *b*a *a*b = a*b$ and $b*a*a*b*b*a = b*a$.

$a*b*b*a*a*b = a*b*a*b = a*b$. Also $b*a*a*b*b*a = b*a*b*a = b*a$. hence 'a*b' is invertible is N.

Theorem 3:- In a semi near ring (N,+,*) if the following identities hold

(a) $x*x \approx x$, (b) $x + x*y + x \approx x$, (c) $x + y*x + x \approx x$, then the following holds

(1) $x + x \approx x$, (2) $x*y +y*x + x*y \approx x*y$, (3) $y*x + x*y + y*x \approx y*x$, (4) $x*y*x + x*y + x*y*x \approx x*y*x$, (5) $y + y*x + y \approx y$, (6) $y + x*y + y \approx y$, (7) $y*x*y + y*x + y*x*y \approx y*x*y$, (8) $y*x*y + x*y*x + y*x*y \approx y*x*y$,

(9) $x*y*x + y*x*y + x*y*x \approx x*y*x$, (10) $x + x*y*x + x \approx x$,

(11) $y + y*x*y + y \approx y$, (12) $x + y*x*y + x \approx x$, (13) $y + x*y*x + y \approx y$.

Proof:- by using Theorem(1) identities 1,2 and 3 are clear to prove (4) we have,

$$(a*b*a + a*b + a*b*a) = (a*b*a + a*b + a*b*a)*(a*b*a + a*b + a*b*a) = (a*b*a + a*b + a*b*a)(a*b*a + a*b + a*b*a)$$

$$= a*b*a*a*b*a + a*b*a*a*b + a*b*a*a*b*a + a*b*a*b*a + a*b*a*b + a*b*a*b*a + a*b*a*a*b*a + a*b*a*a*b + a*b*a*a*b*a$$

$$= a*b*a + a*b + a*b*a + a*b*a + a*b + a*b*a +a*b*a + a*b + a*b*a$$

$$= a*b*a + a*b*a*b + a*b*a + a*b*a + a*b*a*b + a*b *a + a*b*a + a*b*a*b + a*b*a$$

$$= a*b*a + a*b*a + a*b*a = a*b*a + a*b*a = a*b*a.$$

(5) $y + y*x + y \approx y$ as $b + b*a + b = (b + b*a + b)*(b + b*a + b) = b$. (6) $y + x*y + y \approx y$ as $b + a*b + b = (b + a*b + b)*(b + a*b + b) = b$.

(7) $y*x*y + y*x + y*x*y \approx y*x*y$

$$b*a*b + b*a + b*a*b = (b*a*b + b*a + b*a*b)(b*a*b + b*a + b*a*b)$$

$$= b*a*b*b*a*b + b*a*b*b*a + b*a*b*b*a*b + b*a*b*a*b + b*a*b*a + b*a*b*a*b+ b*a*b*b*a*b + b*a*b*b*a + b*a*b*b*a*b$$

$$= b*a*b + (b*a*b)*a + (b*a*b) + (b*a*b) + b*a + (b*a*b) + b*a*b*a*b + b*a*b*a + b*a*b$$

$$= b*a*b + b*a +b*a*b + b*a*b + (b*a*b)a + b*a*b$$

$$= b*a*b + b*a + b*a*b = b*a*b.$$

(8) $y*x*y + x*y*x + y*x*y \approx y*x*y$

$b*a*b + a*b*a + b*a*b = (b*a*b + a*b*a + b*a*b)*(b*a*b + a*b*a + b*a*b)$

$= b*a*b *b*a*b + b*a*b*a*b \ *a + b*a*b*b*a*b + a*b*a*b*a*b + a*b*a.a*b*a + a*b*a.b*a*b + b*a*b.b*a*b + b*a*b*a*b*a + b*a*b.b*a*b$

$= b*a*b + (b*a*b)a + (b*a*b) + a*b + (a*b)*a + a*b + b*a*b + (b*a*b)*a + (b*a*b)$

$= b*a*b + a*b + b*a*b = b*a*b$. (9) $x*y*x$

$+ y*x*y +x*y*x \approx x*y*x$

$a*b*a + b*a*b + a*b*a = (a*b*a + b*a*b + a*b*a)(a*b*a + b*a*b + a*b*a)$

$= a*b*a*a*b*a + a*b*a*b*a*b + a*b*a*a*b*a + b*a*b*a*b*a + b*a*b*b*a*b + b*a*b*a*b*a$

$+ a*b*a*a*b*a + a*b*a*b*a*b + a*b*a*a*b*a$

$= a*b*a + (a*b*a)*b + a*b*a + b*a + (b*a)* \ b + (b*a) + a*b*a + (a*b*a)*b + a*b*a$

$= a*b*a + b*a + a*b*a = a*b*a + b*a *b*a + a*b*a = a*b*a + b*(a*b*a) + a*b*a = a*b*a$,

as $x + y*x + x \approx x$.

(10) $x + x*y*x + x \approx x$ as

$a + a*b*a + a = (a + a*b*a + a)(a + a*b*a + a)$

$= a + a*b*a + a + a*b*a + a*b*a + a + a*b*a + a$

$= a + a*b*a+ a + a*b*a + a + a*b*a+ a$

$= a + a*b*a+ a + a*b*a+ a = a + a*b*a+ a = a$

(11) $y + y*x*y + y \approx y$

$b + b*a*b + b = b + b*a*b + b* b + b*a*b + b$

$= b + b*a*b+ b + b*a*b + b*a*b*a*b + b*a*b + b + b*a*b+ b$

$= b + b*a*b + b*a*b + b + b*a*b+ b$

$= b + b*a*b+ b + b*a*b+ b = b + b*a*b+ b = b$

(12) $x + y*x*y + x \approx x$ as

$a + b*a*b + a = (a + b*a*b + a)(a + b*a*b + a)$

$= a + a*b + a + b*a + b*a + b*a + a + a*b + a$

$= a + b*a + a + a*b + a = a + a*b + a = a.$

(13) $y + x*y*x + y \approx y$

$b + a*b*a + b = (b + a*b*a + b)(b + a*b*a + b)$

$= b + b*a + b + a*b + a*b*a + a*b + b + b*a + b$

$= b + a*b + a*b*a + a*b + b + b*a + b$

$= b + a*b + (a*b)*a + a*b + b + b*a + b$

$= b + a*b + b + b*a + b = b + a*b + b = b$

Theorem 4:- In a semi near ring $(N,+,*)$ if the following identities hold

(a) $x*x \approx x$, (b) $x + x*y + x \approx x$, (c) $x + y*x + x \approx x$, then the following are equivalent.

(1) $x*y +y*x + x*y \approx x*y$, (2) $y*x + x*y + y*x \approx y*x$,

Theorem 5:- In a semi near ring $(N,+,*)$ if the following identities hold

(a) $x*x \approx x$, (b) $x + x*y + x \approx x$, (c) $x + y*x + x \approx x$, then the following are equivalent.

(1) $y*x*y + x*y*x + y*x*y \approx y*x*y$, (2) $x*y*x + y*x*y + x*y*x \approx x*y*x$,

Proof:- proof is Obvious

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Lemma 6:- If N is a semi near ring which satisfies (i) $a*a = a$; (ii) $a + a*b + a = a$, $a + b*a + a = a$, then '$a*b$' is invertible in N.

Proof:- By using theorem1, we have $a*b + b*a + a*b = a*b$ and $b*a + a*b + b*a = b*a$.
Now, $a*b * b*a * a*b = a*b$ and $b*a * a*b * b*a = b*a$.
$a*b*b*a*a*b = a*b*a*b = a*b$. Also $b*a*a*b*b*a = b*a*b*a = b*a$. hence '$a*b$' is invertible is N.

Now we have obtained certain results on regular semi near ring using Green's equivalences L , R , D which are defined by

$(a, b) \in L^{+} \Leftrightarrow a + b = a ; b + a = b$, $(a, b) \in L^{*} \Leftrightarrow a * b = a ; b * a = b$

$(a, b) \in R^{+} \Leftrightarrow a + b = b ; b + a = a$ $(a, b) \in R^{*} \Leftrightarrow a * b = b ; b * a = a$

$(a, b) \in D^{+} \Leftrightarrow a + b + a = a ; b + a + b = b$. $(a, b) \in D^{*} \Leftrightarrow a * b * a = a ; b * a * b = b$

Lemma 7:- In a regular semi near ring N if for a, b $\in$S with $(a, a*b) \in L^{+}$, then $a*b + a + a*b = a*b$,

as $(a, a*b) \in L^{+} \Rightarrow a + a*b = a ; a*b + a = a*b$.

Now, $a*b + a = a*b + a + a*b = a*b$, and also $a + a*b + a = a$

Lemma 8:- In a regular semi near ring N if for a, b $\in$S with $(a, b*a) \in L^{+}$, then $b*a + a + b*a = b*a$.

as $(a, a*b) \in L^{+} \Rightarrow a + b*a = a ; b*a + a = b*a$.

Now, $b*a + a = b*a + a + b*a = b*a$, and also $a + b*a + a = a$

Lemma 9:- In a regular semi near ring N if for a, b $\in$S with (b, a$*$b) $\in L^+$, then b + a$*$b = b , as a$*$b + b = a$*$b. a$*$b + b + a$*$b = a$*$b and also b + a$*$b + b = b.

hence (b, a$*$b) $\in L^+ \Rightarrow$ a$*$b + b + a$*$b = a$*$b.

Lemma 10:- In a regular semi near ring N if for a, b $\in$S with (b, b$*$a) $\in L^+$, then b + a$*$b = b, as b$*$a + b = b$*$a. b$*$a + b = b$*$a + b + b$*$a = b$*$a.

Hence b$*$a + b + b$*$a = b$*$a , and b + b$*$a +b = b$*$a.

From the above it is observed that the relation 'L' is an equivalence relation which is compatiable under $*$. It is also observed that every element in 'L' has an inverse in N

Theorem 11:- In a semi near ring (N, +,$*$) the following are equivalent, (1) x + x$*$y +x $\approx$ x ; x$*$y + x + x$*$y $\approx$ x$*$y.

(2) D^+ is the least distributive congruence such that x$*$y + y$*$x + x$*$y $\approx$ x$*$y.

Proof :- (1) $\Rightarrow$ (2), a$*$b + b$*$a + a$*$b = a$*$b$*$a + (a$*$b$*$a)$*$b + a$*$b$*$a = a$*$b$*$a = a$*$b.

Also, a$*$b + b$*$a + a$*$b = (a$*$b$*$a)b + (a$*$b$*$a) + (a$*$b$*$a)b = (a$*$b$*$a)b = a$*$b.

Or

(1) $\Rightarrow$ (2), as, a + a$*$b + a = a, $\Rightarrow$ a L^+ a$*$b.

and a$*$b + a + a$*$b = a$*$b and also a + a$*$b = a$*$b and a$*$b + a = a. $\Rightarrow$ a R^+a$*$b.

hence a D^+ a$*$b $\Rightarrow$ b$*$a D^+ a$*$b $\Rightarrow$ a$*$b + b$*$a + a$*$b = a$*$b; b$*$a + a$*$b + b$*$a = b$*$a.

(2) $\Rightarrow$ (1), a$*$b + b$*$a + a$*$b = a$*$b $\Rightarrow$ a$*$b D^+ b$*$a, Claim :- a + a$*$b + a = a ; a$*$b + a + a$*$b = a$*$b

Since a$*$b D^+ b$*$a $\Rightarrow$ a. a$*$b D^+ a$*$b$*$a $\Rightarrow$ a$*$b D^+ a$*$b$*$a $\Rightarrow$ a$*$b + a$*$b$*$a +a$*$b = a$*$b.

Hence a + a$*$b + a = a.

Now claim :- a$*$b + a + a$*$b = a$*$b. Since a$*$b D^+ b$*$a

$\Rightarrow$ a$*$ a$*$b D^+ a$*$b$*$a $\Rightarrow$ a$*$b D^+ a$*$b$*$a $\Rightarrow$ a$*$b + a$*$b$*$a + a$*$b = a$*$b $\Rightarrow$ a$*$b$*$a$*$b +

$a*b*a + a*b*a*b = a*b*a*b$

$\Rightarrow (a*b*a)*b + a*b*a + (a*b*a)*b = (a*b*a)*b$. hence (2)

$\Rightarrow$ (1) holds.

hence (1) and (2) are equivalent.

Def 4: A Semi near ring (N,+, $*$) is called commutative if $x + y \approx y + x$ and $x*y \approx y*x$.

Def 5:- A Semi near ring with set of Idempotents is called **Boolean semi nearring**

Def 6:- A commutative semi near ring with set of Idempotents is called a **Boolean semilattice**.

Lemma 12:- In a commutative semi near ring N, the following holds

(1) $x*y + x + x*y \approx x$, as

$a*b + a + a*b = a*b + a = a*b + a*a = a*(b + a) = a + a*(b + a) + a$

(2) $y*x + y + y*x \approx y*x$, as

$b*a + b + b*a = b*a + a = a + b*a + a*a = a + (b + a)*a + a*a = a*a + (b + a)*a + a*a = a + (b + a)*a + a = a$.

Theorem 13:- For **Boolean semi nearring** N, the following are equivalent,

(1) $x*x \approx x$; $x + x*y + x \approx x$

(2) $x + x \approx x$; $y + y*x + y \approx y$.

Proof:- (1) $\Rightarrow$ (2)

Let (1) holds,

Now, $a + a = (a + a)*(a + a) = a + a*(a + a) + a = a$.

also, $b + b*a + b = b + a*b + b = b + b*(b + a) + b = b$.

(2) $\Rightarrow$ (1) , Now $(a + a) = (a + a)*(a + a) = a + a*(a + a) + a = a$.

But $a + a = a \Rightarrow a*a = a$, and also $a + a*b + a = a + a*(a + b) + a = a$

Hence (1) and (2) are equivalent.

Theorem 14:- For **Boolean semi nearring** N, the following are equivalent

(1) $x*y*x + x*y + x*y*x \approx x*y*x$

(2) $x*y + y*x + x*y \approx x*y$.

Proof:- It is clear.

Theorem 15:- For **Boolean semi nearring** N the following are equivalent

(1) $x*y + x + x*y \approx x*y$; $y + x*y + y \approx y$. (2) $(x + y)*(y + x)*(x + y) \approx x*y$.

Proof:- $(1) \Rightarrow (2)$

$$(a+b)*(b+a)*(a+b) = (a*b + a*a + b*b + b*a)*(a+b)$$

$$= a*b*a + a*b*b + a*a*a + a*a*b + b*b*a + b*b*b + b*a*a + b*a*b$$
$$= a*b*a + a*b + a + a*b + b*a + b + b*a + b*a$$
$$= a*b + a + a*b + b + a*b$$
$$= a*b + b + a*b = a*b$$

$(2) \Rightarrow (1)$

Now $a*b + a + a*b = a*b + a + a*b + a*b + a + a*b$
$$= (a*b + a*a + b*b + b*a)*(a+b)$$
$$= (a+b)*(b+a)*(a+b) = a*b.$$

Theorem 16:- For **<u>Boolean semi nearring</u>**
N the following are equivalent,

(1) $(x+y)*(y+x)*(x+y) \approx x*y$
(2) $x*y + x + x*y \approx x*y$; $y + x*y + y \approx y$. (3) $x*y +$
$y*x + x*y \approx x*y$

Proof:- proof is Obvious.

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