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Original Article

Non-Degeneracy and Uniqueness of Periodic Solutions for Fifth-Order Non-Linear Differential Equations

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Abstract - *In this paper, the non-degeneracy of a class of fifth-order linear differential equations are investigated by Wirtinger inequality. In addition, the non-degenerate results are used to obtain the existence and uniqueness of periodic solutions for the fifth-order non-linear differential equations with super-linear terms.*

Keywords - *Existence and Uniqueness, Non-degeneracy, Periodic solution, Super-linear.*

1. Introduction

Since Isaac Newton and Johannes Kepler investigated motion of planets in the seventeenth century, the issues of the existence of periodic solutions have gained a great deal of scholars' attention. Existence and uniqueness, as two important properties of periodic solutions, reflect the regularity and balance of the development of things and play an important role in the modelling of practical problems such as neural networks, ecology and many other fields. [1,2] Consequently, it is of great significance to analyse the existence and uniqueness of periodic solutions for differential equations.

For a considerable duration, many researchers have devoted themselves in studying the existence and uniqueness of periodic solutions for differential equations, including sub-linear differential equations, [3] semi-linear differential equations [4] and super-linear differential equations. [5] It can be seen from the foregoing papers that most of them deal with the existence of periodic solutions, while there exists relatively few works on the existence and uniqueness of periodic solutions. An essential approach to the investigation of the existence and uniqueness of periodic solutions is to view the differential equation as a linear perturbation, and thereby dividing the problem into two parts: one is to consider the non-degeneracy of periodic solutions for the linear equations, and the other is to investigate how such periodic solutions appear under the perturbation. Further, the non-degenerate results for linear differential equations are applied to study the existence and uniqueness of periodic solutions to non-linear differential equations. And the non-degeneracy of the equations here means that it has no non-trivial solutions.

The notion of non-degeneracy for linear differential equations, proposed by Lasota and Opial, [6] can be traced back to 1964. They considered the non-degeneracy of the following differential equation

$$X''(t)+a(t)x(t)=0, \tag{1}$$

Where $a(t) \in L^1(\mathbb{R}\ /T\mathbb{Z})$, $T$is a positive constant. The non-degeneracy of equation (1) signifies that (1) has only a trivial solution $x(t) \equiv 0$. Afterwards, Fonda and Mawhin [7] in 1989 dealt with the existence of periodic solutions for a non-linear second-order differential equation by the non-degenerate results of equation (1). Subsequently, Ortega and Zhang [8] improved the results of [6] in 2005. They obtained the non-degeneracy for equation (1) if $a(t) \in L^p(\mathbb{R}/T\mathbb{Z})$ with $1 \leq p \leq +\infty$. Besides, they employed the results of non-degeneracy of equation (1) to prove the existence and uniqueness of periodic solutions for the following second-order super-linear differential equation

$$X''(t)+[x]^{\sigma}_{+} = h(t)+S,$$

Where $\sigma \in (1,\infty)$, $[x]_+ = max\{x,0\}$, $h \in L^1(\mathbb{R}/T\mathbb{Z})$ and $\int_0^T h(t)dt = 0$, $s$is a constant.

The above papers pay attention to periodic solutions for the second-order differential equations. In the past few decades, many scholars have concentrated on periodic solutions for high-order differential equations and achieved noteworthy achievements. [9-12] Li and Zhang [9] in 2009 considered the non-degeneracy under the conditions of periodic boundary values for the following fourth-order linear differential equation

$$x^{(4)}(t) = a(t)x(t), \tag{2}$$

Where $a(t) \in L^p(\mathbb{R}/T\mathbb{Z})$. They further established the existence and uniqueness of periodic solutions for a fourth-order super-linear differential equation by utilizing the non-degenerate results of (2). After that, Torres et al. [10] in 2013 studied the non-degeneracy for 2n-order linear differential equation

$$\begin{cases} x^{(2n)}(t) + \sum_{m=1}^{2n-1} a_m x^{(m)}(t) = a(t)x(t),\ t,\ x \in \mathbb{R} \\ x^{(i)}(0) = x^{(i)}(T), i = 0,1,2,\cdots,2n-1 \end{cases}, \tag{3}$$

Where $a_m \in \mathbb{R}$, $a(t) \in L^p(\mathbb{R}/T\mathbb{Z})$. In addition, they considered the existence and uniqueness of periodic solutions for the associated 2n-order non-linear differential equations with a super-linear term by the non-degenerate results of equation (3).

The differential equations discussed above are all even-order and there exists rather few odd-order differential equations. More recently, in 2022, Yao et al. [13] handled with the non-degeneracy for the following third-order linear differential equation

$$x'''(t) + a_2 x''(t) + a_1 x'(t) = a_0(t)x(t),$$

Where $a_1, a_2 \in \mathbb{R}$, $a_0(t) \in L^1(\mathbb{R}/T\mathbb{Z})$. And they demonstrated the existence and uniqueness of periodic solutions for third-order non-linear differential equations under super-linear conditions.

Inspired by the papers of [13,14], in this paper, the non-degeneracy for the fifth-order linear differential equations are first considered

$$x^{(5)}(t) + a_4 x^{(4)}(t) + a_3 x'''(t) + a_2 x''(t) + a_1 x'(t) = a_0(t)x(t), \tag{4}$$

Where $a_i \in \mathbb{R}, i = 1,2,3,4, a_0(t) \in L^p(\mathbb{R}/T\mathbb{Z})$. In addition, with the help of the results of non-degeneracy of equation (4), the existence and uniqueness of periodic solutions for the following super-linear differential equation are discussed

$$x^{(5)}(t) + a_4 x^{(4)}(t) + a_3 x'''(t) + a_2 x''(t) + a_1 x'(t) = f(x(t)) - s + h(t), \tag{5}$$

Where $s \in \mathbb{R}$, $h \in L^1(\mathbb{R}/T\mathbb{Z})$, $\int_0^T h(t)dt = 0$, $f \in C(\mathbb{R}, \mathbb{R})$ is a monotonic function and the non-linear term $f(x)$ grows super-linearly as $x \to \infty$. Finally, an example is given to verify the validity of the theorem.

It is worth mentioning that this work is an improvement and generalization of [13]: Theorem 2.1 in [13] does not consider the case of $\bar{a}_0 := \frac{1}{T}\int_0^T a_0(t)dt < 0$. Theorem 2.1 in this paper investigates the non-degeneracy as $\bar{a}_0 \neq 0$ (i.e. $\bar{a}_0 > 0$ and $\bar{a}_0 < 0$), which has a wider range of applications.

2. Non-Degeneracy of Equation (4)

In this section, the conditions of non-degeneracy for equation (4) is given by Wirtinger inequality. For convenience, we define

$$\|x'\|_p := \left(\int_0^T |x'(t)|^p dt\right)^{\frac{1}{p}}, \quad \|a_0\| := \max_{t\in[0,T]} |a_0(t)|,$$

where $1 \le p \le +\infty$. Next, we recall two lemmas from [15,16].

Lemma 2.1. (Wirtinger inequality [15, Theorem 1.3]) Let $x \in \widetilde{H}_T^M(\mathbb{R})$. Then we have

$$\int_0^T |x(t)|^2 dt \le C_M \int_0^T \left|x^{(M)}(t)\right|^2 dt,$$

where $\widetilde{H}_T^M(\mathbb{R}) := \left\{x \in H_{loc}^M(\mathbb{R}), x(t+T) = x(t), \int_0^T x(t)dt = 0, \forall t \in \mathbb{R}\right\}$, and $C_M := \left(\frac{T}{2\pi}\right)^{2M}$ is the best constant for this inequality.

Lemma 2.2. ([15, Lemma 2.3]) Let $x \in C_T^n(\mathbb{R})$. Then we have

$$\|x'\|_p \le \left(\frac{T}{\pi_p}\right)^{(n-1)} \|x^{(n)}\|_p,$$

where $\pi_p = 2\int_0^{\frac{p-1}{p}} \left(1 - \frac{s^p}{p-1}\right)^{-\frac{1}{p}} ds = \frac{2\pi(p-1)^{-\frac{1}{p}}}{p\, sin\left(\frac{\pi}{p}\right)}$ with $p > 1$, $C_T^n(\mathbb{R}) := \{x \in C^n(\mathbb{R}) : x(t+T) = x(t), \forall t \in \mathbb{R}\}$.

In particular, if $p = 2$, then $\pi_2 = \pi$, that is, $\|x'\|_2 \le \left(\frac{T}{\pi}\right)^{(n-1)} \|x^{(n)}\|_2$.

We obtain equation (4) is non-degenerate employing the above two lemmas.

Theorem 2.1. Assume that the coefficient $a_0(t) \in L^p(\mathbb{R}/T\mathbb{Z})$ satisfying $\bar{a}_0 \neq 0$. Besides, suppose that one of the following conditions holds:

(i) For $a_4 \neq 0$, one has

$$\|a_0\| < \left(|a_4| - |a_2|\left(\frac{T}{\pi}\right)^2\right)\left(\frac{2\pi}{T}\right)^4 \tag{6}$$

(ii) For $a_4 = 0$, one has

$$\|a_0\| < |a_2|\left(\frac{2\pi}{T}\right)^2 \tag{7}$$

Then equation (4) is non-degenerate in$x \in W_T^{5,p}(\mathbb{R})$, where $W_T^{5,p}(\mathbb{R}) := \left\{W_{loc}^{5,p}(\mathbb{R}) : x(t+T) \equiv x(t), \forall t \in \mathbb{R}\right\}$,$W_{loc}^{5,p}(\mathbb{R}) := \left\{x \middle| D^i x \in L^p(\mathbb{R}), i = 0,1,\cdots,5\right\}$ is a Sobolev space and $D^i x$ denotes the i-th order weak derivative of x.

Proof. Assume that$x \in W_T^{5,p}(\mathbb{R})$ is a non-trivial solution of equation (4). Let us write $x = \bar{x} + \tilde{x}$, where $\tilde{x} := x - \bar{x}$ and $\int_0^T \tilde{x}(t)\, dt = 0$. Substituting $x = \bar{x} + \tilde{x}$ into (4), one obtains

$$\tilde{x}^{(5)}(t) + \sum_{i=1}^4 a_i \tilde{x}^{(i)}(t) = a_0(t)\bar{x} + a_0(t)\tilde{x}(t) \tag{8}$$

Integrating equation (8) over $[0, T]$, one arrives at

$$\int_0^T \tilde{x}^{(5)}(t)dt + \sum_{i=1}^4 a_i \int_0^T \tilde{x}^{(i)}(t)dt = \bar{x}\int_0^T a_0(t)\, dt + \int_0^T a_0(t)\tilde{x}(t)dt.$$

Since $\int_0^T \tilde{x}^{(5)}(t)dt = 0$, $\int_0^T \tilde{x}^{(i)}(t)dt = 0$, $i = 1,2,3,4$ and $\bar{a}_0 \neq 0$, it is clear that

$$\bar{x} = -\frac{\int_0^T a_0(t)\tilde{x}(t)dt}{\bar{a}_0 T} \tag{9}$$

Multiplying equation (8) by $(\bar{x} - \tilde{x}(t))$ and integrating it over $[0, T]$, one has

$$\bar{x}\int_0^T \tilde{x}^{(5)}(t)dt - \int_0^T \tilde{x}^{(5)}(t)\tilde{x}(t)dt + \sum_{i=1}^4 a_i \bar{x}\int_0^T \tilde{x}^{(i)}(t)dt$$
$$-\sum_{i=1}^4 a_i \int_0^T \tilde{x}^{(i)}(t)\tilde{x}(t)dt = |\bar{x}|^2\int_0^T a_0(t)\, dt - \int_0^T a_0(t)|\tilde{x}(t)|^2 dt. \tag{10}$$

Besides, employing integration by parts, one yields

$$\int_0^T \tilde{x}^{(4)}(t)\tilde{x}(t)dt = \int_0^{T\int} \left|\tilde{x}'||^2 d\right|$$

$$\int_0^{T\int \int_0^T |\tilde{x}'(t)|^2 dt} \tilde{x}'d$$

$$\int_0^T \tilde{x}^{(i)}(t)\tilde{x}(t)dt = 0,\ i = 1,3,5. \tag{11}$$

Substituting (11) into (10), and it follows from $\tilde{x}(0) = \tilde{x}(T)$ that

$$a_4 \int_0^{T\bar{\int}_0 |\bar{x}|^2 \int_0^T a_0(t)|\tilde{x}(t)|^2 dt_2 \int_0^T |\tilde{x}'(t)|^2 dt} \left|\tilde{x}'||^2 d\right| \tag{12}$$

In what follows, depending on the value of a_4, the following three cases are considered:

Case 1: If $a_4 > 0$, it can be obtained from $\bar{a}_0 > 0$, Lemmas 2.1 and 2.2 that

$$a_4 \int_0^{T\int \int_0^T a_0(t)|\tilde{x}(t)|^2 dt_2 \int_0^T |\tilde{x}'(t)|^2 dt} \left|\tilde{x}'||^2 d\right|$$
$$\le \int_0^T |a_0(t)||\tilde{x}(t)|^2 dt + |a_2|\int_0^T |\tilde{x}'(t)|^2 dt$$
$$\le \left(\|a_0\|\left(\frac{T}{2\pi}\right)^4 + |a_2|\left(\frac{T}{\pi}\right)^2\right)\int_0^{T\int} \left|\tilde{x}'||^2 d\right| \tag{13}$$

From (6) one gets $\int_0^{T\int} \left|\tilde{x}''||^2 d\right|$, i.e. $\tilde{x}'(t) \equiv c$, $c$is a constant. Since $\int_0^T \tilde{x}'(t)dt = \tilde{x}(T) - \tilde{x}(0) = 0$, we have$\tilde{x}'(t) \equiv c \equiv 0$, that is $\tilde{x}(t) \equiv c_1$, $c_1$is a constant. And because $\int_0^T \tilde{x}(t)dt = 0$, it is clear that $\tilde{x}(t) \equiv c_1 \equiv 0$. Combining with (9) we approach $\bar{x} = 0$. Therefore, we have $x(t) = \bar{x} + \tilde{x}(t) \equiv 0$ which contradicts the assumption. Thus, equation (4) is non-degenerate.

Case 2: If $a_4 < 0$, multiplying both sides of (12) by -1, one obtains

$$-a_4 \int_0^{T\bar{\int}_0|\bar{x}|^2 \int_0^T a_0(t)|\tilde{x}(t)|^2 dt_2 \int_0^T |\tilde{x}'(t)|^2 dt} \left|\tilde{x}'||^2 d\right|.$$

Similar to (13), from $\bar{a}_0 < 0$, Lemmas 2.1 and 2.2, one sees that

$$|a_4| \int_0^{T\int \left(\|a_0\|\left(\frac{T}{2\pi}\right)^4 + |a_2|\left(\frac{T}{\pi}\right)^2\right) \int_0^{T\int} \left|\tilde{x}'||^2 d\right|} \left|\tilde{x}'||^2 d\right|.$$

It is evident from (6) that $\int_0^{T\int} \left|\tilde{x}'||^2 d\right|$. Analogous to the proof of Case 1, one obtains that equation (4) is non-degenerate.

Case 3: If $a_4 = 0$, it follows from (12) that

$$-a_2 \int_0^T \left|\tilde{x}'(t)\right|^2 dt = -\bar{a}_0 T|\bar{x}|^2 + \int_0^T a_0(t)|\tilde{x}(t)|^2 dt.$$

Similar to the discussion of Case 1 and Case 2, one concludes

$$|a_2| \int_0^T \left|\tilde{x}'(t)\right|^2 dt \le \|a_0\| \left(\frac{T}{2\pi}\right)^2 \int_0^T \left|\tilde{x}'(t)\right|^2 dt,$$

that is,

$$\left(\|a_0\| \left(\frac{T}{2\pi}\right)^2 - |a_2|\right) \int_0^T \left|\tilde{x}'(t)\right|^2 dt \ge 0.$$

According to (7) one has $\int_0^T |\tilde{x}'(t)|^2 dt = 0$, i.e. $\tilde{x}'(t) \equiv c_2$, c_2 is a constant. And since $\int_0^T \tilde{x}'(t)dt = 0$, one gets $\tilde{x}'(t) \equiv c_2 \equiv 0$. One has $\bar{x} = 0$ due to (9). Consequently, one sees that $x(t) = \bar{x} + \tilde{x}(t) \equiv 0$ contradicting the assumption. And hence it is known that (4) is non-degenerate.

3. Uniqueness of Periodic Solution for Equation (5)

In this section, the existence and uniqueness of periodic solutions for super-linear differential equation (5) are established by means of the non-degenerate results of equation (4). In fact, integrating equation (5) over $[0, T]$, one obtains

$$\int_0^T x^{(5)}(t)dt + \sum_{i=1}^4 a_i \int_0^T x^{(i)}(t)dt = \int_0^T f(x(t))dt - sT + \int_0^T h(t)dt^{\cdot}$$

Since $x(0) = x(T)$ and $\int_0^T h(t)dt = 0$, it is clear that

$$s = T^{-1} \int_0^T f(x(t))dt = f(x(t_*)) \in \Re(f) := \{f(u): u \in \mathbb{R}\}^, \tag{14}$$

where $t_* \in (0, T)$. Consider the fifth-order differential equation

$$y^{(5)}(t) + a_4 y^{(4)}(t) + a_3 y'''(t) + a_2 y''(t) + a_1 y'(t) + g(t, y) = q(t)\,, \tag{15}$$

where $a_i, i = 1,2,3,4$are constants, $g \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$, and for all t and y one has $g(t + T, y) = g(t, y)$.

Define the measurable functions

$$\mu_+(t) = \underset{y \to -\infty}{limsup}\, g(t, y),\ \ \mu_-(t) = \underset{y \to +\infty}{liminf}\, g(t, y),\ \ t \in \mathbb{R}.$$

It is straightforward to see that $\mu_+, \mu_- \in (\mathbb{R}, \mathbb{R} \cup \{-\infty, \infty\})$. Let

$$L_y = y^{(5)}(t) + a_4 y^{(4)}(t) + a_3 y^{'^{'^{2}}{}^{'^{'^{1}}}}\ .$$

In order to derive the existence and uniqueness of periodic solutions to equation (5), the following definitions and lemmas are first provided.

Definition 3.1. ([8]) Given $\sigma \in [1, \infty)$, and $\mathcal{A}, \mathcal{B} \in [0, \infty)$, we say that $f$satisfies the condition $\mathcal{C}(\sigma; \mathcal{A}, \mathscr{B})$ if

$$\left|\frac{f(x_1) - f(x_2)}{x_1 - x_2}\right|^\sigma \le \mathcal{A}\left(\frac{f(x_1) + f(x_2)}{2}\right) + \mathscr{B},$$

for every $x_1, x_2 \in \mathbb{R}$, $x_1 \neq x_2$.

Lemma 3.1. ([14], Theorem 1.1) Assume that $g(t, y)$ is bounded below for $y \ge 0$ and bounded above for $y \le 0$. Besides, suppose that the following conditions hold:

(P_1) The solutions of $L_y = 0$ are constants.

(P_2) There exists α,β such that $|g(t, y)| \le g(t, y) + \alpha|y| + \beta$ for all $(t, y) \in \mathbb{R} \times \mathbb{R}$.

(P_3) $\int_0^T \mu_-(t)dt < \int_0^T q(t)dt < \int_0^T \mu_+(t)dt$.

Then there exists a constant $\varepsilon > 0$ such that equation (15) has at least one periodic solution if $\alpha \le \varepsilon$.

From Theorem 2.1 and Lemma 3.1, the following conclusions are obtained.

Theorem 3.1. Assume that the following conditions hold:
$(P_4) f: \mathbb{R} \to \mathbb{R}$ is bounded above for $x \geq 0$and bounded below for $x \leq 0$,$s \in int\, \Re\,(f)$.
(P_5) There exists $a, b \geq 0$ such that $|f(x)| \leq f(x) + a|x| + b$.
(P_6) $f \in \mathcal{C}(\sigma; \mathcal{A}, \mathcal{B})$is a strictly increasing function and f satisfies one of the following conditions:

(j)
$$\mathcal{A}s + \mathscr{B} < \frac{(M_1(\sigma,n))^\sigma}{T}, \tag{17}$$
where $M_1(\sigma, n) := \left(|a_4| - |a_2|\left(\frac{T}{\pi}\right)^2\right)\left(\frac{2\pi}{T}\right)^4$.

(jj)
$$\mathcal{A}s + \mathscr{B} < \frac{(M_2(\sigma,n))^\sigma}{T}, \tag{18}$$
where $M_2(\sigma, n) := |a_2|\left(\frac{2\pi}{T}\right)^2$.
Then there exists a constant $c_0 > 0$ such that equation (5) has a unique periodic solution if $a \leq c_0$.

Proof. Step 1. Claim that equation (5) has at least one periodic solution.
Comparing equation (5) with (15), one yields
$$g(t, y) = -f(x(t)),\ \ q(t) = -s + h(t). \tag{19}$$

It is clear that (P_2) holds. And it follows from equation (14) and (19) that
$$-sT = -\int_0^T f(x(t))dt = \int_0^T g(t, y)dt = \int_0^T q(t)dt.$$

Thus, (P_3) is satisfied and one only needs to show that (P_1) is held in the following.
Assume that $x(t)$ is a periodic solution of the homogeneous linear differential equation L_y, then one gets
$$x^{(5)}(t) + a_4 x^{(4)}(t) + a_3 x^{'2\,'1\,'}\,. \tag{20}$$

Multiplying both sides of the equation (20) by $x(t)$ and integrating it over $[0, T]$, it is evident that
$$\int_0^T x^{(5)}(t)x(t)\, dt + \sum_{i=1}^4 a_i \int_0^T x^{(i)}(t)x(t)\, dt = 0. \tag{21}$$

Similar to (11), applying integration by parts, (21) can be simplified as
$$a_4 \int_0^{T \int_2 \int_0^T |x'(t)|^2 dt = 0} \left|x^{'}||^2 d\right|.$$
From Lemma 2.2, one arrives at
$$\left(|a_2|\left(\frac{T}{\pi}\right)^2 - |a_4|\right)\int_0^{T\int} \left|x^{'}||^2 d\right|.$$

It follows from (14), (16) and (17) that $|a_2|\left(\frac{T}{\pi}\right)^2 - |a_4| < 0$. Thus $\int_0^{T\int} \left|x'||^2 d\right|$. And one obtains $x(t) \equiv c_3$,c_3 is a constant. Hence, (P_1) holds. It is not difficult to show that equation (5) has at least one periodic solution from Lemma 3.1.

Step 2. Claim that equation (5) has at most one periodic solution.
Let $x_1(t)$ and $x_2(t)$ be two different periodic solutions of equation (5) and then one has
$$x_j{}^{(5)}(t) + \sum_{i=1}^4 a_i x_j{}^{(i)}(t) = f(x_j(t)) - s + h(t),\, j = 1,2. \tag{22}$$

Integrating equation (22) over $[0, T]$, one gets
$$\int_0^T f(x_j(t))dt = sT,\, j = 1,2.$$

Let $z(t) := x_1(t) - x_2(t)$ be the difference of two solutions. Clearly, $z(t) \neq 0$.Then the difference of equation (22) gives
$$z^{(5)}(t) + \sum_{i=1}^4 a_i z^{(i)}(t) = f(x_1(t)) - f(x_2(t)). \tag{23}$$

Let $I := \{t \in \mathbb{R}: z(t) \neq 0\}$, which is a non-empty open subset of $\mathbb{R}$. The function
$$a_0(t) := \frac{f(x_1(t)) - f(x_2(t))}{x_1(t) - x_2(t)} \tag{24}$$

is well defined for all $t \in I$. It is easily seen to yield that $a_0(t) \in C(I)$. As a matter of convenience, we define $a_0(t) = 0$ on complement $J := \mathbb{R}\backslash I$. Then $a_0(t)$ is well-defined on $\mathbb{R}$. It is obvious that $a_0(t)$ is measurable. As $f(x(t))$ is strictly increasing, one has $a_0(t) > 0$ for all $t \in I$. From (16) one obtains

$$|a_0(t)|^\sigma \leq \mathcal{A}\square\left(\frac{f(x_1(t))+f(x_2(t))}{2}\right)+\mathscr{B}. \tag{25}$$

It follows from (25) that

$$\|a_0\|_\sigma^\sigma = \int_{I\cap[0,T]} |a_0(t)|^\sigma dt$$
$$\leq \int_{I\cap[0,T]}\left(\mathcal{A}\square\left(\frac{f(x_1(t))+f(x_2(t))}{2}\right)+\mathscr{B}\right)dt + \int_{J\cap[0,T]}\left(\mathcal{A}\square\left(\frac{f(x_1(t))+f(x_2(t))}{2}\right)+\mathscr{B}\right)dt$$
$$= \frac{\mathcal{A}}{2}\left(\int_0^T f(x_1(t))dt + \int_0^T f(x_2(t))dt\right)+\mathscr{B}T$$
$$= (\mathcal{A}\square s+\mathscr{B})T.$$

Thus, one deduces $\|a_0\|_\sigma \leq \left((\mathcal{A}\ \ s+\mathcal{B})T\right)^{\frac{1}{\sigma}}$. From (17) and (18), one arrives at $\|a_0\|_\sigma < M_1(\sigma,n)$ and $\|a_0\|_\sigma < M_2(\sigma,n)$, respectively. According to $\bar{a}_0 \neq 0$, one has $z(t) \equiv 0$ from Theorem 2.1, which contradicts $x_1(t) \neq x_2(t)$. Therefore, equation (5) has at most one periodic solution.

Combining the two steps, there exists a constant $c_0 > 0$ such that equation (5) has a unique periodic solution if $a \leq c_0$.
In what follows, an example is given to illustrate the results.

Example 3.1. Consider the following fifth-order super-linear differential equation

$$x^{(5)}(t)+a_4x^{(4)}(t)+a_3x'''(t)+a_2x''(t)+a_1x'(t)=\exp(x(t))-s+\sin\gamma t, \tag{26}$$

Where a_i, $i = 1,2,3,4$, s, γ are constants, $s \in (0,\frac{N}{2\pi})$, $N := max\{N_1,N_2\}$, here $N_2 := \gamma|a_2|\left(\frac{2\pi}{T}\right)^2$, $N_1 := \gamma\left(|a_4| - |a_2|\left(\frac{T}{\pi}\right)^2\right)\left(\frac{2\pi}{T}\right)^4$.

Comparing equation (5) and (26), one obtains $f(x(t)) = exp(x(t))$ and $\Re(f) = (0,\infty)$, $h(t) = sin\,\gamma\,t$, $T = \frac{2\pi}{\gamma}$. Besides, $|exp(x)| \leq exp(x)+1$, here $a = 0, b = 1$. Thus (P_4) and (P_5) hold. From [9] it is known that $f(x(t)) = exp(x(t)) \in \mathcal{C}(1;1,0)$. Hence, $\sigma = 1, \mathcal{A} = 1, \mathscr{B}\square = 0$.

$$s = \mathcal{A}\square s+\mathscr{B}\square < \frac{N_1}{2\pi} = \frac{\gamma}{2\pi}\left(|a_4| - |a_2|\left(\frac{T}{\pi}\right)^2\right)\left(\frac{2\pi}{T}\right)^4$$

and

$$s = \mathcal{A}\square s+\mathscr{B}\square < \frac{N_2}{2\pi} = \frac{\gamma}{2\pi}|a_2|\left(\frac{2\pi}{T}\right)^2$$

hold and satisfy (17) and (18). Thus (P_6) is held. From Theorem 3.1, there exists a constant $c_0 > 0$ such that equation (5) has a unique periodic solution.

4. Conclusion

In this paper, the non-degeneracy of a class of fifth-order linear differential equations with $\bar{a}_0 \neq 0$ (i.e. $\bar{a}_0 > 0$ and $\bar{a}_0 < 0$) are obtained. And the most important aspect is that the obtained results of non-degeneracy are applied to establish the existence and uniqueness of periodic solutions for the corresponding non-linear differential equations, demonstrating the significance and value of studying the non-degeneracy of equations.

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