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Original Article

On the Clique-Edge Graph of a Graph

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Abstract *- In this paper, we analyse the concept of the clique-edge graph, CE(G), which is defined as the edge intersection graph of all cliques within a given graph G. We discuss the impact of various binary operations on the structure and properties of the clique-edge graph, providing a detailed analysis of how these operations influence CE(G). Additionally, we investigate the connectedness of CE(G), offering insights into the conditions under which the clique-edge graph remains connected or becomes disconnected.*

Keywords *- Clique-Edge Graphs, Composition, Connectedness, Join, Triangle graph.*

1. Introduction

Graph operators, in particular, intersection graphs, play a vital role in the study of various structural properties and dynamics of graphs and networks. Intersection graphs have been receiving attention in graph theory for some time. The line graph $L(G)$ was the first intersection graph to be defined in the literature. The notion of 'line graph' as a 'graph operator' was introduced by Krausz [6]. The line graph $L(G)$ of graph G has all the edges (i.e. K_2 subgraphs) of G as its vertices and two vertices of $L(G)$ are adjacent if the corresponding edges of G are adjacent. Since then, many other graph operators, such as clique graph, total graph, etc, and their dynamics have been studied [7]. This notion led to the definition of another graph operator, triangle graph $T(G)$, which was introduced independently several times under different names and in different contexts. [2, 12, 1, 8] $T(G)$ has triangles (K_3 subgraphs) of G as vertices and two vertices of $T(G)$ are adjacent if the corresponding K_3 share an edge. The Cycle Graph, $Cy(G)$, is a generalization of $T(G)$. $Cy(G)$ was introduced by Gervacio, in [5]. It has all induced cycles of G as its vertices, and two vertices of $Cy(G)$ are adjacent if the corresponding induced cycles share an edge. In [9] and [10], several classes of graphs, such as cycle periodic, cycle expanding and inverse cycle graphs, are studied. In this paper, we define a new graph operator named Clique-Edge graph, denoted by $CE(G)$. The clique-edge graph, $CE(G)$ of graph G, is the edge intersection graph of all cliques of G. Triangles can be considered as cliques of order three. This concept can be generalized to cliques of order n. The clique-edge graph is thus a generalization of the triangle graph. It can also be viewed as a generalization of the clique graph introduced by Hamelink in [3] and studied by several authors, including Hedge et al. in [4]. The motivation for this definition is the close relationship with the well-studied classes- triangle graphs and clique graphs. We deal with some results on the clique-edge graph. The second section gives a definition, examples and some basic results that follow from the definition. In the third section, we discuss about the connectedness of clique-edge graphs of some classes of graphs. The fourth section examines the effect of some binary operations on clique-edge graphs. All the graphs considered here are undirected and simple. For all basic concepts and notations not mentioned in this paper, we refer to [11].

Fig. 1 G and CE(G)

2. The Clique-Edge Graph

The proposed approach consists of five phases, namely; hand region segmentation, morphological processing, contour simplification and fingertip detection and is illustrated in Figure 2.

In this section, we shall define *CE(G)* and prove some results on *CE(G)*.

Definition 2.1. [11] A clique of a graph *G* is a maximal complete subgraph of *G*.

Definition 2.2. [7] The clique graph *C(G)* of graph *G* is the intersection graph of all cliques of *G*.

Definition 2.3. Let *G* be a graph. The clique-edge graph of *G*, *CE(G)* has its vertices the cliques of *G* and two vertices of *CE(G)* are adjacent if the corresponding cliques have some common edge. (Figure 1)

Theorem 2.1. *The clique-edge graph of a graph is a subgraph of its clique graph*.

Proof.

First, note that both *CE(G)* and *C(G)* have the cliques of *G* as their vertices. Two vertices of *C(G)* are adjacent if they have a nonempty intersection, i.e. they have at least one common vertex. Two vertices of *CE(G)* are adjacent if they have a common edge.

Thus, any edge of *CE(G)* is

also an edge of *C(G)*. Hence, $CE(G) \subseteq C(G)$.

In what follows, we denote paths, cycles, wheels and complete graphs by P_n, C_n, W_n and K_n, respectively (Figure 2). The square of a graph G^2 is obtained by joining vertices in G, which are at a distance of at most two.

Theorem 2.2. $CE(P_n) \cong \overline{K}_{n-1}$.

Proof.

Each edge forms a clique in P_n, and they are the only cliques in P_n. So, there are n -1 cliques in P_n, each of the form K_2. Hence, no two of them can have a common edge. Therefore, $CE(P_n)$ has $n-1$ vertices, all of which are isolated vertices. Thus, $CE(P_n) \cong \overline{K}_{n-1}$.

Theorem 2.3. $CE(C_n) \cong \overline{K}_n$ *for* $n \geq 4$ *and* $CE(C_3) \cong \overline{K}_1$.

Proof.

The only clique in C_3 is K_3. So, clearly $CE(C_3) \cong \overline{K}_1$. For $n \geq 4$, the cliques C_n are of the form K_2 and they are n in number. Also, no two of them can share an edge. Thus, $CE(C_n)$ has n isolated vertices. Hence, $CE(C_n) \cong \overline{K}_n$.

Theorem 2.4. $CE(P_n^2) \cong P_{n-2}$.

Proof.

Let $P_n = v_1 v_2 \ldots v_n$. The edges of P_n^2 are of the form $v_i v_{i+1}$, i = 1, 2, . . ., $n-1$ and $v_i v_{i+2}$, i = 1, 2, . . . $n-2$. Hence, three consecutive vertices in P_n will correspond to a clique K_3. The sub graph H_i induced by $\{v_i, v_{i+1}, v_{i+2}\}$, i = 1, 2, . . ., $n-2$ will form a clique.

There will be $n-2$ such cliques. Also, two cliques H_i and H_{i+1} will share an edge in common for each i = 2, 3, . . ., $n-1$. Therefore, $CE(P_n^2)$ will have $n-2$ vertices, and every pair of consecutive vertices will be adjacent. Thus, $CE(P_n^2) \cong P_{n-2}$.

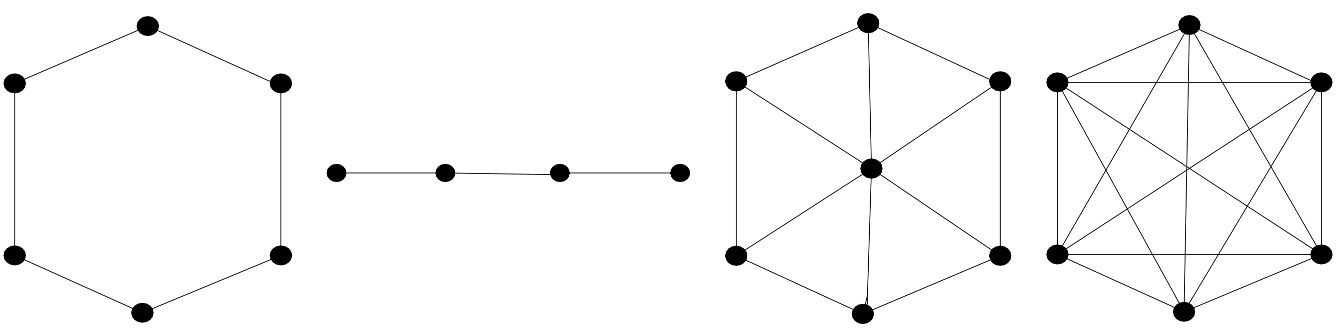

Fig. 2 **C_6** **P_4** **W_7** **K_6**

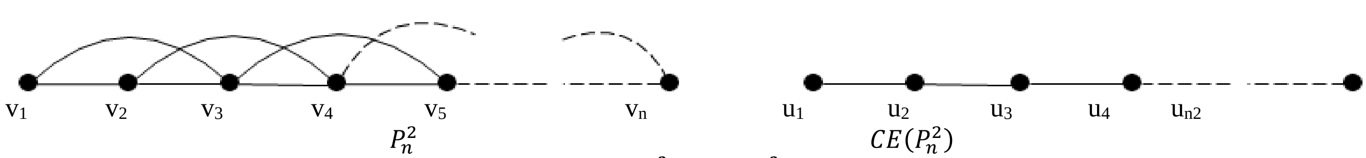

Fig. 3 P_n^2 **and** $CE(P_n^2)$

Theorem 2.5. $CE(C_n^2) \cong K_1$ *for n = 3, 4, 5;* $CE(C_6^2) \cong Q_3$ *and* $CE(C_n^2) \cong C_n$ *for* $n \geq 7$.

Proof. It is clear that for $n = 3, 4, 5$, and $CE(K_n) \cong K_1$ as K_n contains only one clique. Hence, $CE(C_n^2) \cong K_1$. From the Fig 4, $CE(C_6^2) \cong Q_3$.

Now, let $n \geq 7$ and C_n: $v_1 v_2 \ldots v_n v_1$. If n is an even number, then $C_n^2 \cong C_n \cup C_n' \cup C_n''$, where C_n' : $v_1 v_3 v_5 \ldots v_n v_1$ and C_n'': $v_2 v_4 v_6 \ldots v_{n-1} v_2$. If n is an odd number, then $C_n^2 \cong C_n \cup C_n'''$, where C_n''' : $v_1 v_3 v_5 \ldots v_n v_2 v_4 v_6 \ldots v_{n-1} v_1$. In both cases, since $n \geq 7$, the only cliques are triangles T_i induced by $v_i v_{i+1} v_{i+2}$ for $i = 1, 2, \ldots, n-2$, T_{n-1} induced by $v_{n-1}v_nv_1$ and T_n induced $v_nv_1v_2$. So, there are n cliques, and each T_i and T_{i+1} share a common edge. Hence, $CE(C_n^2) \cong C_n$.

Theorem 2.6. $CE(W_n) \cong C_{n-1}$

Proof. The only cliques W_n are triangles, and there are $n-1$ such cliques. Also, each consecutive cliques share a common edge. Hence, $CE(W_n) \cong C_{n-1}$.

Theorem 2.7. $CE(Petersen^2) \cong K_1$.

Proof. $Petersen^2 \cong K_{10}$. Hence, $CE(Petersen^2) \cong K_1$.

3. Connectedness

In this section, we examine the connectedness of $CE(G)$. In contrast to the case of clique graphs, the clique-edge graph of a connected graph need not be connected.

Theorem 3.1. *The clique-edge graph of a triangle-free graph is totally disconnected*.

Proof. For a triangle-free graph, any clique is either a single vertex, denoted as K_1, or an edge, denoted as K_2 . As a result, no two cliques in such a graph can share an edge. Therefore, the clique-edge graph, which represents the connections between these cliques, is totally disconnected.

G *CE(G)*

Fig. 4 C_6^2 **and** $CE(C_6^2)$

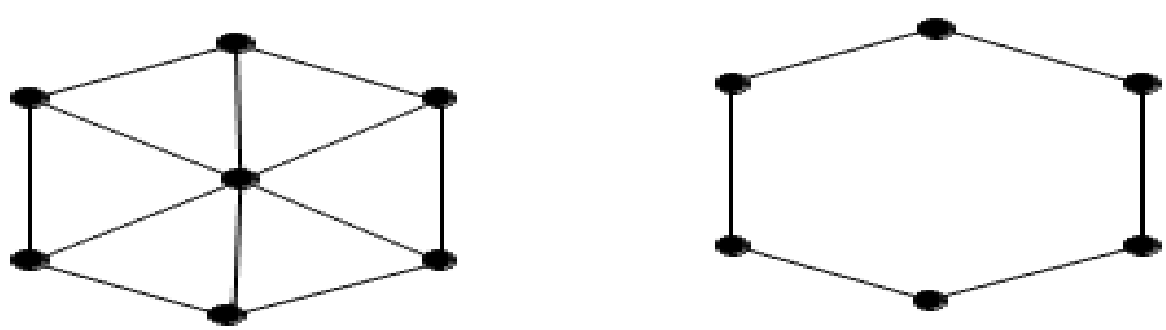

Fig. 5 W_7 **and** $CE(W_7)$

Theorem 3.2: *If a graph G has a cut-vertex, then the clique-edge graph CE(G) is disconnected.*

Proof: Assume that the graph G has a cut-vertex v. When v is removed from G, the graph splits into at least two components, say G_1 and G_2.

Consider the subgraphs $[G_1, v]$ and $[G_2, v]$, where $[G_i, v]$ represents the subgraph induced by G_i and the vertex v. These two subgraphs share only the vertex. Consequently, any clique from $[G_1, v]$ cannot share an edge with any clique from $[G_2, v]$. Therefore, in the clique-edge graph *CE(G)*, any vertex v_1, which corresponds to a clique of $[G_1, v]$, cannot be connected by a path to any vertex v_2, which corresponds to a clique of $[G_2, v]$. Thus, *CE(G)* is disconnected.

Remark 3.3: *The converse of this theorem is not true. That CE(G) is disconnected does not imply that G has a cut-vertex. For, $CE(C_5)$ is disconnected. But C_5 does not have a cut-vertex.*

Theorem 3.4: *Let G be a connected graph with connected CE(G). Then for any two vertices, V_1 and V_2 of CE(G), min $D(V_1, V_2) \leq 2d(V_1, V_2) + 1$, where $D(V_1, V_2) = \{d(v_1, v_2) : v_1 \in V_1, v_2 \in V_2\}$.*

Proof: Consider two vertices, V_1 and V_2 of. They correspond to two cliques, V_1 and V_2 of G. Let $d(V_1, V_2) = r$. Let $U_0U_1U_2 \ldots U_r$ be the shortest path connecting $U_0 = V_1$ and $U_r = V_2$. Each U_i, $i = 0, 1, 2, \ldots, r$ corresponds to a clique in *G*, and since U_{i-1} and U_i being adjacent in *CE(G)* , they share a common edge $e_i = u_i^1u_i^2$ for $i = 1, 2, \ldots, r$. Then $u_i^1 \in U_{i-1}$ and $u_i^2 \in U_i$. Thus, $u^2_{i-1} \in U_{i-1}$ and $u_i^1 \in U_{i-1}$. Since each U_i is a clique, u^2_{i-1} and u_i^1 are adjacent in *G*. Therefore $u_1^1u_1^2u_2^1u_2^2 \ldots u_r^1u_r^2$ forms a walk of length $2r + 1$ in *G*. Hence min $D(V_1, V_2) \leq 2r + 1 = 2d(V_1, V_2) + 1$.

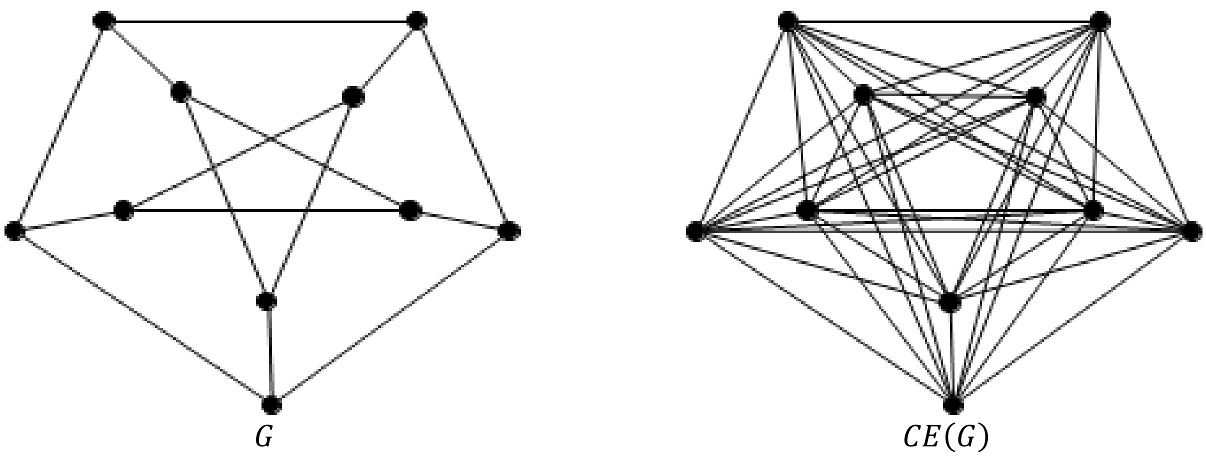

Fig. 6 Petersen graph and its square

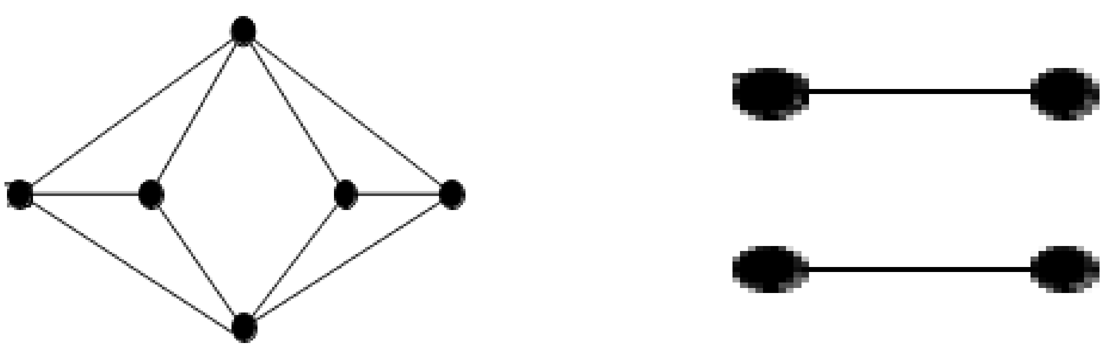

Fig. 7 A connected graph with disconnected *CE(G)*

4. Operations on Clique-Edge Graphs

It would be interesting to examine the effect of some binary operations in relation to clique-edge graphs. Let G_1 and G_2 be vertex disjoint graphs. We consider the following operations.

Definition 4.1. Union: $G_1 \cup G_2$ is the graph such that $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$.

Definition 4.2. Join: $G_1 + G_2$ is defined as $V(G_1 + G_2) = V(G_1 \cup G_2)$ and $E(G_1 + G_2) = E(G_1 \cup G_2) \cup [V(G_1) \times V(G_2)]$, where $V(G_1) \times V(G_2)$ represents the set of unordered pairs (v_1, v_2), with $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$.

Definition 4.3. Cartesian Product: $G_1 \times G_2$ is the graph where $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and for $v_1, w_1 \in V(G_1)$ and $v_2, w_2 \in V(G_2)$, (v_1, v_2) and (w_1, w_2) are adjacent vertices in $G_1 \times G_2$ precisely when $(v_1, w_1) \in E(G_1)$ and $(v_2, w_2) \in E(G_2)$.

Definition 4.4. Composition: $G_1(G_2)$ is defined as $V(G_1(G_2)) = V(G_1) \times V(G_2)$ and $E(G_1(G_2)) = \{(v_1, v_2)(w_1, w_2) \,/\, \textit{either } (v_1, w_1) \in E(G_1) \textit{ or } v_1 = w_1 \textit{ and } (v_2, w_2) \in E(G_2)\}$.

Theorem 4.1. *Let G_1 and G_2 be graphs with disjoint vertex sets. Then $CE(G_1 \cup G_2) \cong CE(G_1) \cup CE(G_2)$.*
Proof. Trivial.
Theorem 4.2. *Let G_1 and G_2 be connected graphs with disjoint vertex sets. Then $CE(G_1 + G_2) \cong C(G_1) \times C(G_2)$, where $C(G)$ is the clique graph of G.*

Proof. $CE(G_1 + G_2)$ contains all the edges of G_1 and G_2 and also all possible edges between G_1 and G_2. Therefore, a clique in G_1 together with a clique in G_2, will form a clique in $CE(G_1 + G_2)$. Corresponding to a pair of cliques (C_{G1}, C_{G2}) where C_G is a clique in G, there will be a clique in $CE(G_1 + G_2)$. Two such cliques will be edge intersecting in $CE(G_1 + G_2)$ if the corresponding cliques in G_1 and G_2 are intersecting. Consequently, $CE(G_1 + G_2) \cong C(G_1) \times C(G_2)$.

Theorem 4.3. *Let $G_1(n_1, m_1)$ and $G_2(n_2, m_2)$ be graphs with disjoint vertex sets. Then $CE(G_1 \times G_2) \cong n_2CE(G_1) \cup n_1CE(G_2)$.*

Proof. $G_1 \times G_2$ contains n_2 copies of G_1 and n_1 copies G_2. Also, no new cliques are formed under the operation cartesian product. Hence the result.

Theorem 4.4. $CE[G_1(G_2)] \cong L(G_1)$ if and only if G_1 is triangle-free and G_2 is complete.

Proof. $G_1(G_2)$ can be obtained by replacing each vertex of G_1 with a copy of G_2 and each edge u_iu_j of G_1 by all possible edges between the of G_2 corresponding to the vertices u_i and u_j.
Since G_2 is complete and G_1 is triangle free, corresponding to every edge of G_1, there is a clique in $G_1(G_2)$. Two such cliques are edge intersecting whenever the corresponding edges are incident. Hence $CE[G_1(G_2)] \cong L(G_1)$. Conversely, assume $CE[G_1(G_2)] \cong L(G_1)$. If G_2 is not complete, the number of cliques in $G_1(G_2)$ will be greater than the number of edges of G_1, which is a contradiction. If G_1 has a triangle corresponding to each triangle, there will be a clique in $G_1(G_2)$. Therefore, the number of cliques in $G_1(G_2)$ will be less than the number of edges in G_1, which is a contradiction. Hence the result.

5. Conclusion and Future Works

In conclusion, we have analysed the concept of the clique-edge graph, *CE(G)*, which is constructed as the edge intersection graph of all cliques within a given graph *G*. Our detailed investigation of the impact of various binary operations on the structure and properties of *CE(G)* reveals significant changes in its composition and characteristics.

These operations can lead to alterations in clique connectivity and edge relationships, directly influencing the structure of *CE(G)*. Furthermore, our exploration of the connectedness of *CE(G)* has provided a comprehensive understanding of the conditions under which the clique-edge graph remains connected or becomes disconnected. Through this analysis, we offer insights into how the underlying graph *G* and the applied binary operations affect the overall connectedness of *CE(G)*, contributing to a deeper understanding of its structural dynamics. Future studies could focus on characterizing clique-edge graphs, analyzing parameters like radius, diameter, and domination number, exploring the convergence of the iterated *CE* operator, and investigating the clique-edge graphs of specific graph classes.

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