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Original Article

Kosambi-Carton-Chern (KCC) Theory for Jacobi Stability Analysis in Certain Systems

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Abstract *- KCC theory offers a robust geometric framework for analyzing the stability of dynamical systems described by second-order differential equations. Its use of KCC invariants and Jacobi stability analysis provides insights into the behavior of nonlinear systems across various fields, including cosmology, mechanics, biology, and control theory. By transforming stability analysis into a geometric problem, the KCC theory enables a deeper understanding of the conditions under which systems maintain or lose stability, thereby offering practical insights into real-world applications. The main ideas of the KCC theory are examined in this study, along with how it is used for Jacobi stability analysis in specific systems. Jacobi stability for various dynamic systems, including the Rititake, Rossler, Chua circuit, RF, and tumor growth models, as well as the KCC theory and its constituent parts, is explained.*

Keywords *- KCC- geometric theory, Jacobian stability, Deviation curvature tensor, Cartan tensor, Finsler connection*.

2010 AMS Subject Classification: *53B40,53C60,53C70,58B20.*

1. Introduction

The Kawaguchi–Chern–Cartan (Kawaguchi-Chern-Cartan) theory, initially developed in the context of Finsler geometry, has found applications beyond its original scope. It serves as a powerful tool for studying the dynamics and stability of differential systems, particularly those that exhibit nonlinear behavior.

In mathematical modeling, stability analysis plays a crucial role in understanding the behavior of complex systems. Kosambi-Carton-Chern (KCC) theory, a significant advancement in this field, offers a comprehensive framework for analyzing stability in certain systems. The primary objectives of this study are to examine the fundamental aspects of KCC theory and its utilization in Jacobi stability analysis for specific systems.

Synge [1], Knebelman [2], and Douglas [3] pioneered the contemporary geometry of the second-order differential equations (SODE) in the 1920s. KCC geometric theory first appeared in the papers of Kosambi [4], Cartan [5], and Chern [6] in the years 1933–1939. This is why the term KCC is abbreviated. Sixty years later, Antonelli, Ingarden, and Matsumoto rediscovered and developed this theory [7,9,10]. Following this, several applications in biology, chemistry, engineering, and physics were shown [11,12,13,14,15,16]. Additionally, new ideas and contemporary methods related to KCC theory in black hole theories are available in [17,18, 19]. Boehmer, Harko, and Sabău [8] examined Jacobi stability linked to classical stability, along with several applications in astrophysics, gravitation, and cosmology, is highly significant and helpful.

2. Novelty and Significance

Kosambi-Carton-Chern (KCC) theory emerges as a synthesis of ideas from various mathematical disciplines, including differential geometry, algebraic topology, and dynamical systems theory. KCC theory offers a powerful tool for exploring the stability properties of dynamical systems governed by differential equations.

The foundational concept of KCC theory revolves around the notion of Jacobi fields. Jacobi fields represent infinitesimal perturbations along trajectories of dynamical systems. By studying the behavior of Jacobi fields, one can gain insights into the stability or instability of the underlying system.

3. Preliminaries

We discuss some preliminary concepts that are further useful:

Definition 3. 1 (Finsler Space): A Finsler function L of n-dimensional manifold M^n is called Finsler space, F^n = (M^n , L).

Definition 3.2 (Finsler Metric): Let (M^n , L) be a Finsler space. The Hessian of L^2 with respect to the tangent space coordinates, y defines the components g_{ij} of a zero homogeneous symmetric tensor field (0, 2).

$g_{ij}(x,y) = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2$(x, y).

Definition 3 3 (Cartan Tensor): Let (M^n , L) be a Finsler space. The 3rd derivative of $L^2$with respect to the tangent space, co-ordinates y defines the components C_{ijk} of minus one homogeneous symmetric (0,3)-tensor field

$C_{ijk} = \frac{1}{4}\dot{\partial}_i\dot{\partial}_j L^2$(x, y) = $\frac{1}{2}\dot{\partial}_{k}g_{ij}$

The associate Cartan tensor C^i_{jk} is defined as

$C^i_{jk} = g^{ih}C_{jhk}$

Definition 4.4 (Finsler Connections): The function F^i_{jk} , $N^i_j$and V^i_{jk} are connections coefficients of Finsler connection and denote as the triad

FΓ = (F^i_{jk} , N^i_j, V^i_{jk})

Definition5. 5 (Berwald Connection): The Berwald connection [7] is denoted as BΓ = (G^i_{jk} , G^i_j,0) and determined from the following axioms:

(i) L-metrical: $L_{|i}$ = 0,
(ii) (h) h- torsion T = 0,
(iii) Deflection when D = 0,
(iv) (v) hv-torsion P^1= 0,
(v) V- connection is flat when V^i_{jk} = 0.

4. Key Components of KCC Theory

The KCC theory originates from the geometric study of second-order differential equations (SODEs), offering a framework that parallels the geometric structure observed in Riemannian geometry but with a focus on systems described by nonlinear equations.

Jacobi Stability Analysis: Central to KCC theory is the analysis of Jacobi stability. This involves examining the evolution of the Jacobi fields along the trajectories of the dynamical system. A system is Jacobi stable if its fields remain bounded over time, indicating a stable behavior. Conversely, the unbounded Jacobi fields signify an instability in the system.

Geometric Interpretation: KCC theory offers a geometric interpretation of the stability analysis. By considering the curvature and torsion of the underlying manifold, the intrinsic geometric properties that influence the stability of the dynamical system can be discerned.

Topological Aspects: Another distinguishing feature of KCC theory is its consideration of topological aspects in stability analysis. By exploring the topology of the phase space, KCC theory provides a deeper understanding of stability phenomena, including bifurcations and phase transitions.

5. Method and Discussion

The methodology employed in examining the fundamental aspects of KCC theory and its utilization in Jacobi stability analysis for specific systems involves a structured, step-by-step approach. The process is based on both theoretical formulations and practical applications to specific dynamical systems.

5.1. KCC-Theory

The second-order differential equations will identify a non-linear connection on TM and a Finsler connection, specifically the Berwald connection generated by HTM [20]. We shall obtain the system's invariants from the Berwald connection.

Let's review the fundamental ideas and findings of the KCC theory [8, 9, 5, 6, 10, 11, and 4]. Imagine a n-dimensional manifold M, along with its bundle of tangents, TM. Consider

x = $(x^1, x^2, \cdots\cdots, x^n)$

y = $\left(\frac{dx^1}{dt}, \frac{dx^2}{dt}, \cdots \quad \frac{dx^n}{dt}\right) = \left(\frac{dx}{dt}\right)$

Where t is local 2n+1 co-ordinate in an open, connected subset Ω of the real (2n+1) dimensional Euclidean space $R^n \times R^n \times R^1$. Let us examine the second-order differential equation in the following form:

$$\frac{d^2x^i}{dt^2} + 2G^i\,(\text{x, y, t}) = 0, \text{ for i = 1, 2,}\cdots\cdots n. \tag{1}$$

Where $G^i \in G^\infty$ with initial conditions $((x)_0, (y)_0 t_0\ ) \in \Omega$. Equation (1) is the same as the equation of Finsler space of the Eular-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial Y^I}\right) - \frac{\partial L}{\partial x^i} = F_i, \text{ for i = 1,2,}\cdots\cdots n. \tag{2}$$

L = Lagrangian force

F_i= External force

The intrinsic geometric properties described in equation (1) for non-singular coordinate transformations

$$\bar{x}^{\,i} \quad = f^i(x^1, x^2, \cdots\cdots, x^n\ ), \quad \text{for i = 1, 2,}\cdots\cdots n. \tag{3}$$

$\bar{t}$ = t is given by the five KCC invariants [4]. Carton [5], and Chen [6].

The KCC -covariant derivative of a contravariant vector field from equation (3) is given by

ξ= ξ^i (x) $\frac{\partial}{\partial x^i}$ on $\Omega \subseteq R^n \times R^n$ from [8, 21].

$$\frac{D\xi^i}{dt} = \frac{d\xi^i}{dt} + \mathrm{N}_j^i\,\xi^i \tag{4}$$

N_j^i = Co-efficient of the non-linear connection

This non-linear connection is described as a dynamical covariant derivative ∇^{N} [12]. Over the manifold M for two vector fields v, w, the covariant derivatives ∇^{∇}

$$\nabla_v^{\mathrm{N}} = [\ v^i \frac{\partial}{\partial x^i} w^i + \mathrm{N}_j^i(x,y) w^j] \frac{\partial}{\partial x^i} \tag{5}$$

Now substituting $\xi^i = y^i$ in equation (5) and using equation (1) gives

$$\frac{Dy^i}{dt} = \mathrm{N}_j^i\, y^i - 2G^i = -\epsilon^i \tag{6}$$

ϵ^i is contravariant vector field is called the first KCC- invariant represents external force [13].

The variation of trajectories x^i(t) along the curve $x^i = x^i(t)$in the equation (1) as

$\bar{x}^{\,i}\,(t) = x^i(t) + \xi^i(t)_\eta$

describe x and $\bar{x}$ are solutions of equation (1). We enable to see that

G (t,x+$\eta\xi, \dot{x + \eta\xi}$) – G(t,x,$\dot{x}$) = 0. While applying the mean value theorem, the variational equation is defined as [2,11,12]

$$\frac{d^2\xi^i}{dt^2} + 2\mathrm{N}_j^i \frac{d\xi^i}{dt} + 2\frac{\partial G^i}{\partial x^j}\xi^j = 0 \tag{7}$$

After using the equation (4) in equation (7) then

$$\frac{d^2\xi^i}{dt^2} = P_j^i \xi^j \tag{8}$$

$$\text{Where } P^i_j = -2\frac{\partial G^i}{\partial x^j} - 2G^l G^i_{jl} + y^l \frac{\partial N^i_j}{\partial x^l} + N^i_l N^l_j + \frac{\partial N^i_j}{\partial t} \tag{9}$$

Now, $G^i_{jl} = \frac{\partial N^i_j}{\partial t}$ is the Berwald connection [8,9] and P^i_j is the second KCC-Invariant of equation (1).The equation (1) describes the geodesic, while the equation (8) describes the Jacobi field equation. The idea is that the generalization of a Riemannian of Finsler Manifold's geodesics is known as Jacobi stability for SODE. For KCC theory, this validates the term Jacobi stability. The system in equation (1) has three, four, and five invariants, which are defined as [8, 9].

$$P^i_{jk} = \frac{1}{3}\left(\frac{\partial P^i_j}{\partial y^k} - \frac{\partial P^i_k}{\partial y^j}\right),\ P^i_{jkl} = \frac{\partial P^i_{jk}}{\partial y^l},\ D^i_{jkl} = \frac{\partial G^i_{jk}}{\partial y^l} \tag{10}$$

The third, fourth, and fifth invariants are the torsion tensor, Riemann curvature tensor, and Douglas curvature tensor, respectively. As an alternative, we provide the third and fourth invariants with the following definition [9]:

$$B^i_{jk} = \frac{\delta N^i_j}{\delta x^k} - \frac{\delta N^i_k}{\delta x^j}, \tag{11}$$

$$B^i_{jkl} = \frac{\partial B^i_{kl}}{\partial y^i} \tag{12}$$

$$\text{Where } \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j_i \frac{\partial}{\partial y^j} \tag{13}$$

6. Jacobi Stability of the Dynamical System

Jacobi stability, named after the German mathematician Carl Gustav Jacob Jacobi, is a concept used in the dynamical systems in the context of differential equations. It concerns the stability of solutions to ordinary differential equations (ODEs) around fixed points or equilibrium points.

In the context of nonlinear dynamical systems, stability analysis around equilibrium points are critical for understanding long-term behavior. The Jacobian matrix plays a fundamental role in this analysis.

The trajectories of the curve $x^i = x^i$(t) of $\frac{d^2x^i}{dt^2} + 2G^i$ (x, y, t) = 0, for i = 1,2,$\cdots\cdots n.$ in equation (1), defining a canonical inner product of Euclidean space ($R^n, <, >$). Let the second KCC-invariant vector ξ hold the following conditions:

$$\xi(0) = 0, \dot{\xi}(0) = W \neq 0 \tag{14}$$

Now, an adapted inner product $<<.,.>>$ for the KCC-invariant tensor ξ is

$<< X, Y >> := \frac{1}{<W,W>} . < X, Y >$ for X, Y in R^n. Now, $||W||^2 := < W, W > = 1$, shows that t $\approx 0^+$ is the trajectories of equation (1) [14,15,16].

If all eigenvalues of $P^i_j(0)$ or $P^i_j|t_0$ have negative real parts, then jacobi equilibrium at (x(t_0), $\dot{x}(t_0)$) indicating that small l perturbations around this point will decay over time, and the system will return to equilibrium. If any eigenvalue $P^i_j(0)$ are positive real parts, and the equilibrium point is unstable. This implies that small perturbations will grow over time, leading to trajectories that diverge. If any eigenvalue is purely imaginary, further analysis is needed, typically involving higher-order terms or specialized techniques, to determine the stability. This is often the case in systems with center manifold behavior, where linear stability analysis alone is insufficient.

Jacobi stability analysis provides a valuable tool for understanding the qualitative behavior of dynamical systems around their equilibrium points. KCC theory of Jacobi stability analyzes the different dynamic systems. Here are some descriptions of examples:

6.1. Jacobi Stability of the Rikitake System

The Rikitake system [22], named after Japanese mathematician Morikazu Rikitake, is a simple model of a dynamo, which is a device that generates electric currents from the movement of conductive fluids. The Rikitake system is often used as a prototype for studying dynamo theory and as an example in the field of dynamical systems. To analyze the Jacobi stability of the Rikitake system, we would first need to define the system's equations of motion. The Rikitake system is described by a set of ordinary differential equations.

The Rikitake system typically consists of two coupled differential equations representing the evolution of two state variables.

$\frac{dx}{dt}$ = a(y-x)-b z, $\frac{dy}{dt}$ = x(1-z)-y, $\frac{dz}{dt}$ = x y-c z

where x, y, and z are the state variables, and a, b, and c are parameters of behavior of the system.

To analyze this system, one would typically linearize the equations of motion around a particular equilibrium point and then examine the eigenvalues of the resulting linearized system. The stability of the equilibrium point is determined by the sign of the real parts. However, it's worth noting that the Rikitake system is a highly nonlinear dynamical system, and the analysis of its stability can be quite complex. For the values of the parameters a, b, and c, the system can exhibit a wide range of behaviors, including stable fixed points, periodic orbits, and chaotic dynamics [22].

6.2. Jacobi Stability of the Rössler System

The Rössler system [23,24] is a well-known set of three coupled ordinary differential equations that exhibits chaotic behavior. It was introduced by German biochemist Otto Rössler in 1976. The equations describing the Rössler system are:

$\frac{dx}{dt}$ = -y-z

$\frac{dy}{dt}$ = x + ay

$\frac{dz}{dt}$ = z(x-c)

To analyze it, we need to consider the linearized system around an equilibrium point. Let us denote the equilibrium point as x_0 , y_0 , z_0 .We linearize the system by computing:

$$\begin{bmatrix} -\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} - & -\frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{bmatrix}$$

where f_1=−y−z, f_2= x+ ay, and f_3= b+ z(x− c).

Then, we evaluate this Jacobian matrix at the equilibrium point x_0 , y_0 , z_0 and compute its eigenvalues.

However, it is important to note that the Rössler system is known for its chaotic behavior, which means that it does not possess stable equilibrium points in the traditional sense. Instead, it exhibits complex dynamics such as strange attractors.

6.3. Jacobi Stability of the Modified Chua Circuit System

The modified Chua's circuit is a nonlinear electronic circuit exhibiting chaotic behavior. It is derived from the original Chua's circuit by introducing modifications to its parameters or adding additional components. Chua's circuit is a simple electronic circuit featuring a piecewise linear resistor, and it is known for its ability to generate chaotic oscillations.

The equations describing the modified Chua's circuit system can vary depending on the specific modifications made to the original circuit. However, a common set of equations for the modified Chua's circuit can be represented as:

$\frac{dx}{dt}$ = α(y-x-f(x))

$\frac{dy}{dt}$ = x- y + z

$\frac{dz}{dt}$ = βy

Here, x, y, and z are the state variables, while α and β are parameters.

However, it is important to note that Chua's circuits, including the modified versions, are primarily known for their chaotic behavior. Therefore, stability analysis in the traditional sense may not be as meaningful, as these systems typically exhibit complex dynamics such as chaotic attractors and sensitive dependence on initial conditions.

6.4. Jacobi Stability of the Tumor Growth Model

The tumor growth model describes the dynamics of tumor growth over time. There are various mathematical models used to represent tumor growth, ranging from simple exponential growth models to more complex models incorporating factors such as nutrient supply, angiogenesis (the formation of new blood vessels), and immune system interactions.

One commonly used model for tumor growth is the Gompertz model, which is a type of exponential growth model that slows down over time. The Gompertz model is described by the following ordinary differential equation:

$$\frac{dV}{dt} = -\alpha V \, l_n \left(\frac{V}{V_0}\right)$$

Here, V represents the volume of the tumor, t represents time, α is a parameter related to the growth rate, and V_0 is a reference volume. To analyze it, we first need to find its equilibrium points. In this case, the equilibrium points occur when the growth rate given by

$$-\alpha V \, l_n \left(\frac{V}{V_0}\right) = 0, \text{ for } V = V_0$$

To linearize the equilibrium point V = V_0, we compute the Jacobian matrix J of the system evaluated at this point. The Jacobian matrix is given by:

$$J = \begin{bmatrix} -\alpha l_n \left(\frac{V_0}{V_0}\right) & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The eigenvalues for this Jacobian matrix are all zero. This means that equilibrium point V = V_0

is neutrally stable according to the linearized analysis.

However, it is important to note that the Gompertz model is a simplification of tumor growth dynamics and does not capture all the complexities of real tumor growth. In reality, tumor growth is influenced by various factors such as nutrient supply, intercellular interactions, and immune responses. Therefore, while the linear stability analysis provides some insights into the behavior of the model around equilibrium, it may not fully capture the dynamics of tumor growth in real biological systems. More sophisticated models and analyses are often needed to accurately capture tumor growth dynamics and study the effects of interventions such as chemotherapy or immunotherapy.

6.5. Jacobi stability of RF System

The Rabinovich-Fabrikant system [25] is a famous set of ordinary differential equations that exhibits chaotic behavior. It was introduced by Michael V. Rabinovich and Anatoly I. Fabrikant in 1979. The system is often used as a model in the study of chaotic systems and nonlinear dynamics. The Rabinovich-Fabrikant system is given as:

$$\frac{dx}{dt} = y\,(\mathcal{Z}-1+x^2) + \gamma x$$

$$\frac{dy}{dt} = x\,(3\mathcal{Z}+1-x^2) + \gamma x$$

$$\frac{dz}{dt} = -2\mathcal{Z}(\alpha + xy)$$

The equilibrium points of the Rabinovich-Fabrikant system must first be determined by solving the ensuing system of equations and setting the right-hand side of each equation to zero in order to examine the system's Jacobi stability. Next, in order to assess the stability of the system, we linearize it around each equilibrium point. However, it is important to note that the Rabinovich-Fabrikant system is known for its chaotic behavior, which means that it does not possess stable equilibrium points in the traditional sense. Instead, it exhibits complex dynamics such as strange attractors and sensitivity. Therefore, the chaotic structure of the system hampers the interpretation of stability in terms of attraction, even if the linearized stability of equilibrium points in the Rabinovich-Fabrikant system may be analyzed using Jacobi stability analysis.

7. Applications of KCC Theory

Kosambi-Carton-Chern theory finds applications across various scientific domains, including physics, biology, and engineering. In particular, KCC theory has been instrumental in studying the stability of dynamical systems exhibiting complex behavior, such as chaotic and multi-stable systems.

7.1. Examples of Systems Analyzed Using KCC Theory

Celestial Mechanics: KCC theory has been applied to analyze the stability of celestial orbits in the solar system, shedding light on the long-term stability of planetary motion.

Biological Systems: In biology, KCC theory has been utilized to investigate the stability of ecological networks and population dynamics, offering insights into the resilience of ecosystems.

Control Theory: In engineering applications, KCC theory has been employed in control theory to assess the stability of feedback control, aiding in the design of robust control strategies.

8. Future research

In future research, the combination of KCC theory with numerical methods could provide even more powerful tools for analyzing complex systems where analytical solutions are difficult to obtain.

9. Conclusion

The Kosambi-Carton-Chern (KCC) theory stands as a challenging framework for Jacobi stability analysis in certain dynamical systems. By leveraging concepts from geometry, topology, and dynamical systems theory, KCC theory suggests an all-inclusive approach to understanding stability phenomena. Its applications span diverse scientific disciplines, making it a valuable tool for researchers seeking to unravel the intricacies of complex systems.

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