International Journal of Mathematics Trends and Technology
ISSN: 2231-5373/ https://doi.org/10.14445/22315373/IJMTT-V70I11P101

Volume 70 Issue 11, 1-4, November 2024

Original Article

Metrizability of the Strong φ-metric Space

Stela Çeno[1], Dhurata Valera[2]

[1,2]Department of Mathematics, Faculty of Natural Science, University of Elbasan "Aleksandër Xhuvani", Albania.

[1]Corresponding Author : stela.ceno@uniel.edu.al

Received: 15 August 2024 Revised: 28 September 2024 Accepted: 14 November 2024 Published: 27 November 2024

Abstract *- After presenting a strong φ-metric space and proving that this space is regular and normal, in this article, we prove the Stone-type theorem in a strong φ-metric space. Also, it is used Bing metrization theorem for access to a satisfactory condition of metrizability.*

Keywords *- Strong φ-metric, Strong φ-metric space, Metrizability, Topological space, Regular metric space.*

1. Introduction

This article is a consecutive of previous work [1], where the strong φ-metric space is proved to be a regular and a normal space. In this paper, we are committed to studying the metrizability of the strong φ-metric space. The definitions and results in this section are given to support the proof.

Definition1.1. [2] Let X be a topological space and $B = \{B_m: m \in M\}$ be a family of subsets of X. Then:

a) If, for $\forall x \in X$, exists a neighbourhood U of x such that $B_m \cap U = \emptyset$ for at most one $m \in M$, then B is called *discrete*.

b) If for $\forall x \in X$, exists a neighbourhood U of x such that $\{m \in M: A_m \cap U \neq \emptyset\}$ is finite, then B is called *locally finite*.

c) If $B = \bigcup_{i \in \mathbb{N}} B_i$, where every B_i is locally finite, then B is called σ-*locally finite*.

d) If $B = \bigcup_{i \in \mathbb{N}} B_i$, where every B_i is discrete, then B is called σ-*discrete*.

e) If $X = \bigcup_{m \in M} B_m$ then B is called a *cover* of X.

f) A cover $A = \{A_i: i \in I\}$ of subsets of X is called a *refinement* of the cover B_i, if for each $i \in I$ there exists $m \in M$ such that $A_i \subset B_m$.

Definition 1.2. [2] Let (X, τ) be a topological space. Then:

a) X is regular space if for any closed subsets $A \subset X$ and for $x \in X \setminus A$, there exist two disjoint open sets U and V containing A and X, respectively.

b) X is normal space if, for any two disjoint closed subsets A and B of X, there exists two disjoint open sets U and V, containing A and B, respectively.

Definition 1.3. [3,4] Let (X, τ) be a topological space.

a) A subset U of X is called sequentially open if each sequence $\{x_n\} \subset X$ converging to a point $x \in U$, then there exists $N \in \mathbb{N}$ such that $\{x_n\} \in U$ for all $n \geq N$.

b) A subset U of X is called sequentially closed if no sequence in U converges to a point not in U.

c) X is called semi-metrizable if there exists a function $d: X \times X \to [0, \propto[$ such that for all $x, y \in X$:

i) $d(x, y) = 0 \Leftrightarrow x = y$;

ii) $d(x,y) = d(y,x)$

iii) $x \in \bar{A} \Leftrightarrow d(x,A) = inf\{d(x,y): y \in A\} = 0$ for any $A \subset X$.

d) (X,τ) is said to be metrizable if there exists a metric on X whose topology is the same as topology τ.

Theorem 1.4. [2] (The Stone Theorem) Every open cover of a metrizable space has an open refinement which is both locally finite and σ-discrete.

Theorem 1.5. [5] (The Bing Metrization Theorem) A topological space is metrizable if and only if it is regular and has a σ-discrete base.

In [1], as it is mentioned at the beginning, we have presented the definition of the strong φ-metric and proved two theorems regarding strong φ-metric space, as it follows:

Definition 1.6. The function $d_s: X \times X \to \mathbb{R} \geq 0$, is called a *strong φ-metric* and satisfies the following conditions:

a) $d_s(x,y) = 0 \Leftrightarrow x = y$;

b) $d_s(x,y) = d_s(y,x)$;

c) $d_s(x,z) \leq K d_s(x,y) + d_s(y,z) + \varphi(x,y,z)$, $\forall x,y,z \in X$, and $K \geq 1$, with $\varphi: X \times X \times X \to \mathbb{R} \geq 0$ a function fulfilling:
   i) $\varphi(x,y,z) = 0$ if $x = z$ or $y = z$;
   ii) $\varphi(x,y,z) = \varphi(y,x,z)$;
   iii) $\forall \varepsilon > 0, \exists \delta > 0$ such that $\varphi(x,y,z) < \varepsilon$, whenever $d_s(x,y) < \delta$ or $d_s(y,z) < \delta$, $\forall x,y,z \in X$.

The ordered pair (X,d_s) is called a *strong φ-metric space*.

Theorem 1.7. Every strong φ-metric space (X,d_s), is regular.

Theorem 1.8. Every strong φ-metric space (X,d_s), is normal.

2. Main Results

Theorem 2.1. (Stone-type theorem) In a strong φ-metric space (X,d_s), every open cover of X has an open refinement, which is σ-locally finite and σ-discrete at the same time.

Proof. Let $\{\mathcal{U}_m : m \in M\}$ be an open cover of X. By the Zermelo theorem on well-ordering [2], we can take a well-ordering relation < on M. Define the families $\mathcal{V}_i = \{\mathcal{V}_{m,i} : m \in M\}$ of subsets of X by letting $\mathcal{V}_{m,i} = \bigcup_{c \in \mathcal{C}} B_s(c, \frac{1}{2^i})$ where $\mathcal{C}$ is the set of all points $c \in X$ satisfying the following conditions:

i) m is the smallest element of M such that $c \in \mathcal{U}_m$.

ii) $c \notin \mathcal{V}_{t,j}$ for all $j < i$ and for all $t \in M$.

iii) $B_s(c, \frac{5}{2^i}) \subset \mathcal{U}_m$.

Apparently, the sets $\mathcal{V}_{m,i}$ are open, and by the third statement (iii), we have $\mathcal{V}_{m,i} \subset \mathcal{U}_m$. For each $x \in X$, take the smallest $m \in M$ such that $x \in \mathcal{U}_m$ and a natural number i such that $B_s(c, \frac{5}{2^i}) \subset \mathcal{U}_m$. It implies that $x \in \mathcal{C}$ if an only if $x \notin \mathcal{V}_{t,j}$ for all $t \in M$ and $j < i$. Then $x \in \mathcal{V}_{m,i}$. Thus, we have either $x \in \mathcal{V}_{t,j}$ for all $j < i$ and all $t \in M$ or $x \in \mathcal{V}_{m,i}$. This proves that $\mathcal{V} = \bigcup_{i \in \mathbb{N}} \mathcal{V}_i$ is an open refinement of the cover $\{\mathcal{U}_m : m \in M\}$.

Now, for every $i \in \mathbb{N}$, let $x_1 \in \mathcal{V}_{m_1,i}$ and $x_2 \in \mathcal{V}_{m_2,i}$ with $m_1 \neq m_2$. Let us suppose $m_1 < m_2$. By the definition of $\mathcal{V}_{m_1,i}$ and $\mathcal{V}_{m_2,i}$, there exists $c_1, c_2 \in X$ satisfying conditions (i), (ii), (iii) and $x_1 \in B_s(c_1, \frac{1}{2^i})$, $x_2 \in B_s(c_2, \frac{1}{2^i})$. Again, we have $B_s(c_1, \frac{5}{2^i}) \subset \mathcal{U}_{m_1}$ and $c_2 \notin \mathcal{U}_{m_1}$ and this implies $d_s(c_1,c_2) \geq \frac{5}{2^i}$. But we have:

$$d_s(c_1,c_2) \le Kd_s(c_1,x_1) + d_s(x_1,x_2) + Kd_s(x_2,c_2) + \varphi(c_1,c_2,x_1) + \varphi(x_1,c_2,x_2)$$

$$d_s(x_1,x_2) \ge \frac{5}{2^i} - Kd_s(c_1,x_1) - Kd_s(x_2,c_2) - \varphi(c_1,c_2,x_1) - \varphi(x_1,c_2,x_2). \qquad (2.1)$$

Again for $\frac{1}{2^{i+1}} > 0$ there $\exists \beta_1, \beta_2 > 0$ such that $\varphi(c_1,c_2,x_1) < \frac{1}{2^{i+1}}$ whenever $d_s(c_1,x_1) < \frac{\beta_1}{K}$ and $\varphi(x_1,c_2,x_2) < \frac{1}{2^{i+1}}$, whenever $d_s(x_2,c_2) < \frac{\beta_2}{K}$. Let $\beta = min\{\frac{\beta_1}{K}, \frac{\beta_2}{K}, \frac{1}{2^i}\}$. Then $d_s(c_1,x_1) < \beta$, $d_s(x_2,c_2) < \beta$ and $\varphi(c_1,c_2,x_1) < \frac{1}{2^{i+1}}$, $\varphi(x_1,c_2,x_2) < \frac{1}{2^{i+1}}$. Then (2.1) gives:

$$d_s(x_1,x_2) \ge \frac{5}{2^i} - 2\beta - 2\frac{1}{2^{i+1}} \ge \frac{5}{2^i} - 2\frac{1}{2^i} - \frac{1}{2^i} = \frac{1}{2^{i-1}}.$$

To prove that the families $\mathcal{V}_i$ are σ-discrete, suppose there exists $x \in X$ such that $x_1, x_2 \in B_s(x, \frac{1}{2^{i+1}})$. Then we have $d_s(x,x_1) < \frac{1}{2^{i+1}}$, $d_s(x,x_2) < \frac{1}{2^{i+1}}$ and:

$$\frac{1}{2^{i-1}} < d_s(x,x_2) \le Kd_s(x_1,x) + d_s(x,x_2) + \varphi(x_1,x_2,x). \qquad (2.2)$$

Now for $\frac{1}{K(2^{i+1})} > 0$ there exists $\beta' > 0$ such that $\varphi(x_1,x_2,x) < \frac{1}{2^{i+1}}$ whenever $d_s(x_2,x) < \beta'$. If $\delta = min\left\{\beta', \frac{1}{K(2^{i+1})}, \frac{1}{2^{i+1}}\right\}$, then $d_s(x_2,x) < \delta$ and $\varphi(x_1,x_2,x) < \frac{1}{2^{i+1}}$. The inequality (2.2) gives:

$$\frac{1}{2^{i-1}} < d_s(x_1,x_2) < K\frac{1}{K(2^{i+1})} + \delta + \frac{1}{2^{i+1}} \le 2\frac{1}{2^{i+1}} + \frac{1}{2^{i+1}} < \frac{1}{2^i} + \frac{1}{2^i} = \frac{1}{2^{i-1}}.$$

This is a contradiction, and hence, it proves that each ball of radius $\frac{1}{2^{i+1}}$ meets at most one element of $\mathcal{V}_i$ that is $\mathcal{V} = \bigcup_{i\in\mathbb{N}} \mathcal{V}_i$ is σ-discrete. Let $i \in \mathbb{N}$ then for all $t \in M$, $i \ge j + k$ and $c \in \mathcal{C}$ implies $c \notin \mathcal{V}_{t,j}$. Now if $B_s(x, \frac{1}{2^{i-1}}) \subset \mathcal{V}_{t,j}$, then $c \notin B_s(x, \frac{1}{2^{k-1}})$ and $d_s(x,c) \ge \frac{1}{2^{k-1}}$.

Again, $j + k \ge k + 1$ and $i \ge k + 1$ implies $\frac{1}{2^{j+k}} \le \frac{1}{2^{k+1}}$ and $\frac{1}{2^i} \le \frac{1}{2^{k+1}}$. Next, suppose there exists $y \in B_s(x, \frac{1}{2^{j+k}}) \cap B_s(c, \frac{1}{2^i})$. Then:

$$d_s(x,c) \le Kd_s(x,y) + d_s(y,c) + \varphi(x,c,y) \qquad (2.3)$$

For $\frac{1}{2^k} > 0$ there $\exists \alpha > 0$ such that $\varphi(x,c,y) < \frac{1}{2^k}$ whenever $d_s(x,y) < \frac{\alpha}{K}$. Let $\gamma = min\left\{\frac{\alpha}{K}, \frac{1}{2^{j+k}}\right\}$. Then: $d_s(x,y) < \gamma$ and $\varphi(x,c,y) < \frac{1}{2^k}$. Therefore from (2.3) we obtain:

$$\frac{1}{2^{k-1}} \le d_s(x,c) < \gamma + \frac{1}{2^i} + \frac{1}{2^k} \le \frac{1}{2^{j+k}} + \frac{1}{2^{k+1}} + \frac{1}{2^k} \le \frac{1}{2^{k+1}} + \frac{1}{2^{k+1}} + \frac{1}{2^k} = \frac{1}{2^{k-1}}.$$

Which concludes $B_s\left(x, \frac{1}{2^{i+k}}\right) \cap B_s\left(c, \frac{1}{2^i}\right) = \emptyset$ and this implies $B_s\left(x, \frac{1}{2^{i+k}}\right) \cap \mathcal{V}_{m,i} = \emptyset$ for $i \ge j + k$ and $m \in M$ with $B_s\left(x, \frac{1}{2^{k-1}}\right) \subset \mathcal{V}_{t,j}$. Since $\mathcal{V}$ is a refinement of $\{\mathcal{U}_m : m \in M\}$, so for each $x \in X$, there exists *l*,*j* and *t* such that $B_s\left(x, \frac{1}{2^t}\right) \subset \mathcal{V}_{t,j}$ and thus, there exists *k*,*j* and *t* such that $B_s\left(x, \frac{1}{2^{k-1}}\right) \subset \mathcal{V}_{t,j}$. Then the ball $B_s\left(x, \frac{1}{2^{i+k}}\right)$ meets at most $(j + k - 1)$ members of $\mathcal{V}$. This proves that $\mathcal{V}_i$ is locally finite, that is $\mathcal{V}$ is σ-locally finite.

Corollary 2.2. Let is (X, d_s) a strong φ-metric space. Then *X* has σ-discrete base.

Proof. For every $i \in \mathbb{N}$, let $\mathcal{A}_i = \left\{B_s\left(x, \frac{1}{i}\right) : x \in X\right\}$. Then $\mathcal{A}_i$ is an open cover of *X*. By Theorem 2.1, there exists an open σ-discrete refinement $\mathcal{B}_i$ of $\mathcal{A}_i$. Put $\mathcal{B} = \bigcup_{i\in\mathbb{N}} \mathcal{B}_i$. Then $\mathcal{B}$ is a σ-discrete base of *X*.

Corollary 2.3. Every strong φ-metric space is metrizable.

Proof. From the proven theorem, a strong φ-metric space is a normal space [1], and Corollary 2.3. it follows that *X* is regular space with a σ-discrete base. Then from Bing Metrization Theorem (Theorem 1.5.), *X* is metrizable.

3. Conclusion

The main conclusion of this paper is the fact that a strong φ-metric space is metrizable.

References

[1] S. Çeno, and Dh. Valera, "*Introduction to Strong Φ-Metric and Some Basic Properties," 4th International Conference on Engineering, Natural and Social Science,* Konya, Turkey, 2024.

[2] Ryszard Engelking, *General Topology,* Sigma Series in Pure Mathematics, Heldermann Verlag, Berlin, 1989. [Publisher Link]

[3] Frank Siwiec, "On Defining a Space by a Weak-Base," *Pacific Journal of Mathematics,* vol. 52, pp. 233-245, 1974. [CrossRef] [Google Scholar] [Publisher Link]

[4] S.P. Franklin, "Spaces in which Sequences Suffice," *Fundamenta Mathematicae,* vol. 57, pp. 107-115, 1965. [Google Scholar]

[5] R.H. Bing, "Metrization of Topological Spaces," *Canadian Journal of Mathematics,* vol. 3, pp. 175-186, 1951. [CrossRef] [Google Scholar] [Publisher Link]

[6] Mehmet Kir, and Hükmi Kiziltunc, "On Some Well-Known Fixed Point Theorems in B-Metric Spaces," *Turkish Journal of Analysis and Number Theory*, vol. 1, no. 1, pp. 13-16, 2013. [CrossRef] [Google Scholar] [Publisher Link]