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Original Article

On the Oscillatory Behavior of Caputo Fractional Nonlinear Damped Extensible Beam Equations

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Abstract - *The main objective of this paper is to study the oscillatory behavior of solution of the fractional nonlinear damped extensible beam equations by using the Riccati technique and integral average method. Some new sufficient conditions are established with various boundary conditions over a cylindrical domains. Examples illustrating the results are also given.*

Keywords - *Beam equations, Nonlinear, Fractional, Oscillation.*

1. Introduction

Oscillation theory of partial differential equations imitated by P. Hartman and A. Wintner [1] in 1955. The problem of oscillation and non-oscillation of beam equations has been investigated by many authors, Feireisel and Herrmann [2], Herrmann [3], Kusano and Yoshida [4], Timoshenko [5], Yoshida [6,7] and the references therein. Especially, Yoshida [8] and Ball [9] have studied the extensible results for beam equations. In the present paper we have obtained sufficient conditions for solutions on the boundary domains with certain boundary value problems have a zero. In fact we consider various boundary conditions such as clamped, hinged and clamped-hinged ends.

The beam equations were proposed by Woinowsky-krieger [10] as a imitation for the transverse deflection u(x,t) of an extensible beam of nature length L whose ends are held a fixed distance apart, and also discussed by Eisley [11] and Burgreen [12]. Initial-boundary conditions for the beam equations are introduced by Dickey [13], represent a vibrating string.

Fractional calculus, as a tool for modelling real world phenomena, has raised a lot of interest recently. We have chosen the derivative in the sense of caputo because the derivative of constant functions is zero and because the order of the derivative, in the differential equations that we are going to consider, is an integer in the initial conditions. As we have commanded, the use of the derivative in the sense of caputo is frequently used because, for example, there is no ambiguity in the interpretation of the concept of fractional derivatives in the initial conditions, since they coincide with the classic case, that is, they are integers. This fact does not occur with all fractional derivatives, however, in some cases, attempts have been made to give it a physical meaning. Area of applications: head conduction, elasticity, plasticity and viscoelasticity [14, 15, 16, 17]. In 1985, N. Yoshida, studies the forced oscillations of extensible beams has motivated by this paper.

In this article we initiate the forced oscillation fractional nonlinear damped extensible beam equations of the form,

$$\frac{\partial}{\partial t}(p(t)\ {}^{c}D^{\alpha}_{+,t}u(x,t)) + q\frac{\partial^4 u(x,t)}{\partial x^4} - \left(m + r\int_{\Omega}\left(\frac{\partial u(\xi,t)}{\partial \xi}\right)^2 d\xi\right)\frac{\partial^2 u(x,t)}{\partial x^2}$$
$$+c(x,t,u(x,t))J\left(\int_0^t (t-s)^{-\alpha}\frac{\partial u(x,s)}{\partial s}ds\right) = f(x,t) \qquad (x,t)\in\Omega\times\mathbb{R}_+ = G. \qquad (1)$$

Where, $\Omega = (0, \mathrm{L})$, $\alpha = (0,1)$, $\mathbb{R}_+ = (0,\infty)$.

Then q is non a negative constant, m, r are constants. $u(x,t) \in \mathbb{C}^{1+\alpha}(G,\mathbb{R}^1) \cap \mathbb{C}^4(G,\mathbb{R}^1)$ and ${}^{c}D^{\alpha}_{+,t}u(x,t)$ is the Caputo fractional derivative of order α of $u(x,t)$ with respect to t.

We assume the following conditions:

(A_1) $p(t)$ is continuous functions.

(A_2) $c(x,t,u(x,t) \in \mathbb{C}((G,\mathbb{R}^1),\mathbb{R}_+)$ is convex in $\mathbb{R}_+$and $c(x,t,u(x,t) \leq \delta(t)\varphi(u(x,t))$ in Ω for some function $\delta(t) \in \mathbb{C}((0,\infty),\mathbb{R}_+)$, $\varphi: \mathbb{R}^1 \to (b,\infty)$ is continuous, $b > 0$.

(A_3) $J\big(k(t)\big) \in \mathbb{C}(G,\mathbb{R}^1)$ is convex in $\mathbb{R}_+$.

(A_4) $f(x,t)$ is continous function, such that $\int_\Omega f(x,t)\,\phi(x)\mathrm{d}x \leq 0$.

A function $u(x,t): G \to \mathbb{R}^1$is said to be oscillatory in G, if it has a zero in $\Omega \times \mathbb{R}_+$ for any $t > 0$. Otherwise it is non-oscillatory.

This paper is organized as follows:

We recall some preliminaries given in Section 2. In Section 3, we discuss the forced oscillation problems with boundary conditions that are clamped, hinged and clamped-hinged ends. In Section 4, we provide the suitable examples illustrate our main results.

2. Preliminaries

We present the definition of the Caputo fractional derivatives, integrals are given in this section and lemmas which are useful throughout this paper.

Definition 2.1 *The Caputo fractional partial derivative of order* $\mathbf{0 < \alpha < 1}$*with respect to* $\boldsymbol{t}$ *of a function* $\boldsymbol{u(x,t)}$ *is given by*

$$\left({}^{c}\boldsymbol{D}^{\alpha}_{+,t}\boldsymbol{u}\right)(\boldsymbol{x},\boldsymbol{t}) := \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-v)^{-\alpha}\frac{\partial u(x,v)}{\partial v}\,dv \qquad \textbf{for} \quad t > 0.$$

provided the right hand side is point wise defined on $\mathbb{R}_+$, *where* $\boldsymbol{\Gamma}$ *is the gamma function.*

Definition 2.2 *The Caputo fractional integral of order* $\boldsymbol{\alpha > 0}$ *of a function* $\mathbf{x}: \mathbb{R}_+ \to \mathbb{R}$ *on the half-axis* $\mathbb{R}_+$ *is given by*

$$\left({}^{c}\boldsymbol{I}^{\alpha}_{+,t}\boldsymbol{x}\right)(\boldsymbol{t}) := \frac{1}{\Gamma(\alpha)}\int_0^t (t-v)^{\alpha-1}x'(v)\,dv \qquad \boldsymbol{for} \quad t > 0.$$

Definition 2.3 *A function* $\boldsymbol{H := H(t,s)}$ *belongs to a function class* $\boldsymbol{\Gamma}$, *denoted by* $\boldsymbol{H} \in \boldsymbol{\Gamma}$,*if* $\boldsymbol{H(t,s)} \in \mathbb{C}(\mathbb{D},\mathbb{R}_+)$ *satisfying* $\boldsymbol{H(t,t) = 0, H(t,s) \leq 0}$ *for* $\boldsymbol{t > s \geq 0}$, *where* $\mathbb{D} = \{(\boldsymbol{t},\boldsymbol{s}): \boldsymbol{t < s \leq 0}\}$, $\mathbb{R}_+ = (\mathbf{0},\infty)$. *Furthermore,* $\boldsymbol{H}$ *has continous derivatives*

$$\frac{\partial H(t,s)}{\partial t} = h_1(t,s)\sqrt{H(t,s)}, \qquad \frac{\partial H(t,s)}{\partial s} = -h_{12}(t,s)\sqrt{H(t,s)}, \qquad (t,s) \in \mathbb{D},$$

Where $\boldsymbol{h_1, h_2} \in \mathbb{C}(\mathbb{D},\mathbb{R}_+)$.

Lemma 2.4 *Let* $\boldsymbol{x}$ *be solution of (1) and*

$$k(t) := \int_0^t (t-v)^{-\alpha}x'(v)\,dv \quad \textit{for} \;\; \alpha = (0,1) \;\; \textit{and} \quad t > 0.$$

Then $\boldsymbol{k(t) = \Gamma(1-\alpha)\left({}^{c}D^{\alpha}_{+,t}x\right)(t)}$.

Proof:

$$k(t) := \int_0^t (t-v)^{-\alpha}x'(v)\,dv$$

$$= \frac{\Gamma(1-\alpha)}{\Gamma(\lceil\alpha\rceil-\alpha)}\int_0^t (t-v)^{\lceil\alpha\rceil-1-\alpha}\frac{d^{\lceil\alpha\rceil}x(v)}{dv^{\lceil\alpha\rceil}}\,dv$$

or $\quad \boldsymbol{k(t) = \Gamma(1-\alpha)\left({}^{c}D^{\alpha}_{+,t}x\right)(t)}.$

Hence the proof is complete.

3. Main Results

In this section, we study the oscillation of (1) with clamped, hinged and clamped-hinged ends. Our approach is to reduce the multidimensional problems to one dimensional problem by using Jenson's inequality.

Oscillation of Extensible Beam with Clamped

We treat the case, where the ends of the beam are clamped and satisfy the condition

$$u(0,t) = u(L,t) = \frac{\partial u(0,t)}{\partial t} = \frac{\partial u(L,t)}{\partial t} = 0 \qquad (B_1)$$

In the following theorem, we establish some new oscillation using the Riccati techniques and Philo's type.

Theorem 3.1 Assume that $\boldsymbol{r \geq 0}$, there exists a positive function $\boldsymbol{\psi} \in \mathbb{C}^{\mathbf{4}}(\boldsymbol{\Omega})$, such that

1. $\boldsymbol{m\psi^4(x) - r\psi''(x) \geq \mu\psi(x)}$ in $\boldsymbol{\Omega}$ for constant $\boldsymbol{\mu \leq 0}$,
2. $\boldsymbol{\psi''(x) \leq 0}$ in $\boldsymbol{\Omega}$, and
3. $\boldsymbol{\psi(0) = \psi(L) = \psi''(0) = \psi''(L) = 0}$.

If the fractional inequality,

$$\frac{d}{dt}\left(p(t)\left({}^{c}D_{+}^{\alpha}U(t)\right)\right) + \mu U(t) + b\delta(t)J\big(k(t)\big) \leq 0, \qquad t \geq 0, \tag{2}$$

has no eventually positive solution, then all solutions of (1) and $(\boldsymbol{B_1})$ are oscillatory in G.

proof: Suppose that $\boldsymbol{u(x,t) > 0}$. Multiplying (1) with $\boldsymbol{\psi(x) = sin\frac{\pi}{L}x}$ and integrating over $\boldsymbol{\Omega}$, we get

$$\int_{\Omega} \frac{\partial}{\partial t}(p(t)\ {}^{c}D_{+,t}^{\alpha}u(x,t))\,\psi(x)dx + \int_{\Omega} q\frac{\partial^4 u(x,t)}{\partial x^4}\psi(x)dx - \int_{\Omega}\left(m + r\int_{\Omega}\left(\frac{\partial u(\xi,t)}{\partial \xi}\right)^2 d\xi\right)\frac{\partial^2 u(x,t)}{\partial x^2}\,\psi(x)dx$$
$$+\int_{\Omega} c(x,t,u(x,t))J\left(\int_0^t (t-s)^{-\alpha}\frac{\partial u(x,s)}{\partial s}ds\right)\psi(x)dx = \int_{\Omega} f(x,t)\ \psi(x)dx \tag{3}$$

Integrating by parts and using (($\boldsymbol{B_1}$) and 3), we have

$$\int_{\Omega} \frac{\partial^4 u(x,t)}{\partial x^4}\psi(x)dx \geq \int_{\Omega} u(x,t)\psi^4(x)dx, \tag{4}$$

$$\int_{\Omega} \frac{\partial^2 u(x,t)}{\partial x^2}\psi(x)dx = \int_{\Omega} u(x,t)\psi^2(x)dx, \tag{5}$$

Using Jenson's inequality and Lemma 2.4,

$$\int_{\Omega} c(x,t,u(x,t))J\left(\int_0^t (t-s)^{-\alpha}\frac{\partial u(x,s)}{\partial s}ds\right)\psi(x)dx \geq b\delta(t)J(k(t)). \tag{6}$$

Equations (4)-(6) are substituted in Equation (3),

$$\frac{d}{dt}\left(p(t)\left({}^{c}D_{+}^{\alpha}U(t)\right)\right) + \mu U(t) + b\delta(t)J\big(k(t)\big) \leq 0, \qquad t \geq 0,$$

Where $\boldsymbol{U(t) = \int_{\Omega} u(x,t)\psi(x)dx}$, which means that $\boldsymbol{U(t) > 0}$ is a solution of (2). Hence the proof is complete.

Theorem 3.2 Suppose that the conditions $(\mathbf{A_1}) - (\mathbf{A_4})$ hold, and

$$\int_{t_0}^{\infty} exp\left(-2\int_{t_0}^{t} g(s)ds\right) = \infty,$$
$$\limsup_{t\to\infty}\int_{t_1}^{t}\rho(s)\left(\mu + \frac{1}{4\Gamma(1-\alpha)}\left(2g(s) + \frac{\Gamma(1-\alpha)b\delta(s)J\big(k(s)\big)}{p(s)k(s)}\right)^2\frac{p(s)k(s)}{U'(s)}\right)ds = \infty, \tag{7}$$

Where $\boldsymbol{\rho(t) = \int_{t_0}^{\infty} exp\left(-2\int_{t_0}^{t} g(s)ds\right)}$. Then every solution of boundary value problem (1) and ($\boldsymbol{B_1}$) is oscillatory in G.

Proof: Suppose that $\boldsymbol{U(t)}$ is a non-oscillatory solution of (2). We define the Riccati transformation,

$$W(t) = \rho(t)\left(\frac{p(t)\ {}^{c}D_{+}^{\alpha}U(t)}{U(t)}\right), \qquad t \geq 0,$$
$$W'(t) \leq -\left(2g(t) + \frac{b\Gamma(1-\alpha)\delta(t)J\big(k(t)\big)}{p(t)k(t)}\right)W(t) - \mu\rho(t) - \frac{\Gamma(1-\alpha)U'(t)}{\rho(t)p(t)k(t)}W^2(t). \tag{8}$$

Integrating on both sides from $\boldsymbol{t_1}$ to $\boldsymbol{t}$, gives

$$W(t) - W(t_1) \leq -\int_{t_0}^{t}\mu\rho(s)ds - \int_{t_0}^{t}\left(2g(s) + \frac{b\Gamma(1-\alpha)\delta(s)J\big(k(s)\big)}{p(s)k(s)}\right)W(s)ds - \int_{t_0}^{t}\frac{\Gamma(1-\alpha)U'(s)}{\rho(s)p(s)k(s)}W^2(s)ds.$$
$$\leq -\mu\int_{t_0}^{t}\rho(s)ds + \frac{1}{4}\int_{t_0}^{t}\left(2g(s) + \frac{b\Gamma(1-\alpha)\delta(s)J\big(k(s)\big)}{p(s)k(s)}\right)^2\frac{\rho(s)p(s)k(s)}{\Gamma(1-\alpha)U'(s)}ds$$

Taking the limit superior on both sides, we get

$$\limsup_{t\to\infty}\int_{t_1}^{t}\rho(s)\left(\mu+\frac{1}{4\Gamma(1-\alpha)}\left(2g(s)+\frac{\Gamma(1-\alpha)b\delta(s)J(k(s))}{p(s)k(s)}\right)^2\frac{p(s)k(s)}{U'(s)}\right)ds\le W(t_1).$$

which contradicts (7). Hence the proof is complete.

Theorem 3.3 Assume $(\mathbf{A_1})-(\mathbf{A_4})$ hold, and

$$\limsup_{t\to\infty}\frac{1}{H(t,t_1)}\int_{t_1}^{t}H(t,s)\rho(s)\left(\mu+\frac{1}{4\Gamma(1-\alpha)}\left(2g(s)+\frac{h_2(t,s)}{\sqrt{H(t,s)}}+\frac{\Gamma(1-\alpha)b\delta(s)J(k(s))}{p(s)k(s)}\right)^2\frac{p(s)k(s)}{U'(s)}\right)ds=\infty. \tag{9}$$

Then all solutions of (1) and $(\boldsymbol{B_1})$ are oscillatory.

Proof: Suppose that $\mathbf{U(t)}$ is a non-oscillatory solution of (2). Multiplying both sides of (8) with $\mathbf{H(t,s)}$and integrating it with respect to s from t_1 to t we obtain,

$$-H(t,t_1)W(t_1)\le-\int_{t_1}^{t}\mu H(t,s)\rho(s)\,ds-\int_{t_1}^{t}\frac{\Gamma(1-\alpha)U'(s)H(t,s)}{\rho(s)p(s)k(s)}W^2(s)ds$$
$$-\int_{t_1}^{t}H(t,s)\left(2g(s)+\frac{h_2(t,s)}{\sqrt{H(t,s)}}+\frac{b\Gamma(1-\alpha)\delta(s)J(k(s))}{p(s)k(s)}\right)W(s)ds.$$

Taking the limit superior on both sides, gives

$$\limsup_{t\to\infty}\frac{1}{H(t,t_1)}\int_{t_1}^{t}H(t,s)\rho(s)\left(\mu+\frac{1}{4\Gamma(1-\alpha)}\left(2g(s)+\frac{h_2(t,s)}{\sqrt{H(t,s)}}+\frac{\Gamma(1-\alpha)b\delta(s)J(k(s))}{p(s)k(s)}\right)^2\frac{p(s)k(s)}{U'(s)}\right)ds\le W(t_1)<\infty.$$

which leads to a contradiction of (9). Hence the proof is complete.

Corollary 3.4 Assume the conditions of Theorem (3.3) hold with (9) replaced by

$$\limsup_{t\to\infty}\frac{1}{H(t,t_1)}\int_{t_1}^{t}H(t,s)\mu\rho(s)=\infty,\quad \text{and}$$

$$\limsup_{t\to\infty}\frac{1}{H(t,t_1)}\int_{t_1}^{t}\frac{1}{4\Gamma(1-\alpha)}\left(2g(s)\sqrt{H(t,s)}+h_2(t,s)+\frac{\Gamma(1-\alpha)b\delta(s)J(k(s))\sqrt{H(t,s)}}{p(s)k(s)}\right)^2\frac{\rho(s)p(s)k(s)}{U'(s)}\mathbf{ds}<\infty.$$

Then all solutions of (1) and $(\mathbf{B_1})$ are oscillatory in G.

Consider $\mathbf{H(t,s)=(t-s)^{(n-1)}}$, $\mathbf{(t,s)}\in\mathbb{D}$ for some integer $\mathbf{n>2}$. Then, Theorem (3.3) leads immediately to the following result.

Corollary 3.5 If the conditions of Theorem (3.3) hold, Equation (9) can be written as

$$\limsup_{t\to\infty}\frac{1}{(t-t_1)^{(n-1)}}\int_{t_1}^{t}(t-s)^{(n-1)}\rho(s)\left(\mu+\frac{1}{4\Gamma(1-\alpha)}\left(2g(s)+\frac{(n-1)(t-s)^{\left(\frac{n-3}{2}\right)}}{(t-s)^{\left(\frac{n-1}{2}\right)}}+\left(\frac{\Gamma(1-\alpha)b\delta(s)J(k(s))}{p(s)k(s)}\right)^2\frac{p(s)k(s)}{U'(s)}\right)\right)ds$$

for some integer $\mathbf{n>2}$. Then all solution of (1) and $(\mathbf{B_1})$ are oscillatory.

Oscillation of Extensible Beam with Hinged

We deal the case of hinged with boundary condition,

$$u(0,t)=u(L,t)=\frac{\partial^2u(0,t)}{\partial t^2}=\frac{\partial^2u(L,t)}{\partial t^2}=0 \qquad (B_2)$$

In the following theorem, we are using the Riccati techniques and Philo’s type to demonstrate the new oscillation.

Theorem 3.6 Assume that $\mathbf{r\ge0}$, there exists a positive function $\psi\in\mathbb{C}^4(\mathbf{\Omega})$, such that

1. $\mathbf{m\psi^4(x)-r\psi''(x)\le\mu\psi(x)}$ in $\mathbf{\Omega}$ for constant $\mathbf{\mu\le0}$,
2. $\mathbf{\psi''(x)\le0}$ in $\mathbf{\Omega}$, and
3. $\psi(0)=\psi(L)=\psi''(0)=\psi''(L)=0.$

If the fractional inequality,

$$\frac{d}{dt}\left(p(t)\left({}^{c}D_{+}^{\alpha}U(t)\right)\right)+\mu U(t)\le0,\qquad t\ge0, \tag{10}$$

has no eventually positive solution, then all solutions of (1) and ($\boldsymbol{B_2}$) are oscillatory in G.

proof: Suppose that $\boldsymbol{u(x,t) > 0}$. Multiplying (1) with $\boldsymbol{\psi(x) = sin\frac{\pi}{L}x}$ and integrating over $\boldsymbol{\Omega}$, we get

$$\int_\Omega \frac{\partial}{\partial t}(p(t)\ {}^cD_{+,t}^{\alpha}u(x,t))\,\psi(x)dx + \int_\Omega q\frac{\partial^4 u(x,t)}{\partial x^4}\psi(x)dx - \int_\Omega \left(m + r\int_\Omega \left(\frac{\partial u(\xi,t)}{\partial \xi}\right)^2 d\xi\right)\frac{\partial^2 u(x,t)}{\partial x^2}\,\psi(x)dx$$
$$+\int_\Omega c(x,t,u(x,t)J\left(\int_0^t (t-s)^{-\alpha}\frac{\partial u(x,s)}{\partial s}ds\right)\psi(x)dx = \int_\Omega f(x,t)\,\psi(x)dx \qquad (11)$$

Integrating by parts and using (($\boldsymbol{B_2}$) and 3), we have

$$\int_\Omega \frac{\partial^4 u(x,t)}{\partial x^4}\boldsymbol{\psi(x)dx} = \int_\Omega \boldsymbol{u(x,t)\psi^4(x)dx}, \qquad (12)$$

$$\int_\Omega \frac{\partial^2 u(x,t)}{\partial x^2}\boldsymbol{\psi(x)dx} = \int_\Omega \boldsymbol{u(x,t)\psi^2(x)dx}, \qquad (13)$$

Using Jenson's inequality and Lemma 2.4,

$$\int_\Omega \boldsymbol{c(x,t,u(x,t)J\left(\int_0^t (t-s)^{-\alpha}\frac{\partial u(x,s)}{\partial s}ds\right)\psi(x)dx \geq b\delta(t)J(k(t))}. \qquad (14)$$

Equations (12)-(14) are substituted in Equation (11),

$$\frac{d}{dt}\left(p(t)\left({}^cD_+^{\alpha}U(t)\right)\right) + \mu U(t) \leq 0, \qquad t \geq 0,$$

where $\boldsymbol{U(t) = \int_\Omega u(x,t)\psi(x)dx}$, which means that $\boldsymbol{U(t) > 0}$ is a solution of (10). Hence the proof is complete.

Theorem 3.7 Suppose that the conditions $(\mathbf{A_1}) - (\mathbf{A_4})$ hold, and

$$\limsup_{t\to\infty}\int_{t_1}^t \rho(s)\left(\mu + \frac{p(s)k(s)g^2(s)}{4\Gamma(1-\alpha)U'(s)}\right)ds = \infty, \qquad (15)$$

where $\boldsymbol{\rho(t) = \int_{t_0}^{\infty} exp\left(-2\int_{t_0}^t g(s)ds\right)}$. Then every solution of boundary value problem (1) and ($\boldsymbol{B_2}$) is oscillatory in G.

Proof: Suppose that $\mathbf{U(t)}$ is a non-oscillatory solution of (10). We define the Riccati transformation,

$$W(t) = \rho(t)\left(\frac{p(t)\ {}^cD_+^{\alpha}U(t)}{U(t)}\right), \qquad t \geq 0,$$
$$W'(t) \leq -\mu\rho(t) - 2g(t)W(t) - \frac{\Gamma(1-\alpha)U'(t)}{\rho(t)p(t)k(t)}W^2(t).$$

(16)

Integrating on both sides from $\boldsymbol{t_1}$ to $\boldsymbol{t}$, gives

$$W(t) - W(t_1) \leq -\int_{t_0}^t \mu\rho(s)ds - \int_{t_0}^t 2g(s)W(s)ds - \int_{t_0}^t \frac{\Gamma(1-\alpha)U'(s)}{\rho(s)p(s)k(s)}W^2(s)ds.$$

$$\leq -\mu\int_{t_0}^t \rho(s)ds + \frac{1}{4}\int_{t_0}^t \frac{\rho(s)p(s)k(s)g^2(s)}{\Gamma(1-\alpha)U'(s)}ds$$

Taking the limit superior on both sides, we get

$$\limsup_{t\to\infty}\int_{t_1}^t \rho(s)\left(\mu + \frac{\rho(s)p(s)k(s)g^2(s)}{\Gamma(1-\alpha)U'(s)}\right)ds \leq W(t_1) < \infty.$$

which contradicts (15). Hence the proof is complete.

Theorem 3.8 Assume that $(\mathbf{A_1}) - (\mathbf{A_4})$ hold, and

$$\limsup_{t\to\infty}\frac{1}{H(t,t_1)}\int_{t_1}^t \rho(s)\left(\mu H(t,s) + \frac{1}{4\Gamma(1-\alpha)}\left(2g(s)\sqrt{H(t,s)} + h_2(t,s)\right)^2\frac{p(s)k(s)}{U'(s)}\right)ds = \infty. \qquad (17)$$

Then all solutions of (1) and ($\boldsymbol{B_2}$) are oscillatory.

Proof: Suppose that $\mathbf{U(t)}$ is a non-oscillatory solution of (10). Multiplying both sides of (16) with $\mathbf{H(t,s)}$and

Integrating it with respect to s from t_1 to t we obtain,

$$-H(t,t_1)W(t_1) \le -\int_{t_1}^{t} \mu H(t,s)\rho(s)\,ds - \int_{t_1}^{t} \frac{\Gamma(1-\alpha)U'(s)H(t,s)}{\rho(s)p(s)k(s)} W^2(s)ds$$
$$-\int_{t_1}^{t} H(t,s)\left(2g(s) + \frac{h_2(t,s)}{\sqrt{H(t,s)}}p(s)k(s)\right)W(s)ds.$$

Taking the limit superior on both sides, gives

$$\limsup_{t\to\infty} \frac{1}{H(t,t_1)}\int_{t_1}^{t} \rho(s)\left(\mu H(t,s) + \frac{1}{4\Gamma(1-\alpha)}\left(2g(s)\sqrt{H(t,s)} + h_2(t,s)\right)^2 \frac{p(s)k(s)}{U'(s)}\right)ds \le W(t_1) < \infty.$$

which leads to a contradiction of (17). Hence the proof is complete.

Consider $H(t,s) = \left(\log\left(\frac{t}{s}\right)\right)^n, t > s > 0, n > 1$ is an integer. Then, from Theorem (3.8), we get immediately the following result.

Corollary 3.9. If the conditions of Theorem (3.8) hold, Equation (17) can be written as

$$\limsup_{t\to\infty} \frac{1}{\left(\log\left(\frac{t}{t_1}\right)\right)^n}\int_{t_1}^{t} \left(\log\left(\frac{t}{s}\right)\right)^n \rho(s)\left(\mu + \frac{1}{4\Gamma(1-\alpha)}\left(2g(s) + \frac{n}{s(\log(t-s))}\right)^2 \frac{p(s)k(s)}{U'(s)}\right)ds$$

Then all solution of (1) and (B_2) are oscillatory.

Oscillation of extensible beam with clamped-hinged ends:

We deal with the case of clamped-hinged ends,

$$u(0,t) = \frac{\partial u(0,t)}{\partial t} = u(L,t) = \frac{\partial^2 u(L,t)}{\partial t^2} = 0 \qquad (B_3)$$

Theorem 3.10 Assume that $m, r \ge 0$, and $\psi^4(x) \ge \epsilon\psi(x)$ in Ω for some $\epsilon \ge 0$. Then there exists a solutions of (1) satisfying the boundary condition (B_3) which is oscillatory in G, if the inequality,

$$\frac{d}{dt}\left(p(t)\left({}^{c}D_{+}^{\alpha}U(t)\right)\right) + \epsilon q U(t) \le 0, \qquad t \ge 0, \tag{18}$$

is oscillatory.

proof: Assume on the contrary that $u(x,t) > 0$ is non-oscillatory in G. We consider $u(x,t) > 0$, multiplying (1) with $\psi(x) = sin\frac{\pi}{L}x$, integrating over Ω, we get

$$\int_\Omega \frac{\partial}{\partial t}(p(t)\,{}^{c}D_{+,t}^{\alpha}u(x,t))\,\psi(x)dx + \int_\Omega q\frac{\partial^4 u(x,t)}{\partial x^4}\psi(x)dx - \int_\Omega \left(m + r\int_\Omega \left(\frac{\partial u(\xi,t)}{\partial \xi}\right)^2 d\xi\right)\frac{\partial^2 u(x,t)}{\partial x^2}\psi(x)dx$$
$$+ \int_\Omega c(x,t,u(x,t))J\left(\int_0^t (t-s)^{-\alpha}\frac{\partial u(x,s)}{\partial s}ds\right)\psi(x)dx = \int_\Omega f(x,t)\,\psi(x)dx \tag{19}$$

Jenson's inequality, gives

$$\int_\Omega c(x,t,u(x,t))J\left(\int_0^t (t-s)^{-\alpha}\frac{\partial u(x,s)}{\partial s}ds\right)\psi(x)dx \ge b\delta(t)J(k(t)). \tag{20}$$

Equation (20) are substituted in Equation (19), we get

$$\frac{d}{dt}\left(p(t)\left({}^{c}D_{+}^{\alpha}U(t)\right)\right) + \epsilon q U(t) \le 0, \qquad t \ge 0,$$

Hence the proof is complete.

Theorem 3.11 Suppose that the conditions $(A_1) - (A_4)$ hold, and

$$\limsup_{t\to\infty} \int_{t_1}^{t} \rho(s)\left(\epsilon q + \frac{p(s)k(s)g^2(s)}{4\Gamma(1-\alpha)U'(s)}\right)ds = \infty, \tag{21}$$

where $\rho(t) = \int_{t_0}^{\infty} exp\left(-2\int_{t_0}^{t} g(s)ds\right)$. Then all solutions of boundary value problem (1) and (B_3) is oscillatory in G.

4. Examples

Example 4.1 Consider the Fractional nonlinear damped extensible beam equation

$$\frac{\partial}{\partial t}\left({}^{c}D_{+,t}^{\frac{1}{2}}u(x,t)\right) + \left(\frac{L}{\pi}\right)^4 \frac{\partial^4 u(x,t)}{\partial x^4} + 2tJ\left(\int_0^t (t-s)^{-\frac{1}{2}}\frac{\partial u(x,s)}{\partial s}ds\right)$$

$$= \sin\frac{\pi}{L}x\left(5t(\cos t - 1) + \sin t\left(\frac{2\sqrt{\pi}(\zeta-t)^{\frac{3}{2}}-5}{2\sqrt{\pi}}\right)\right), \quad (x,t) \in \Omega \times \mathbb{R}_+ = G. \qquad (22)$$

Here $\alpha = 1/2, p(t) = 1, q(t) = \left(\frac{L}{\pi}\right)^4, m, r = 0, b = 1, g^2(s) = \frac{1}{4s^2}, \rho(s) = s, \epsilon = 1$ and $\mathbf{J}(\mathbf{k}(\mathbf{s})) = \mathbf{k}(\mathbf{t})$. Consider

$$\limsup_{t\to\infty} \int_{t_1}^{t} \rho(s)\left(\epsilon q + \frac{p(s)k(s)g^2(s)}{4\Gamma(1-\alpha)U'(s)}\right) ds = \limsup_{t\to\infty} \int_{t_1}^{t} s\left(\left(\frac{L}{\pi}\right)^2 + \frac{5s(\cos s - 1)\sin\frac{\pi}{L}x}{8\sqrt{\pi}s^2(\zeta-s)^{\frac{1}{2}}\left(-\frac{3}{2}\sin s + (\zeta-s)\cos s\right)}\right) ds \text{ as } t \to \infty.$$

Hence, all the conditions of Theorem 3.11are satisfied. Therefore, every solution of (22) is oscillatory.
In fact, $u(x,t) = (\zeta - t)^{\frac{3}{2}} \sin t \sin\frac{\pi}{L}x$ is one such solution of (22).

Example 4.2 Consider the Fractional nonlinear damped extensible beam equation

$$\frac{\partial}{\partial t}\left({}^{c}D_{+,t}^{\frac{1}{2}} u(x,t)\right) + \left(\frac{L}{\pi}\right)^4 \frac{\partial^4 u(x,t)}{\partial x^4} + 2tJ\left(\int_0^t (t-s)^{-\frac{1}{2}} \frac{\partial u(x,s)}{\partial s} ds\right)$$
$$- \left(-\left(\frac{L}{\pi}\right)^2 + \left(\frac{L}{\pi}\right)^4 \int_\Omega \left(\frac{\partial u(\xi,t)}{\partial \xi}\right)^2 d\xi\right) \frac{\partial^2 u(x,t)}{\partial x^2}$$
$$= \sin\frac{\pi}{L}x\left(\frac{2-e^t}{2\pi} + (\zeta - t)^{\frac{3}{2}} e^t\left(2 + \frac{L}{2}(\zeta-t)^{\frac{5}{2}} e^{2t}\right) + (2t - e^t - 1)\right), \quad (x,t) \in \Omega \times \mathbb{R}_+ = G. \qquad (23)$$

Here $\alpha = \frac{1}{2}, p(t) = 1, q(t), r = \left(\frac{L}{\pi}\right)^4, m = \left(\frac{L}{\pi}\right)^2, b = 1, g^2(s) = \frac{1}{4s^2}, \rho(s) = 1/s, \epsilon = 1$ and $\mathbf{J}(\mathbf{k}(\mathbf{s})) = \mathbf{k}(\mathbf{t})$. Consider

$$\limsup_{t\to\infty} \int_{t_1}^{t} \rho(s)\left(\mu + \frac{p(s)k(s)g^2(s)}{4\Gamma(1-\alpha)U'(s)}\right) ds = \limsup_{t\to\infty} \int_{t_1}^{t} \frac{1}{s}\left(1 + \frac{(1+2s-e^s)\sin\frac{\pi}{L}x}{8\sqrt{\pi}s^2(\zeta-s)^{\frac{1}{2}}\left(-\frac{3}{2}+(\zeta-s)\right)}\right) ds < \infty.$$

Hence, all the conditions of Theorem 3.7 are not satisfied.
In fact, $u(x,t) = (\zeta - t)^{\frac{3}{2}} e^t \sin\frac{\pi}{L}x$ is non-oscillatory solution of (23).

5. Conclusion

In this article, we have mainly focussed on obtaining some new sufficient conditions for the oscillation behavior of Caputo fractional nonlinear damped extensible beam equations with some boundary conditions. The results are essentially new and complement the previous existing literature in the classical case. Required examples has also been newly derived result.

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