International Journal of Mathematics Trends and Technology
ISSN: 2231-5373/ https://doi.org/10.14445/22315373/IJMTT-V70I10P102

Volume 70 Issue 10, 7-13, October 2024


*Original Article*

# The Marichev – Saigo – Maeda Fractional Calculus Operator Associated with the Product of a General Class of Polynomial, M – Series and Generalized k – Struve Function

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Received: 08 August 2024 Revised: 18 September 2024 Accepted: 07 October 2024 Published: 26 October 2024

***Abstract*** *- In this paper, researcher develop three theorems by using Marichev – Saigo – Maeda fractional calculus operator, applied the product of the Srivastava polynomial, M – Series and k – Struve function with the help of some lemma. The results are presented in terms of the Generalized k – Wright function. Also obtained some known and intriguing special cases.*

***Keywords*** *- MSM fractional integral operator, M-series, k-Struve function.*

## 1. Introduction

The Wright function is widely used in the partial differential equation of fractional order which is amicable and broadly treated in papers by many authors including Gorenflo et. al. [1]. For $\zeta_i$, $\tau_j \in \mathbb{R} \setminus \{0\}$ and $a_i$ , $b_j \in \mathbb{C}$, i = (1, p); j = (1, q) the generalized form of Wright function defined by Wright [2, 3 – 6] as following

$$ {}_p\Psi_q(z) = {}_p\Psi_q\left[\begin{matrix} (a_i , \zeta_i)_{1,p} \\ (b_j , \tau_j)_{1,q} \end{matrix} ; z\right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(a_i + n\zeta_i) z^n}{\prod_{j=1}^{q} \Gamma(b_j + n\tau_j) n!}, z \in \mathbb{C} \quad \ldots(1.1)$$

Where $\Gamma z$ is the Euler gamma function [7]. The condition for the existence of (1.1) with its illustration in terms of Mellin – Barnes integral and the H – function obtained by Kilbas et al. [8]. The generalized form of the above Wright function (1.1) was given by Gehlot and Prajapati [9], as the generalized k – Wright function defines as

$$ {}_p\Psi_q^k(z) = {}_p\Psi_q^k\left[\begin{matrix} (a_i , \zeta_i)_{1,p} \\ (b_j , \tau_j)_{1,q} \end{matrix} ; z\right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma_k(a_i + n\zeta_i) z^n}{\prod_{j=1}^{q} \Gamma_k(b_j + n\tau_j) n!}, z \in \mathbb{C} \quad \ldots(1.2)$$

Where $k \in \mathbb{R}^+$ and $(a_i + n\zeta_i)$, $(b_j + n\tau_j) \in \mathbb{C} \setminus k\mathbb{Z}^-$ for all $n \in \mathbb{N}_0$. The generalized k – gamma function [10] is defined as

$$\Gamma_k(z) = \int_0^{\infty} e^{-\frac{t^k}{k}} t^{z-1} dt ; \quad (\Re(z) > 0, k \in \mathbb{R}^+) \quad \ldots(1.3)$$

and

$$\Gamma_k(z) = \lim_{n\to\infty} \frac{n!\, k^n (nk)^{\frac{z}{k}-1}}{(z)_{n,k}}, \quad k \in \mathbb{R}^+, z \in \mathbb{C} \setminus k\mathbb{Z}^- \quad \ldots(1.4)$$

Also

$$\Gamma_k(z) = (k)^{\frac{z}{k}-1} \Gamma\left(\frac{z}{k}\right) \quad \ldots(1.5)$$

Where $(z)_{n,k}$ is the k – Pochammer symbol introduced by Daiz and Pariguan [10] defined for complex z ∈ ℂ and k ∈ ℝ as

$$(\mathrm{z})_{\mathrm{n,k}} = \begin{cases} 1 & (n=0) \\ \mathrm{z}(\mathrm{z}+1)(\mathrm{z}+2\mathrm{k})\ldots\ldots(\mathrm{z}+(\mathrm{n}-1)\mathrm{k}) & (n \in \mathrm{N}) \end{cases} \quad \ldots(1.6)$$

On taking k = 1, the generalized k – Wright function (1.2) diminishes to the generalized Wright function (1.1) Saigo [11] defined the fractional integral operator with the Gauss hypergeometric function as kernel, which are a remarkable generalization of the Riemann–Liouville and Erdelyi – Kober fractional calculus operator [12].

For $\xi, \tau, \beta \in \mathbb{C}$ and $x \in \mathbb{R}^+$ with $\Re(\xi) > 0$, the left – hand and the right – hand sided generalized fractional integral operator connected with Gauss hypergeometric function are defined as below:

$$(\mathrm{I}_{0^+}^{\xi,\tau,\beta} f)(x) = \frac{\mathrm{x}^{-\xi-\tau}}{\Gamma(\xi)} \int_0^x (x-t)^{\xi-1} {}_2F_1(\xi+\tau, -\beta; \xi; 1-\frac{\mathrm{t}}{\mathrm{x}}) f(t)dt \quad \ldots(1.7)$$

and

$$(\mathrm{I}_{-}^{\xi,\tau,\beta} f)(x) = \frac{1}{\Gamma(\xi)} \int_x^\infty (t-x)^{\xi-1} {}_2F_1(\xi+\tau, -\beta; \xi; 1-\frac{\mathrm{x}}{\mathrm{t}}) f(t)dt \quad \ldots(1.8)$$

respectively. Here ${}_2F_1(\xi, \tau; \beta; z)$ is the Gauss hypergeometric function [12] defined for $z \in \mathbb{C}$, $|z| < 1$ and $\xi, \tau \in \mathbb{C}$, $\beta \in \mathbb{C} \setminus \mathbb{Z}_0^-$ by

$${}_2F_1(\xi, \tau; \beta; z) = \sum_{n=0}^{\infty} \frac{(\xi)_n (\tau)_n z^n}{(\beta)_n \; n!} \qquad \text{where } (z)_n = (z)_{n,1} \quad \ldots(1.9)$$

By substituting $\tau = -\xi$ and $\tau = 0$ in equation (1.7), we get corresponding R–L and Erdelyi – Kober fractional operator respectively. Marichev (2) was introduced and studied fractional calculus operators which are the generalization of the Saigo operator, later generalized by Saigo and Maeda (13). For $\xi, \xi', \tau, \tau', \beta \in \mathbb{C}$ and $x \in \mathbb{R}^+$ with $\Re(\beta) > 0$,the left-hand and the right-hand sided MSM fractional integral operator associated with third Appell function $F_3$ are defined as

$$(\mathrm{I}_{0^+}^{\xi,\xi',\tau,\tau',\beta} f)(x) = \frac{\mathrm{x}^{-\xi}}{\Gamma(\beta)} \int_0^x \frac{(x-t)^{\beta-1}}{\mathrm{t}^{\xi'}} F_3(\xi, \xi', \tau, \tau', \beta, 1-\frac{\mathrm{t}}{\mathrm{x}}, 1-\frac{\mathrm{x}}{\mathrm{t}}) f(t)dt \quad \ldots(1.10)$$

and

$$(\mathrm{I}_{-}^{\xi,\xi',\tau,\tau',\beta} f)(x) = \frac{\mathrm{x}^{-\xi'}}{\Gamma(\beta)} \int_x^\infty \frac{(t-x)^{\beta-1}}{\mathrm{t}^{\xi}} F_3(\xi, \xi', \tau, \tau', \beta, 1-\frac{\mathrm{x}}{\mathrm{t}}, 1-\frac{\mathrm{t}}{\mathrm{x}},) f(t)dt \quad \ldots(1.11)$$

The third Appell function (6) is defined by

$$F_3(\xi, \xi', \tau, \tau', \beta, x, y) = \sum_{m,n=0}^{\infty} \frac{(\xi)_m (\xi')_n (\tau)_m (\tau')_n x^m y^n}{(\beta)_{m+n} \; n!\, m!}, \max\{|x|, |y|\} < 1 \quad \ldots(1.12)$$

The Srivastava polynomial defined by Srivastava [14-15] in the following manner

$$\mathrm{S}_{\mathrm{w}}^{\mathrm{u}}[\mathrm{x}] = \sum_{s=0}^{(w/u)} \frac{(-W)u_{,s}}{s!} A_{w,s} x^s \qquad \mathrm{w} = 0,1,2,\ldots \quad \ldots(1.13)$$

Where $w$ is an arbitrary positive integer and the coefficient $A_{w,s}$ $(w, s) > 0$ is arbitrary constant real or complex. This polynomial provides a large number spectrum of the well-known polynomial as one of its particular cases on appropriately specializing the coefficient $A_{w,s}$, particularly by setting $u = 1$, $A_{w,s} = \frac{s!}{(-W)u_{,s}}$ for s = k and $A_{w,s} = 0$ for s ≠ k the above polynomial leads to a power function.

$$\mathrm{S}_{\mathrm{w}}^{\mathrm{u}}[\mathrm{x}] = x^k \qquad (\mathrm{k} \in \mathrm{Z}^+ \; with\; k \le w) \quad \ldots(1.14)$$

The generalized $k$-Struve function was defined by Nisar K S [3], [16] as

$$\mathrm{S}_{\mathrm{v,c}}^{\mathrm{k}}(t) = \sum_{\mathrm{n}=0}^{\infty} \frac{(-\mathrm{c})^{\mathrm{n}}}{\Gamma_{\mathrm{k}}\left(\mathrm{nk}+\mathrm{v}+\frac{3\mathrm{k}}{2}\right)\Gamma\left(\mathrm{n}+\frac{3}{2}\right) n!} \left(\frac{\mathrm{t}}{2}\right)^{2\mathrm{n}+\frac{\mathrm{v}}{\mathrm{k}}+1} \quad (\mathrm{k} \in \mathbb{R}^+, \mathrm{c} \in \mathbb{R}; \mathrm{v} > -1) \quad \ldots(1.15)$$

By putting k = 1 and c = 1 in (1.15), it will be reduced to Struve function of order v is defined by [17] as:

$$\mathrm{H}_{\mathrm{v}}(\mathrm{t}) = \sum_{\mathrm{n}=0}^{\infty} \frac{(-1)^{\mathrm{n}}}{\Gamma\left(\mathrm{n}+\mathrm{v}+\frac{3}{2}\right)\Gamma\left(\mathrm{n}+\frac{3}{2}\right) n!} \left(\frac{\mathrm{t}}{2}\right)^{2\mathrm{n}+\mathrm{v}+1} \quad \ldots(1.16)$$

To study more about Struve function, their generalization and properties the revered reader is call to consider references [4-6, 18-23]. In 2008 the mathematician Manoj Sharma [24] introduced the M – series as:

$${}_pM_q^{\omega}(z) = {}_pM_q^{\omega}(a_1, \ldots, a_p; b_1, \ldots, b_q; z)$$

$$= \sum_{m=0}^{\infty} \frac{(a_1)_m \ldots (a_p)_m \; z^m}{(b_1)_m \ldots (b_q)_m \; \Gamma(\omega m+1)} \qquad \ldots(1.17)$$

Where $z, \omega \in \mathbb{C}$, $\Re(\omega) > 0$, and $(a_i)_m$ $(i = \overline{1,p})$, $(b_j)_m$ $(j = \overline{1,q})$ are the Pochhammer symbols. The Series (1.17) is defined when none of the parameters $b_j$ $(j = \overline{1,q})$ is a negative integer or zero; if any numerator parameter $a_i$ is a negative integer or zero, then the series terminates to a polynomial in z. By using the ratio test it is evident that the series (1.17) is convergent for all z, when q > p, it is convergent for $|z| < 1$ when p = q + 1, Divergent When p > q + 1. In some cases, the series is convergent for z = 1 and z = -1. Let us consider

$$\omega = \sum_{j=1}^{p} a_j - \sum_{j=1}^{q} b_j \qquad \ldots(1.18)$$

When p = q + 1, the series is absolutely convergent for |z| = 1. if $R(\omega) < 0$, convergent for z = -1, if $0 \le \Re(\omega) < 1$ and divergent for |z| = 1, if $1 \le \Re(\omega)$. The following MSM integral operator are required here [13, p.394], [also see in 16] to obtain the MSM fractional integration of generalized k – Struve function.

Lemma 1: Let $\xi, \xi', \tau, \tau', \beta, \eta \in \mathbb{C}$ such that $\Re(\xi) > 0$

(i) If $\Re(\eta) > 0$ max $\{0, \Re(\xi' - \tau'), \Re(\xi + \xi' + \tau - \beta)\}$

$$\left(I_{0^+}^{\xi,\xi',\tau,\tau',\beta} t^{\eta-1}\right)(x) = \frac{\Gamma(\eta)\Gamma(-\xi'+\tau'+\eta)\,\Gamma(-\xi-\xi'-\tau+\beta+\eta)}{\Gamma(\tau'+\eta)\,\Gamma(-\xi-\xi'+\beta+\eta)\,\Gamma(-\xi'-\tau+\beta+\eta)}\, x^{-\xi-\xi'+\beta+\eta-1} \qquad \ldots(1.19)$$

(ii) If $\Re(\eta) > 0$ max $\{\Re(\tau), \Re(-\xi-\xi'-\beta), \Re(-\xi-\tau'+\beta)\}$, then

$$\left(I_{-}^{\xi,\xi',\tau,\tau',\beta} t^{-\eta}\right)(x) = \frac{\Gamma(-\tau+\eta)\Gamma(\xi+\xi'-\beta+\eta)\,\Gamma(\xi+\tau'-\beta+\eta)}{\Gamma(\eta)\,\Gamma(\xi-\tau+\eta)\,\Gamma(\xi+\xi'+\tau'-\beta+\eta)}\, x^{-\xi-\varsigma'+\beta-\eta} \qquad \ldots(1.20)$$

## 2. Main Results

Theorem 1: Let $\xi, \xi', \tau, \omega, \tau', \beta, \eta \in \mathbb{C}$ and $k \in \mathbb{R}^+$, be such that $\Re(\omega) > 0$, $\Re(\beta) > 0$, $\Re(\frac{\sigma}{k}) >$ max $\{0, \Re(\xi' - \tau'), \Re(\xi + \xi' + \tau - \beta)\}$.Also let $c \in \mathbb{R}$; v > -1, then for t > 0

$$\{I_{0^+}^{\xi,\xi',\tau,\tau',\beta}\left(t^{\frac{\sigma}{k}-1} S_{v,c}^{k}(t) S_w^u[t^{\mu}]\, {}_pM_q^{\omega}(t^{\rho})\right)\}(x) = \frac{k^{\beta+\frac{1}{2}}\, x^{-\xi-\xi'+\beta+\frac{\sigma}{k}+\frac{v}{k}}}{2^{\frac{v}{k}+1}} \sum_{s=0}^{(w/u)} \frac{(-W)u,s}{s!} A_{w,s}\, x^{s\mu} \times \sum_{m=0}^{\infty} \frac{(a_1)_m \ldots (a_p)_m \; (x^{\rho})^m}{(b_1)_m \ldots (b_q)_m \; \Gamma(\omega m+1)}$$

$${}_3\Psi_5^{k}\left[\begin{matrix}(\sigma+v+\mu sk+m\rho k+k,\ 2k), (-k\xi'+k\tau'+\sigma+v+\mu sk+m\rho k+k, 2k) \\ (k\tau'+\sigma+v+\mu sk+m\rho k+k, 2k),\ (-k\xi-k\xi'+k\beta+\sigma+v+\mu sk+m\rho k+k, 2k)\end{matrix}\right.$$
$$\left.\begin{matrix}(-k\xi-k\xi'-k\tau+k\beta+\sigma+v+\mu sk+m\rho k+k, 2k) \\ (-k\xi'-k\tau+k\beta+\sigma+v+\mu sk+m\rho k+k, 2k), \left(v+\frac{3k}{2}, k\right), \left(\frac{3k}{2}, k\right)\end{matrix} \middle| \left(-\frac{cx^2k}{4}\right)\right] \ldots(2.1)$$

**Proof:** By using the definition of (1.13), (1.15) (1.17) and taking the left-hand sided MSM fractional integral operator inside the summation the left-hand side of (2.1) becomes

$$= \sum_{s=0}^{(w/u)} \frac{(-W)u,s}{s!} A_{w,s} \times \sum_{m=0}^{\infty} \frac{(a_1)_m \ldots (a_p)_n}{(b_1)_m \ldots (b_q)_m \; \Gamma(\omega m+1)}$$

$$\times \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma_k\left(nk+v+\frac{3k}{2}\right)\Gamma\left(n+\frac{3}{2}\right) n!\, 2^{2n+\frac{v}{k}+1}} \left(I_{0^+}^{\xi,\xi',\tau,\tau',\beta}\{t^{\frac{\sigma}{k}+\frac{v}{k}+\mu s+m\rho+2n+1-1}\}\right)$$

Making use of lemma (1.19), we obtain

$$= \frac{x^{-\xi-\xi'+\beta+\frac{\sigma}{k}+\frac{v}{k}}}{2^{\frac{v}{k}+1}} \sum_{s=0}^{(w/u)} \frac{(-W)u,s}{s!} A_{w,s}\, x^{\mu s} \sum_{m=0}^{\infty} \frac{(a_1)_m \ldots (a_p)_m \; (x^{\rho})^m}{(b_1)_m \ldots (b_q)_m \; \Gamma(\omega m+1)}$$

$$\times \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{\sigma}{k}+\frac{v}{k}+m\rho+2n+\mu s+1\right)}{\Gamma_k\left(nk+v+\frac{3k}{2}\right)\Gamma\left(n+\frac{3}{2}\right) n!}$$

$$\times \frac{\Gamma\left(-\xi-\xi'-\tau+\beta+\frac{\sigma}{k}+\frac{v}{k}+\mu s+m\rho+2n+1\right)\Gamma\left(-\xi'+\tau'+\frac{\sigma}{k}+\frac{v}{k}+\mu s+m\rho+2n+1\right)}{\Gamma\left(\tau'+\frac{\sigma}{k}+\frac{v}{k}+\mu s+m\rho+2n+1\right)\Gamma\left(-\xi-\xi'+\beta+\frac{\sigma}{k}+\frac{v}{k}+\mu s+m\rho+2n+1\right)\Gamma\left(-\xi'-\tau+\beta+\frac{\sigma}{k}+\frac{v}{k}+\mu s+m\rho+2n+1\right)}\left(\frac{-cx^2k}{4}\right)^n$$

Now using (1.5) on above term, then we get

$$=\frac{x^{-\xi-\xi'+\beta+\frac{\sigma}{k}+\frac{v}{k}}}{2^{\frac{v}{k}+1}k^{-\beta-\frac{1}{2}}}\sum_{s=0}^{(w/u)}\frac{(-W)u,s}{s!}A_{w,s}\,x^{\mu s}\times\sum_{m=0}^{\infty}\frac{(a_1)_m\ \ldots\ (a_p)_m\ \ (x^{\rho})^m}{(b_1)_m\ \ldots\ (b_q)_m\ \ \Gamma(\omega m+1)}$$

$$\sum_{n=0}^{\infty}\frac{\Gamma_k(\sigma+v+\mu sk+m\rho k+k+2nk)}{\Gamma_k(k\tau'+\sigma+v+\mu sk+m\rho k+k+2nk)\,\Gamma_k(-k\xi-k\xi'+k\beta+\sigma+v+\mu sk+m\rho k+k+2nk)}$$

$$\times\frac{\Gamma_k(-k\xi'+k\tau'+\sigma+v+\mu sk+m\rho k+k+2nk)\,\Gamma_k(-k\xi-k\xi'-k\tau+k\beta+\sigma+v+\mu sk+m\rho k+k+2nk)}{\Gamma_k(-k\xi'-k\tau+k\beta+\sigma+v+\mu sk+m\rho k+k+2nk)\,\Gamma_k\left(nk+v+\frac{3k}{2}\right)\Gamma_k\left(nk+\frac{3k}{2}\right)n!}\left(\frac{-cx^2k}{4}\right)^n$$

Using the definition of (1.2) in the above term we at once arrive at the desired result (2.1).

**Theorem 2:** Let $\xi, \xi', \omega, \tau, \tau', \beta, \eta \in \mathbb{C}$ and $k \in \mathbb{R}^+$, be such that $\Re(\omega) > 0$, $\Re(\beta) > 0$, $\Re(\frac{\sigma}{k}) > \max\{\Re(\tau), \Re(-\xi-\xi'-\tau'), \Re(-\xi-\tau'+\beta)\}$.Also let $c \in \mathbb{R}$; $v > -1$, then for $t > 0$

$$\{I_{-}^{\xi,\xi',\tau,\tau',\beta}\left(t^{\frac{\sigma}{k}-1}S^k_{v,c}(t)S^u_w[t^{\mu}]_p M^{\omega}_q(t^{\rho})\right)\}(x) = \frac{k^{\beta+\frac{1}{2}}\,x^{-\xi-\xi'+\beta+\frac{\sigma}{k}+\frac{v}{k}}}{2^{\frac{v}{k}+1}}$$

$$\sum_{s=0}^{(w/u)}\frac{(-W)u,s}{s!}A_{w,s}\,x^{\mu s}\times\sum_{m=0}^{\infty}\frac{(a_1)_m\ \ldots\ (a_p)_m\ \ (x^{\rho})^m}{(b_1)_m\ \ldots\ (b_q)_m\ \ \Gamma(\omega m+1)}$$

$${}_3\Psi_5^k\left[\begin{matrix}(-k\tau-\sigma-v-\mu sk-m\rho k,-2k),(k\xi+k\tau'-k\beta-\sigma-v-\mu sk-m\rho k,-2k)\\(-\sigma-v-\mu sk-m\rho k,-2k),(k\xi+k\xi'+k\tau'-k\beta-\sigma-v-\mu sk-m\rho k,-2k)\end{matrix}\right.$$
$$\left.\begin{matrix}(k\xi+k\xi'-k\beta-\sigma-v-\mu sk-m\rho k,-2k)\\(k\xi-k\tau-\sigma-v-\mu sk-m\rho k,-2k),\left(v+\frac{3k}{2},k\right),\ \left(\frac{3k}{2},k\right)\end{matrix}\middle|\left(-\frac{cx^2k}{4}\right)\right] \quad \ldots(2.2)$$

**Proof:** By using the definition of (1.13), (1.15), (1.17) and taking the right-hand sided MSM fractional integral operator inside the summation the left-hand side of (2.2) becomes

$$=\sum_{s=0}^{(w/u)}\frac{(-W)u,s}{s!}A_{w,s}\times\sum_{m=0}^{\infty}\frac{(a_1)_m\ \ldots\ (a_p)_m}{(b_1)_m\ \ldots\ (b_q)_m\ \ \Gamma(\omega m+1)}$$

$$\times\sum_{n=0}^{\infty}\frac{(-c)^n}{\Gamma_k\left(nk+v+\frac{3k}{2}\right)\Gamma\left(n+\frac{3}{2}\right)n!\,2^{2n+\frac{v}{k}+1}}\left(I_{-}^{\xi,\xi',\tau,\tau',\beta}\{t^{-(-\frac{\sigma}{k}-\frac{v}{k}-\mu s-m\rho-2n)}\}\right)$$

On applying lemma (1.20), we get

$$=\frac{x^{-\xi-\xi'+\beta+\frac{\sigma}{k}+\frac{v}{k}}}{2^{\frac{v}{k}+1}}\sum_{s=0}^{(w/u)}\frac{(-W)u,s}{s!}A_{w,s}\,x^{\mu s}\times\sum_{m=0}^{\infty}\frac{(a_1)_m\ \ldots\ (a_p)_m\ \ (x^{\rho})^m}{(b_1)_m\ \ldots\ (b_q)_m\ \ \Gamma(\omega m+1)}\times$$

$$\sum_{n=0}^{\infty}\frac{\Gamma\left(-\tau-\frac{\sigma}{k}-\frac{v}{k}-\mu s-m\rho-2n\right)}{\Gamma_k\left(nk+v+\frac{3k}{2}\right)\Gamma\left(n+\frac{3}{2}\right)n!}\,\frac{\Gamma\left(\xi+\xi'-\beta-\frac{\sigma}{k}-\frac{v}{k}-\mu s-m\rho-2n\right)\Gamma\left(\xi+\tau'-\beta-\frac{\sigma}{k}-\frac{v}{k}-\mu s-m\rho-2n\right)}{\Gamma\left(-\frac{\sigma}{k}-\frac{v}{k}-\mu s-m\rho-2n\right)\Gamma\left(\xi-\tau-\frac{\sigma}{k}-\frac{v}{k}-\mu s-m\rho-2n\right)\Gamma\left(\xi+\xi'+\tau'-\beta-\frac{\sigma}{k}-\frac{v}{k}-\mu s-m\rho-2n\right)}\left(\frac{-cx^2k}{4}\right)^n$$

Now making use of k – gamma function (1.5), we get

$$=\frac{x^{-\xi-\xi'+\beta+\frac{\sigma}{k}+\frac{v}{k}}}{2^{\frac{v}{k}+1}k^{-\beta-\frac{1}{2}}}\sum_{s=0}^{(w/u)}\frac{(-W)u,s}{s!}A_{w,s}\,x^{\mu s}\times\sum_{m=0}^{\infty}\frac{(a_1)_m\ \ldots\ (a_p)_m\ \ (x^{\rho})^m}{(b_1)_m\ \ldots\ (b_q)_m\ \ \Gamma(\omega m+1)}\times$$

$$\sum_{n=0}^{\infty}\frac{\Gamma_k(-k\tau-\sigma-v-\mu sk-m\rho k-2nk)}{\Gamma_k(-\sigma-v-\mu sk-m\rho k-2nk)\,\Gamma_k(k\xi-k\tau-\sigma-v-\mu sk-m\rho k-2nk)}$$

$$\times \frac{\Gamma_k(k\xi+k\xi'-k\beta-\sigma-v-\mu sk-m\rho k-2nk)\,\Gamma_k(k\xi+k\tau'-k\beta-\sigma-v-\mu sk-m\rho k-2nk)}{\Gamma_k(k\xi+k\xi'+k\tau'-k\beta-\sigma-v-\mu sk-m\rho k-2nk)\,\Gamma_k\left(nk+v+\frac{3k}{2}\right)\Gamma_k\left(nk+\frac{3k}{2}\right)n!}\left(\frac{-cx^2k}{4}\right)^n$$

Using the definition of (1.2) in the above term we at once arrive at the desired result (2.2).

**Theorem 3:** Let $\xi, \xi', \omega\ \tau, \tau', \beta, \eta \in \mathbb{C}$ and $k \in \mathbb{R}^+$, be such that $\Re(\omega) > 0$, $\Re(\beta) > 0$, $\Re(\frac{\sigma}{k}) > \max\{\Re(\tau), \Re(-\xi-\xi'+\beta), \Re(-\xi-\tau'+\beta)\}$.Also let $c \in \mathbb{R}$; $v > -1$, then for $t > 0$ $I_{-}^{\xi,\xi',\tau,\tau',\beta}\left(t^{-\frac{\sigma}{k}} S_{v,c}^{k}(t) S_w^u[t^\mu]_p M_q^\omega(t^\rho)\right)(x) = \frac{k^{\beta+\frac{1}{2}}x^{-\xi-\xi'+\beta-\frac{\sigma}{k}+\frac{v}{k}+1}}{2^{\frac{v}{k}+1}}$

$$\times \sum_{s=0}^{(w/u)} \frac{(-W)u\,,s}{s!} A_{w,s}\, x^{\mu s} \times \sum_{m=0}^{\infty} \frac{(a_1)_m \ldots (a_p)_m\ (x^\rho)^m}{(b_1)_m \ldots (b_q)_m\ \Gamma(\omega m+1)} \times$$

$${}_3\Psi_5^k\left[\begin{matrix}(-k\tau+\sigma-v-\mu sk-m\rho k-k,-2k),\ (k\xi+k\xi'-k\beta+\sigma-v-\mu sk-m\rho k-k,-2k)\\(\sigma-v-\mu sk-m\rho k-k,-2k),\ (k\xi+k\xi'+k\tau'-k\beta+\sigma-v-\mu sk-m\rho k-k,-2k)\end{matrix}\right.$$

$$\left.\begin{matrix}(k\xi+k\tau'-k\beta+\sigma-v-\mu sk-m\rho k-k,-2k)\\(k\xi-k\tau+\sigma-v-\mu sk-m\rho k-k,-2k),\left(v+\frac{3k}{2},k\right),\ \left(\frac{3k}{2},k\right)\end{matrix}\right|\left(-\frac{cx^2k}{4}\right)\right] \quad \ldots(2.3)$$

**Proof: -** By using the definition of (1.13), (1.15), (1.17) and taking the right-hand sided MSM fractional integral operator inside the summation the left-hand side of (2.3) becomes

$$= \sum_{s=0}^{(w/u)} \frac{(-W)u\,,s}{s!} A_{w,s} \times \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_p)_n}{(b_1)_n \ldots (b_q)_n\ \Gamma(\omega n+1)}$$

$$\times \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma_k\left(nk+v+\frac{3k}{2}\right)\Gamma\left(n+\frac{3}{2}\right)n!\,2^{2n+\frac{v}{k}+1}} \left(I_{-}^{\xi,\xi',\tau,\tau',\beta}\{t^{-(-\frac{v}{k}+\frac{\sigma}{k}-\mu s-3n-1)}\}\right)$$

On applying lemma (1.20), we get

$$= \frac{x^{-\xi-\xi'+\beta-\frac{\sigma}{k}+\frac{v}{k}+1}}{2^{\frac{v}{k}+1}} \sum_{s=0}^{(w/u)} \frac{(-W)u\,,s}{s!} A_{w,s}\, x^{\mu s} \times \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_p)_n\ x^n}{(b_1)_n \ldots (b_q)_n\ \Gamma(\omega n+1)}$$

$$\sum_{n=0}^{\infty} \frac{\Gamma\left(-\tau+\frac{\sigma}{k}-\frac{v}{k}-\mu s-m\rho-2n-1\right)}{\Gamma_k\left(nk+v+\frac{3k}{2}\right)\Gamma\left(n+\frac{3}{2}\right)n!}$$

$$\times \frac{\Gamma\left(\xi+\xi'-\beta+\frac{\sigma}{k}-\frac{v}{k}-\mu s-2n-m\rho-1\right)\Gamma\left(\xi+\tau'-\beta+\frac{\sigma}{k}-\frac{v}{k}-\mu s-m\rho-2n-1\right)}{\Gamma\left(\frac{\sigma}{k}-\frac{v}{k}-\mu s-m\rho-2n-1\right)\Gamma\left(\xi-\tau+\frac{\sigma}{k}-\frac{v}{k}-\mu s-m\rho-2n-1\right)\Gamma\left(\xi+\xi'+\tau'-\beta+\frac{\sigma}{k}-\frac{v}{k}-\mu s-m\rho-2n-1\right)}\left(\frac{-cx^2k}{4}\right)^n$$

Now making use of k – gamma function (1.5) in the above term, we get

$$= \frac{x^{-\xi-\xi'+\beta-\frac{\sigma}{k}+\frac{v}{k}+1}}{2^{\frac{v}{k}+1}k^{-\beta-\frac{1}{2}}} \sum_{s=0}^{(w/u)} \frac{(-W)u\,,s}{s!} A_{w,s}\, x^{\mu s} \sum_{m=0}^{\infty} \frac{(a_1)_m \ldots (a_p)_m\ (x^\rho)^m}{(b_1)_m \ldots (b_q)_m\ \Gamma(\omega m+1)}$$

$$\sum_{n=0}^{\infty} \frac{\Gamma_k(-k\tau+\sigma-v-\mu sk-m\rho k-k-2nk)}{\Gamma_k(\sigma-v-\mu sk-m\rho k-k-2nk)\,\Gamma_k(k\xi-k\tau+\sigma-v-\mu sk-m\rho k-k-2nk)}$$

$$\times \frac{\Gamma_k(k\xi+k\xi'-k\beta+\sigma-v-\mu sk-m\rho k-k-2nk)\,\Gamma_k(k\xi+k\tau'-k\beta+\sigma-v-\mu sk-m\rho k-k-2nk)}{\Gamma_k(k\xi+k\xi'+k\tau'-k\beta+\sigma-v-\mu sk-m\rho k-k-2nk)\,\Gamma_k\left(nk+v+\frac{3k}{2}\right)\Gamma_k\left(nk+\frac{3k}{2}\right)n!}\left(\frac{-cx^2k}{4}\right)^n$$

Using the definition of (1.2) in the above term we at once arrive at the desired result (2.3).

## 3. Special Cases

1. If we take w = 0, $A_{0,0} = 1$, then $S_0^u[x] \to 1$ and taking $k \to 1$, c = 1 in (2.1), the generalized k – Struve functions yield to Struve function of order v, so we get the following result.

$$\{I_{0^+}^{\xi,\xi',\tau,\tau',\beta}(t^{\sigma-1}H_v(t)\,{}_pM_q^{\omega}(t^{\rho})\}(x) = \frac{x^{-\xi-\xi'+\beta+\sigma+v}}{2^{v+1}} \times {}_pM_q^{\omega}(x^{\rho}) = \frac{x^{-\xi-\xi'+\beta+\sigma+v}}{2^{v+1}} \times {}_pM_q^{\omega}(x^{\rho})$$

$$\times\, {}_3\Psi_5\left[\begin{matrix}(\sigma+v+m\rho+1,2),\ (-\xi'+\tau'+\sigma+v+m\rho+1,2),\ (-\xi-\xi'-\tau+\beta+\sigma+v+m\rho+1,2)\\ (\tau'+\sigma+v+m\rho+1,2),(-\xi-\xi'+\beta+\sigma+v+m\rho+1,2),(-\xi'-\tau+\beta+\sigma+v+m\rho+1,2),\left(v+\frac{3}{2},1\right),\left(\frac{3}{2},1\right)\end{matrix}\ \middle|\left(-\frac{cx^2}{4}\right)\right]$$

2. On setting w = 0, $A_{0,0} = 1$, then $S_0^u[x] \to 1$ and taking $k \to 1$, c = 1 in (5.8.2), the generalized k – Struve functions yield to Struve function of order v, so we get the following result.

$$\left\{I_{-}^{\xi,\xi',\tau,\tau',\beta}\left(t^{\sigma-1}H_v(t)\,{}_pM_q^{\omega}(t^{\rho})\right)\right\}(x) = \frac{x^{-\xi-\xi'+\beta+\sigma+v}}{2^{v+1}} \times {}_pM_q^{\omega}(x^{\rho})$$

$$\times\, {}_3\Psi_5\left[\begin{matrix}(-\tau-\sigma-v-m\rho,-2),\ (-\xi+\tau'-\beta-\sigma-v-m\rho,-2),\ (\xi+\xi'-\beta-\sigma-v-m\rho,-2)\\ (-\sigma-v-m\rho,-2),(\xi+\xi'+\tau'-\beta-\sigma-v-m\rho,-2),(\xi-\tau-\sigma-v-m\rho,-2),\left(v+\frac{3}{2},1\right),\ \left(\frac{3}{2},1\right)\end{matrix}\ \middle|\left(-\frac{cx^2}{4}\right)\right]$$

3. On taking $k \to 1$ and c = 1 in (2.3), the generalized k – Struve functions yield to Struve function of order v, so we get the following result.

$$I_{-}^{\xi,\xi',\tau,\tau',\beta}(t^{-\sigma}H_v(t)S_w^u[t^{\mu}]\,{}_pM_q^{\omega}(t^{\rho})\}(x) = \frac{x^{-\xi-\xi'+\beta-\sigma+v+1}}{2^{v+1}}\sum_{s=0}^{(w/u)}\frac{(-W)u,s}{s!}A_{w,s}\,x^{\mu s}\,{}_pM_q^{\omega}(x^{\rho})$$

$${}_3\Psi_5\left[\begin{matrix}(-\tau+\sigma-v-\mu s-m\rho-1,-2),\ (\xi+\xi'-\beta+\sigma-v-\mu s-m\rho-1,-2),\ (\xi+\tau'-\beta+\sigma-v-\mu s-m\rho-1,-2)\\ (\sigma-v-\mu s-m\rho-1,-2),\ (\xi+\xi'+\tau'-\beta+\sigma-v-\mu s-m\rho-1,-2),(\xi-\tau+\sigma-v-\mu s-m\rho-1,-2),\left(v+\frac{3}{2},1\right),\ \left(\frac{3}{2},1\right)\end{matrix}\ \middle|-\frac{cx^2}{4}\right]$$

4. On setting w = 0, $A_{0,0} = 1$, then $S_0^u[x] \to 1$ and consider M – series as 1 in (2.1), Researchers arrive at the known result of [16, eq. (2.1), pp. 596].
5. On setting w = 0, $A_{0,0} = 1$, then $S_0^u[x] \to 1$ and consider M – series as 1 in (2.2), Researchers arrive at the known result of [16, eq. (2.2), pp. 597].
6. On setting w = 0, $A_{0,0} = 1$, then $S_0^u[x] \to 1$ and consider M – series as 1 in (2.3), Researchers arrive at the known result of [16, eq. (2.3), pp. 598].
7. If we consider ${}_pM_q^{\omega}(t^{\rho})$ as unity in (2.1), Researchers arrive at the known result of [25, eq. (2.1), pp.302].
8. If we consider ${}_pM_q^{\omega}(t^{\rho})$ as unity in (2.2), Researchers arrive at the known result of [25, eq. (2.2), pp.303].
9. If we consider ${}_pM_q^{\omega}(t^{\rho})$ as unity in (2.3), Researchers arrive at the known result of [25, eq. (2.3), pp.304].

## 4. Conclusion

Due to the generalization of Riemann – Liouville, Weyl, Erdelyi – Kobder and Saigo's fractional calculus operators. MSM fractional calculus operator have a compelling advantage, that was the reason many authors are yelled as general operators. Now we close out of this paper by highlighting that our results (Theorem 1 – 3) can be deduced as the special cases involving familiar fractional calculus operators as above said. The generalized k – Struve function and M – series defined in (1.15) and (1.17) respectively has the property that several special functions appear to be the special cases. Various special cases involving integral relating to the k – Struve function and M – series have been exposed in the earlier research worked by various authors with different arguments.

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