International Journal of Mathematics Trends and Technology
ISSN: 2231-5373/ https://doi.org/10.14445/22315373/IJMTT-V70I11P103
Volume 70 Issue 11, 16-27, November 2024

Original Article

A Study of Bicomplex Q_m-Normal Families

Tehseen Abas Khan[1], Jyoti Gupta[2], Ravinder Kumar[3]

[1,2]*Department of Mathematics, Bhagwant University, Ajmer, Rajasthan, India.*
[3]*Department of Mathematics, Udhampur College, J&K, India.*

[1]*Corresponding Author : tehseenabas90@gmail.com*

Received: 16 August 2024 Revised: 29 September 2024 Accepted: 15 November 2024 Published: 30 November 2024

Abstract - *In this paper, the theory of Q_m -normal families of meromorphic functions of one complex variable is promoted to bicomplex meromorphic functions. To study the properties of Q_m -normal families in the bicomplex case, we have extended the definition of C_m -point and C_m -sequences from one complex variable to the bicomplex case and obtained its important results. Many results of Q_m -normal families of one complex variable case are seen to hold in the bicomplex case. Moreover, the necessary and sufficient condition for Q_m -normality in the bicomplex case is obtained.*

Keywords - *Bicomplex numbers, Bi-complex meromorphic functions, Normal families, Q_m -normal families, C_m -point and C_m-sequences.*

1. Introduction

The foundation of bicomplex analysis can be traced back to the pioneering work of William Kingdon Clifford [17] in the late 19th century, who introduced the concept of bicomplex numbers as an extension of complex numbers. Clifford's contributions laid the groundwork for further developments in this field. In the early 20th century, the field witnessed significant advancements with the seminal works of Georg Frobenius [18] and Elie Cartan, who made notable contributions to the theory of bicomplex numbers and their algebraic properties. In the latter half of the 20th century, J.D. Riley made significant contributions to bicomplex analysis, particularly through his work titled "Contributions to the Theory of Functions of a Bicomplex Variable". The dawn of the 21st century witnessed renewed interest in bicomplex analysis, with researchers such as K.S. Charak, D. Rochan and Narinder Sharma [7,8,11] delving into the study of normal families of bicomplex holomorphic and meromorphic functions. Their work, along with contributions from others in the field, has led to significant advancements in understanding the geometric, analytic and computational aspects of these families. Today, the exploration of Q_m-normal families continues to be a vibrant area of research, with scholars building upon the foundational works of their predecessors to unravel the intricacies of bicomplex analysis and its applications in various domains. Through collaborative efforts and interdisciplinary approaches, researchers aim to further expand the frontiers of knowledge in this fascinating field.

In recent years, the exploration of Q_m -normal families of bicomplex meromorphic functions has emerged as a captivating frontier within the domain of bicomplex analysis. Building upon foundational works by prominent researchers such as Riley, Charak, Sharma, and others, significant strides have been made in unraveling the intricate properties and behaviors inherent to these families.Today, the exploration of Q_m-normal families continues to be a vibrant area of research, with scholars building upon the foundational works of their predecessors to unravel the intricacies of bicomplex analysis and its applications in various domains. Through collaborative efforts and interdisciplinary approaches, researchers aim to further expand the frontiers of knowledge in this fascinating field. In recent years, the exploration of Q_m -normal families of bicomplex meromorphic functions has emerged as a captivating frontier within the domain of bicomplex analysis. Building upon foundational works by prominent researchers such as Riley, Charak, Sharma, and others, significant strides have been made in unraveling the intricate properties and behaviors inherent to these families. The seminal contributions of Riley, dating back to the early 2000s, laid the groundwork for understanding the fundamental characteristics of bicomplex meromorphic functions and their relevance within the framework of Q_m-normal families. Subsequent research endeavors by K. S Charak and Narinder Sharma, among others, have further deepened the understanding, shedding light on the geometric and analytic aspects of these families. Through their collective efforts, researchers have illuminated the rich interplay between bicomplex analysis and the theory of Q_m-normal families, offering valuable insights into the geometric, analytic, and computational facets of this intriguing area of study. As we embark on this journey of exploration, we stand poised to unearth new vistas of knowledge and innovation guided by the pioneering works of these esteemed scholars. The exploration of bicomplex analysis, spanning from the theory of bicomplex numbers to the study of

Q_m -normal families, has been a journey marked by significant contributions from various researchers across different periods. The concept of Q_m-normal families [4] arises from the desire to understand the behavior of functions in the bicomplex plane that possess distinctive normalization properties. Q_m -normality extends the classical notion of normality, introducing a more nuanced criterion for the convergence of certain families of functions. This extension becomes particularly intriguing when applied to bicomplex analysis, where the interplay of real and imaginary components introduces complexities that diverge from the traditional complex setting.

Moreover, the study of Q_m-normality contributes to the broader understanding of function theory in the bicomplex domain, shedding light on the behavior of functions that transcend the classical confines of complex analysis.-normal families in the bicomplex setting lie in their potential applications in various branches of mathematics and physics. These families provide a versatile framework for modeling phenomena that involve multiple variables and complex interactions. In this study, we explore the details of Q_m-normal families [4] in the context of the bicomplex plane. Our objectives are to research the basic characteristics of these families, as well as their possible applications and linkages with other classes of functions. By undertaking this study, we seek to advance the understanding of function theory in bicomplex analysis and contribute valuable insights to the broader mathematical community.

2. Preliminaries

Here, we introduce some of the basic definitions and results of the theory of bicomplex numbers, which are required for defining new definitions and results. Corrado Segre, in 1892, while studying special algebras, published a paper [14] in which he carried his research on an infinite family of algebra whose elements are called bicomplex numbers, tricomplex numbers,….. n-complex numbers. Bicomplex numbers, also known as tetra-numbers, are defined as follows:
$\mathbb{T} = \{z_1 + z_2 i_2 : z_1, z_2 \in C(i_1)\}$, where the imaginary units i_1, i_2 and $j$are governed by the rules:
$i_1{}^2 = i_2{}^2 = -1, j^2 = 1$ and $i_1 i_2 = i_2 i_1 = j,\ i_1 j = j i_1 = -i_2, i_2 j = j i_2 = -i$
Thus, one can easily see that multiplication of two bicomplex numbers is commutative. In fact, bicomplex numbers have a unique character that forms commutative algebra but not division algebra. It is convenient to write the set of bicomplex numbers as $\mathbb{T} = \{x_0 + x_1 i_1 + x_2 i_2 + x_3 j : x_0, x_1, x_2, x_3 \in \mathbb{R}\}$. It is also important to know that every bicomplex number has the following unique idempotent representation:

$$z_1 + z_2 i_2 = (z_1 - z_2 i_1)e_1 + (z_1 + z_2 i_1)e_2$$

Where $e_1 = \frac{(1+j)}{2}$ and $e_2 = \frac{(1-j)}{2}$. This idempotent representation of bicomplex numbers is very useful because addition, multiplication, and division can be done term-by-term. The operation addition on $\mathbb{T}$ can be defined by the function $\bigoplus: \mathbb{T} \times \mathbb{T} \to \mathbb{T}$ as:
$(x_0 + x_1 i_1 + x_2 i_2 + x_3 j, y_0 + y_1 i_1 + y_2 i_2 + y_3 j) = (x_0 + y_0) + (x_1 + y_1)i_1 + (x_2 + y_2)i_2 + (x_3 + y_3)j$

The operation scalar multiplication on $\mathbb{T}$ is defined by the function $\odot: \mathbb{R} \times \mathbb{T} \to \mathbb{T}$ as:
$(a, x_0 + x_1 i_1 + x_2 i_2 + x_3 j) = (a x_0 + a x_1 i_1 + a x_2 i_2 + a x_3 j)$. Thus, the structure ($\mathbb{T}$, $\bigoplus$, $\odot$) forms a linear space. Therefore, the norm on $\mathbb{T}$ is defined by the function $\|\ \|: \mathbb{T} \to \mathbb{R}_{\geq 0}$ as $||x_0 + x_1 i_1 + x_2 i_2 + x_3 j|| = (x_0{}^2 + x_1{}^2 + x_2{}^2 + x_3{}^2)^{\frac{1}{2}}$.
Thus, the structure ($\mathbb{T}$, $\bigoplus$, $\odot$, $\|\cdot\|$) forms a normed linear space.

2.1. Conjugation Operations in $\mathbb{T}$

The concept of complex conjugation plays a pivotal role in both the algebraic and geometric aspects of C complex functions as well as in the analysis of them. Conjugation operations in the bicomplex plane provide essential tools for analyzing and understanding the structure and behavior of bicomplex holomorphic functions, extending many classical concepts from complex analysis to the more intricate bicomplex setting. In the context of bicomplex numbers, there exist three conjugation operations, each playing a crucial role in understanding the properties and behaviors of bicomplex holomorphic functions. It's not surprising that there exist three distinct conjugations in $\mathbb{T}$. The three types of conjugations on $\mathbb{T}$ are defined as follows:

2.1.1. Complex Conjugation

For a bicomplex number $w = z_1 + j z_2$, where $z_1 = x_1 + yi$and $z_2 = u_1 + vi$ with $i^2 = -1$, the complex conjugate is defined as

$$w^{\bullet}{}_1 = \bar{z}_1 + i_2 \bar{z}_2 = (x - yi) + i_2(u - vi).$$

This operation is performed by taking the complex conjugate of each component z_1 and z_2

2.1.2. Bicomplex Conjugation

Another important conjugation operation in the bicomplex context is the bicomplex conjugate. $w = z_1 + i_2 z_2$ it is defined as: $w *= z_1 - i_2 z_2$

This conjugation changes the sign of the i_2-component.

2.1.3. Mixed Conjugation

Combining both the complex and bicomplex conjugations, we get the mixed conjugate:

$$w^{\oplus} = \bar{z}_1 - j\bar{z}_2 = (x - yi) - i_2(u - vi).$$

This operation involves taking the complex conjugate of each component and changing the sign of the i_2-component.

2.2. Cartesian Set and Discuss in $\mathbb{T}$

2.2.1. Definition

The Cartesian set X determined by X_1 andX_2, which are subsets of$C(i_1)$, is defined as follows: $X_1 \times_e X_2 = \{w = z_1 + z_2 i_2 \in T: P_1(w) \in X_1, P_2(w) \in X_2\}$,Where P_1 and P_2 are the projections on X .

It can be easily seen that if X_1 and $X_2$are domains in$C(i_1)$, then $X_1 \times_e X_2$it is also a domain in $\mathbb{T}$. Then a way to construct some "discus" (of center 0) in is to take $\mathbb{T}$ -Cartesian product of two discs (of center 0) in$C(i_1)$. Let r, r_1, r_2 denote the real numbers such that r>0, r_1>0$ and r_2 >0. Also, let $A_1 = \{z_1 - z_2 i_1 \in T: z_1, z_2 \in C(i_1)\}$and $A_2 = \{z_1 + z_2 i_1 \in T: z_1, z_2 \in C(i_1)$ }. Then, the open discussion with the Centre

$a = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2 = \alpha + i_2 \beta$ of radius r_1 and r_2 is defined as follows [1]:

$D(a; r_1, r_2) = B^1(\alpha - i_1\beta, r_1) \times_e B^1(\alpha + i_1\beta, r_2) = \{w_1 e_1 + w_2 e_2: |w_1 - (\alpha - i\beta)| < r_1, |w_2 - (\alpha + i\beta)| < r_2\}$.

Theorem 2.2.2: Every $z_1 + z_2 i_2 \in \mathbb{T}$ is uniquely represented as $z_1 + z_2 i_2 = P_1(z_1 + z_2 i_2)e_1 + P_2(z_1 + z_2 i_2)e_2$

Proof: Let $Z = z_1 + z_2 i_2$ be any bicomplex number. We want to show that

$$Z = P_1(Z)e_1 + P_2(Z)e_2, \text{ where } P_1(Z) = z_1 + z_2 i_1 \text{ and } P_2(Z) = z_1 - z_2 i_1.$$

We know that every bicomplex number can be expressed as:
$Z = z_1 + z_2 i_2 = (z_1 - z_2 i_1)e_1 + (z_1 + z_2 i_1)e_2$, where $e_1 = (1 + i_2 i_1)/2$ and $e_2 = (1 - i_2 i_1)/2$.
Therefore, from (1.2), we have $Z = z_1 + z_2 i_2 = (z_1 - z_2 i_1)e_1 + (z_1 + z_2 i_1)e_2$.
Let $P_1, P_2 : \mathbb{R} \to \mathbb{C}(i_1)$ be the two projections such that: $P_1(Z) = z_1 + z_2 i_1$ and $P_2(Z) = z_1 - z_2 i_1$
Using the above projections and idempotent basis, Z can be written as $Z = P_1(Z)e_1 + P_2(Z)e_2$

Therefore, $P_1(Z)e_1 + P_2(Z)e_2 = (z_1 + z_2 i_1)(1 + i_2 i_1)/2 + (z_1 - z_2 i_1)(1 - i_2 i_1)/2$
$\to P_1(Z)e_1 + P_2(Z)e_2 = 1/2[(z_1 + z_2 i_1)(1 + i_2 i_1) + (z_1 - z_2 i_1)(1 - i_2 i_1)]$
$\to P_1(Z)e_1 + P_2(Z)e_2 = 1/2[z_1(1 + i_2 i_1) + z_2 i_1(1 + i_2 i_1) + z_1(1 - i_2 i_1) - z_2 i_1(1 - i_2 i_1)]$

Combining like terms and using $i_1^2 = 1$ and $i_2 i_1 = -i_1 i_2$, we get:
$P_1(Z)e_1 + P_2(Z)e_2 = 1/2[2z_1 + z_2 i_1 + z_2 i_1 i_2 i_1 - z_2 i_1 + z_2 i_1 i_2 i_1]$

On further simplification, we get: $P_1(Z)e_1 + P_2(Z)e_2 = 1/2[2z_1 + 2z_2 i_2]$. Since $i_1 i_2 i_1 = -i_2$, therefore $P_1(Z)e_1 + P_2(Z)e_2 = z_1 + z_2 i_2$

Thus, we have shown that every bicomplex number $Z = z_1 + z_2 i_2$ can be uniquely expressed as:
$Z = P_1(Z)e_1 + P_2(Z)e_2$, where $P_1(Z) = z_1 + z_2 i_1$ and $P_2(Z) = z_1 - z_2 i_1$.

This representation of bicomplex numbers is advantageous because it allows for straightforward addition, multiplication, and division term-by-term. Furthermore, it aids in understanding the structure of functions of a bicomplex variable, as we will explore in the next section.

2.3. Basic definition and properties of bicomplex meromorphic functions

It is generally known[15] that in the complex plane, a function f is meromorphic in an open set U if and only if f it is a quotient g/h of two functions that are holomorphic in U where h it is not identically zero in any component of U. This formulation serves as the foundation for how a bicomplex meromorphic function is defined. Based on this definition, K.S Charak et al; defined a bicomplex meromorphic function as follows.

Definition 2.3.1: [11] If a function f is a quotient of two functions g/h that are bicomplex holomorphic in Ω and h is not identically in the null-cone in any component of Ω, then f is said to be a bicomplex meromorphic in the open set $\Omega \subset \mathbb{T}$.

Theorem 2.3.2: [11] Let $f: \Omega \subseteq \mathbb{T} \to \mathbb{T}$ be a bicomplex meromorphic function on the open set $\Omega \subset \mathbb{T}$. Then there exist meromorphic functions $f_{e_1}: \Omega_1 \to C(i_1)$ and $f_{e2}: \Omega_2 \to \mathbb{C}(i_1)$ with $\Omega_1 = P_1(\Omega)$ and $\Omega_2 = P_2(\Omega)$, such that $f(z_1 + z_2 i_2)$ $= f_{e_1}(z_1 - z_2 i_1)e_1 + f_{e_2}(z_1 + z_2 i_1)e_2 \ \forall z_1 + z_2 i_2 \in \Omega$

Based on the above definition of bicomplex meromorphic function, we extend algebraic properties to bicomplex meromorphic functions as follows:

Theorem 2.3.3: If $f^1, f^2: \Omega \to \mathbb{T}$ be two bicomplex meromorphic functions on the open set Ω . Then $f^1 + f^2: \Omega \to \mathbb{T}$ be a bicomplex meromorphic function on the open setΩ.

Proof : Since f^1 and f^2 are bicomplex meromorphic functions. Therefore, by definition of bicomplex meromorphic f^1 and f^2 can be written as

$f^1(z_1 + z_2 i_2) = f^1_{e_1}(z_1 - z_2 i_1)e_1 + f^1_{e_2}(z_1 + z_2 i_1)e_2 \ \forall z_1 + z_2 i_2 \in \Omega \ldots\ldots (1)$

$f^2(z_1 + z_2 i_2) = f^2_{e_1}(z_1 - z_2 i_1)e_1 + f^2_{e_2}(z_1 + z_2 i_1)e_2 \ \forall z_1 + z_2 i_2 \in \Omega \ldots\ldots (2)$

Now, let's define the sum of these two bicomplex meromorphic functions as

$$f_{e_2}(z_1 - z_2 i_1)e_2$$

$f(z_1 + z_2 i_2) = f^1(z_1 + z_2 i_2) + f^2(z_1 + z_2 i_2)$

$= f^1_{e_1}(z_1 - z_2 i_1)e_1 + f^1_{e_2}(z_1 + z_2 i_1)e_2 + f^2_{e_1}(z_1 - z_2 i_1)e_1 + f^2_{e_2}(z_1 + z_2 i_1)e_2$

$= \left[f^1_{e_1}(z_1 - z_2 i_1) + f^2_{e_1}(z_1 - z_2 i_1)\right]e_1 + \left[f^1_{e_2}(z_1 + z_2 i_1) + f^2_{e_2}(z_1 + z_2 i_1)\right]e_2$

$= f_{e_2}(z_1 - z_2 i_1)e_2$ [because the sum of two meromorphic functions is a meromorphic function]

Therefore, $f(z_1 + z_2 i_2) = f_{e_2}(z_1 - z_2 i_1)e_2$ is a bicomplex meromorphic function, where f_{e_i} is meromorphic in Ω_i for I = 1, 2.

Theorem 2.3.4: If $f^1, f^2: \Omega \to \mathbb{T}$ be two bicomplex meromorphic functions on the open set Ω . Then $f^1 . f^2: \Omega \to$ be a bicomplex meromorphic function on the open set Ω.

Proof. Since f^1 and f^2 are bicomplex meromorphic functions on the open setΩ. Therefore, by Theorem 2.2.2, there exist meromorphic functions $f_{e_1}: \Omega_1 \to C(i_1)$ and $f_{e2}: \Omega_2 \to C(i_1)$ with $\Omega_1 = P_1(\Omega)$ and $\Omega_2 = P_2(\Omega)$, such that

$$f(z_1 + z_2 i_2) = f_{e_1}(z_1 - z_2 i_1)e_1 + f_{e_2}(z_1 + z_2 i_1)e_2 \forall z_1 + z_2 i_2 \in \Omega$$
$$f^1(z_1 + z_2 i_2) = f^1_{e_1}(z_1 - z_2 i_1)e_1 + f^1_{e_2}(z_1 + z_2 i_1)e_2 \ \forall z_1 + z_2 i_2 \in \Omega$$
$$f^2(z_1 + z_2 i_2) = f^2_{e_1}(z_1 - z_2 i_1)e_1 + f^2_{e_2}(z_1 + z_2 i_1)e_2 \ \forall z_1 + z_2 i_2 \in \Omega$$

To show that the product of two bicomplex meromorphic functions is a bicomplex meromorphic. Let's define the product of these two bicomplex meromorphic functions as

$$f(z_1 + z_2 i_2) = f^1(z_1 + z_2 i_2).f^2(z_1 + z_2 i_2)$$

From (1) and (2), we have

$f(z_1 + z_2 i_2) = f^2_{e_2}(z_1 + z_2 i_1)e_2]$

$= [(f^1_{e_2}(z_1 + z_2 i_1)).(f^2_{e2}(z_1 - z_2 i_1)]e_2$

$= [(f^1_{e_2}(z_1 + z_2 i_1)).(f^2_{e2}(z_1 - z_2 i_1)]e_2$

$= f_{e_2}(z_1 + z_2 i_1)e_2$ [product of two meromorphic functions is a meromorphic function]

wheref_{e_i} is meromorphic in Ω_i for i = 1, 2.

Theorem 2.3.5 : If $f: \Omega \subseteq \mathbb{T} \rightarrow \mathbb{T}$ be a bicomplex meromorphic function on the open set Ω and c be any complex number. Then $c.f$ is also a bicomplex meromorphic function.

Proof. Since $f: \Omega \subseteq \mathbb{T} \rightarrow \mathbb{T}$ be a bicomplex meromorphic function on the open set Ω. Then f is a quotient $\frac{g(w)}{h(w)}$ of two bicomplex holomorphic functions, we need to show that $c.f(w)$ can be expressed in this form as well. For this, assume that $f(w) = \frac{g(w)}{h(w)}$ where $g(w)$ and $h(w)$ are bicomplex holomorphic functions in Ω and $h(w)$ is not identically zero in any component of Ω where $z_1 + z_2 i_2 \in \Omega$. Therefore, from Theorem 2.1.2, there exist holomorphic functions $f_{e_1}: \Omega_1 \rightarrow C(i_1)$ and $f_{e2}: \Omega_2 \rightarrow C(i_1)$ with $\Omega_1 = P_1(\Omega)$ and $\Omega_2 = P_2(\Omega)$, such that
$f(z_1 + z_2 i_2) = f_{e_1}(z_1 - z_2 i_1)e_1 + f_{e_2}(z_1 + z_2 i_1)e_2 \forall \boldsymbol{z_1 + z_2 i_2 \in \Omega}$

To show that $c.f(w)$ it is a bicomplex meromorphic function, define a new functions $G(w) = c.g(w)$ and $H(w) = h(w)$. Then $G(w)$ and $H(w)$ are bicomplex holomorphic functions. Therefore, $G(w)$ and $H(w)$ can be written as

$$G(w) \quad G(z_1 + z_2 i_2) = G_{e_1}(z_1 - z_2 i_1)e_1 + G_{e_2}(z_1 + z_2 i_1)e_2 \quad (3)$$

$$H(w) \quad H(z_1 + z_2 i_2) = H_{e_1}(z_1 - z_2 i_1)e_1 + H_{e_2}(z_1 + z_2 i_1)e_2 \quad (4)$$

From (1) and (2), we have

$$f(w) = \frac{G(w)}{H(w)} = \frac{G_{e_1}(z_1 - z_2 i_1)e_1 + G_{e_2}(z_1 + z_2 i_1)e_2}{H_{e_1}(z_1 - z_2 i_1)e_1 + H_{e_2}(z_1 + z_2 i_1)e_2}$$

$$= \frac{G_{e_1}(z_1 - z_2 i_1)}{H_{e_1}(z_1 - z_2 i_1)} e_1 + \frac{G_{e2}(z_1 + z_2 i_1)}{H_{e2}(z_1 + z_2 i_1)}$$

$$= F_{e_1}(z_1 - z_2 i_1)e_1 + F_{e_2}(z_1 + z_2 i_1)e_2$$

where F_{e_i} is meromorphic in Ω_i for i = 1, 2.
Hence, scalar multiplication of bicomplex meromorphic function is bicomplex meromorphic.

3. Bicomplex Q_m- Normal Families

In [4] Q_m- Normal families of one complex variable of finite order is developed in detail. We intend to develop the theory for bicomplex variables. The theory of bicomplex functions is currently a topic of interest, as it is closely related to Clifford algebras and multicomplex analysis, making it a significant area of study in recent research. In [6], Riley first talked about the bicomplex functions. Later, we find a monograph by G.B. Price [1], and recently, a monography [16] has been written by Leuna et al. In [8], K.S. Charak and N. sharma have defined bicomplex normal families. For developing bicomplex normal families, they have constructed the bicomplex extended plane $\hat{C}$. In this paper, we shall define Bicomplex Q_m-Normal families and investigate its properties.

Definition 3.1: Let $S = \{f_n\}$ be a sequence of bicomlex meromorphic functions defined in a domain Ω. A point Z_0 of Ω is called a C_0-point of S if there exists a discus $D(Z_0, r_1, r_2) = \{||Z - Z_0|| < r, r = min(r_1, r_2)\} \subseteq \Omega$ such that the sequence S is uniformly spherically convergent in $D(Z_0, r_1, r_2)$. S is said to be a C_0-sequence in Ω if $\forall Z_0 \in \Omega$, S is a C_0-sequence.

To understand the definition, let us consider the sequence of bicomplex meromorphic functions $S = \{f_n(Z)\}$ defined in the domain $\Omega \subseteq \mathbb{T}$ where each $f_n(Z)$ is given by:

$$f_n(Z) = \frac{1}{Z_1 - n} + \frac{1}{Z_2 - n}, n \in N$$

Then for each $n \in N$, $f_n(Z)$ has singularities at $Z_1 = n$ and $Z_2 = n$. However, as $n \rightarrow \infty$, the singularities move towards infinity, and $f_n(Z)$ becomes meromorphic in Ω for large enough n. Therefore, as $n \rightarrow \infty$, the sequence S converges uniformly on compact subsets of Ω. Since S converges uniformly on compact subsets of Ω as $n \rightarrow \infty$, every point $Z_0 \in \Omega$ can be considered a C_0-point of S. Hence, S it is a C_0-sequence Ω. This example illustrates the concept of a C_0-point and a C_0-sequence for a sequence of bicomplex meromorphic functions in the domain $\Omega \subseteq \mathbb{T}$.

Definition 3.2: Let $S = \{f_n\}$ be a sequence of bicomlex meromorphic functions in a domain Ω. A point Z_0 of Ω is called a C_1-point of S if $\exists$ a discus $D(Z_0, r_1, r_2) = \{||Z - Z_0|| < r, r = min(r_1, r_2)\} \subseteq \Omega$ such that each point of $D_0(Z_0, r_1, r_2) = \{0 < ||Z - Z_0|| < r\}$ is a C_0-point of S. S is called a C_1-Sequence Ω if each point of Ω is a C_1-point of S. Suppose we denote by E the set of all non C_0-points of S in Ω. Then $E = \phi$ if S is a C_0-sequence and $E_\Omega^1 = \phi$ if S is a C_1-Sequence. By generalizing this, we see that it is natural to have $E_\Omega^2 = \phi$, $E_\Omega^3 = \phi$ enhancing it we define.

Definition 3.3: Let $S = \{f_n\}$ be a sequence of bicomlex meromorphic functions in a domain Ω and Z_0 a point ofΩ. We say that Z_0 is a C_m- point of S, $m \geq 2$ if there is a discus
$D(Z_0, r_1, r_2) = \{||Z - Z_0|| < r, r = min(r_1, r_2)\} \subseteq \Omega$ such that each point of $D_0(Z_0, r_1, r_2) = \{0 < ||Z - Z_0|| < r\}$ is a C_{m-1}-point of$S$. In this way, C_m-point of S can be defined for each integer$m \geq 0$.

Example of C_m-point Let $S = \{f_n(Z)\}$ be the sequence of bicomplex meromorphic functions defined in the domain $\Omega = \{Z \in T: |Z_1| < 1, |Z_2| < 1\}$ where each $f_n(Z)$ is given by:

$$f_n(Z) = \frac{1}{Z_1 - n} + \frac{1}{Z_2 - n}, n \in N$$

Now, let's take a point$Z_0 = (1,1) \in \Omega$. We want to show that Z_0 is a C_m-point of $S$for$Z$. Let $D(Z_0, r_1, r_2)$ be a bicomplex discus centered at $Z_0$with radii $r_1 = r_2 = 1/2$, i.e. $D(Z_0, 1/2, 1/2) = \{Z \in T: ||Z - Z_0|| < 1/2\}$. We need to check that each point in the annulus $D_0(Z_0, r_1, r_2) = \{Z \in T: 0 < ||Z - Z_0|| < 1/2\}$ is a C_{m-1}-point S. For any$Z \in D_0(Z_0, r_1, r_2)$, let's compute $f_n(Z)$ for some$n$. For simplicity, let's take$n = 2$. $f_2(Z) = \frac{1}{Z_1\text{-}2} + \frac{1}{Z_2\text{-}2}$

Since Z lies in $D_0(Z_0, r_1, r_2)$both $|Z_1$-2$|$ and $|Z_2$-2$|$ are greater than $1/2$, ensuring that $f_2(Z)$is well-defined and meromorphic in $D_0(Z_0, r_1, r_2)$. Thus, Z_0 is a C_1-point of S. Since $Z_0$is a C_1-point of$S$, by induction, it can be shown that$Z_0$is a C_m-point of for all$m \geq 2$. This example illustrates the concept of a C_m-point of a sequence of bicomplex meromorphic functions, where the function behaves well in a neighborhood of the point, even if the point itself is singular.

Definition 3.4: A sequence S is said to be C_m-Sequence in Ω if each point of Ω is a C_m-point of$S$.

Theorem 3.5 : Every C_m-sequence of bicomplex meromorphic function defined in a domain Ω is C_{m-1}-sequence in Ω. But converse is not true.

Proof: C_m-sequence implies C_{m-1}-sequence: Let S be a C_m-sequence in Ω. Then by definition, for every point$Z_0 \in \Omega$, there exists a discus $D(Z_0, r_1, r_2)$ such that each point in $D(Z_0, r_1, r_2)$ is a C_{m-1}-point of S. Therefore, S behaves well enough in a neighborhood of each point to have no singularities up to order $m - 1$. Hence, Sit is also a C_{m-1}-sequence in Ω.

The converse of the theorem is not true. To prove that the converse is not necessarily true, let's consider a specific example:

Let $S = \{f_n(Z)\}$be the sequence of bicomplex meromorphic functions defined in the domain$\Omega \subseteq C^2$, where each $f_n(Z)$ is given by:

$$f_n(Z) = \frac{1}{(Z_1 - n)^2} + \frac{1}{(Z_2 - n)^2}, n \in N$$

Then each $f_n(Z)$has singularities at $Z_1 = n$ and $Z_2 = n$. As n increases, the singularities become stronger (second-order poles). Therefore, for each n, there exists a discus $D_0(Z_0, r_1, r_2)$ centered at (n, n) such that S behaves well enough to be a C_1-sequence. To prove that the sequence $S = \{f_n(Z)\}$does not become a C_2-sequence$n \to \infty$, we need to show that there exist points in Ω where the functions in S do not behave well enough to have no singularities up to order 2. For a sequence to be a C_2-sequence, every point in Ω must be a C_2-point of S. This means that there must exist a discus centered at each point $Z_0 \in \Omega$ such that every point in the annulus $D_0(Z_0, r_1, r_2)$is a C_1-point of S. Now, consider the behavior of S as$n \to \infty$. The functions $f_n(Z)$ have singularities at $Z_1 = n$ and$Z_2 = n$, and as n increases, the singularities become stronger (second-order poles). However $n \to \infty$, the distance between the singularities and any fixed point in Ω also tends to infinity. This means that for any fixed point$Z_0 \in \Omega$, no matter how large n becomes, there will always be a neighborhood aroundZ_0 that contains the singularities of $f_n(Z)$. Therefore, there does not exist a discus centered at Z_0 such that every point in the annulus $D_0(Z_0, r_1, r_2)$ is a C_1-point of$S$. Consequently, S does not become a C_2-sequence as$n \to \infty$. Generalizing in this way, we conclude that every C_m-sequence in a domain Ω is C_{m-1}-sequence, but the converse is not true.

The converse of the theorem is not true. To prove that the converse is not necessarily true, let's consider a specific example:

Let $S = \{f_n(Z)\}$ be the sequence of bicomplex meromorphic functions defined in the domain $\Omega \subseteq C^2$, where each $f_n(Z)$ is given by:

$$f_n(Z) = \frac{1}{(Z_1 - n)^2} + \frac{1}{(Z_2 - n)^2}, n \in N$$

Then, each $f_n(Z)$ has singularities at$Z_1 = n$ and $Z_2 = n$. As n increases, the singularities become stronger (second-order poles). Therefore, for each n, there exists a discus $D_0(Z_0, r_1, r_2)$ centered at (n, n) such that S behaves well enough to be a C_1-sequence. To prove that the sequence $S = \{f_n(Z)\}$ does not become a C_2-sequence$n \to \infty$, we need to show that there exist points in Ω where the functions S do not behave well enough to have no singularities up to order2. For a sequence to be a C_2-sequence, every point in Ω must be a C_2-point of S. This means that there must exist a discus centered at each point $Z_0 \in \Omega$ such that every point in the annulus $D_0(Z_0, r_1, r_2)$ is a C_1-point of$S$. Now, consider the behavior of S as$n \to \infty$. The functions $f_n(Z)$ have singularities at $Z_1 = n$ and$Z_2 = n$, and as n increases, the singularities become stronger (second-order poles). However$n \to \infty$, the distance between the singularities and any fixed point in Ω also tends to infinity. This means that for any fixed point $Z_0 \in \Omega$, no matter how large n becomes, there will always be a neighborhood around Z_0that contains the singularities of$f_n(Z)$. Therefore, there does not exist a discus centered at Z_0 such that every point in the annulus $D_0(Z_0, r_1, r_2)$ is a C_1-point of$S$. Consequently, Sdoes not become a C_2-sequence as$n \to \infty$. Generalizing in this way, we conclude that every C_m-sequence in a domain Ω is C_{m-1}-sequence, but the converse is not true.

Lemma 3.6: If for some integers m≥ 2 , Z_0 is a C_m-point of$S$, then
(I) $Z_0$is aC_{m+1}-point of S
(II) there exists a discussion $D(Z_0, r_1, r_2)$ =$\{||Z - Z_0|| < r, r = min(r_1, r_2)\} \subseteq \Omega$ such that each point of which is a C_m-point of S.
(III) $Z_0$is a C_m-point of every subsequence of S.

Proof. Assume that Z_0 is a C_m-point of S . Then, by definition of C_m-point, there exists a discus $D(Z_0, r_1, r_2)$ =$\{||Z - Z_0|| < r, r = min(r_1, r_2)\} \subseteq \Omega$ such that each point of $D_0(Z_0, r_1, r_2)$ =$\{0 < ||Z - Z_0|| < r\}$ is a C_1-point of$S$. We have to prove that Z_0 is a C_3-point of$S$. Since Z_0 it is a C_2-point of$S$, there exists a disk Δ =$\{||Z - Z_0|| < r\} \subseteq \Omega$ such that each point of Δ_0=$\{0 < ||Z - Z_0|| < r\}$ is a C_1-point of$S$. Now, each C_1-point is a C_0-point by definition.

Assume that the result is true for$m = k$, that is, if Z_0 is a C_k-point of$S$, then Z_0 is a C_{k+1}-point of$S$, there exists a disk such that each point is a C_k-point of S and hence Z_0 is a C_k-point of every sub-sequence.
Now, we have to prove the result for $m = k + 1$. For this, we have to prove that Z_0 is a C_{k+2}-point.
By the assumption, Z_0 it is a C_{k+1}-point. Therefore, there exists a discus

$D(Z_0, r_1, r_2)$ =$\{||Z - Z_0|| < r, r = min(r_1, r_2)\} \subseteq \Omega$ such that each point of $D_0(Z_0, r_1, r_2)$ =$\{0 < ||Z - Z_0|| < r\}$ is a C_k-point of S.Now, by the inductive hypothesis, each C_k-point is a C_{k+1}-point. Therefore, Z_0 is a C_{k+2}- point

Proof of II Assume that Z_0 is a C_m-point of S. Then, by definition of C_m-point, there exists a discus $D(Z_0, r_1, r_2)$ =$\{||Z - Z_0|| < r, r = min(r_1, r_2)\} \subseteq \Omega$ such that each point of $D_0(Z_0, r_1, r_2)$ =$\{0 < ||Z - Z_0|| < r\}$ is a C_{m-1}-point of$S$. Now we want to show that each point of D_0 is also a C_m-point of$S$. For the sake of simplicity, let's take this disk asD_1. Let Z' be a C_{m-1}-point of D_1. Since D_1 is a subset of$D$. Therefore, Z' is also a point of$D$, and thus, Z' is a C_{m-1}-point of$S$.

Since the above reasoning holds for any Z' in D_1, and D_1 is a subset of D, it is concluded that every point in D is a C_{m-1}-point of S. Since each point of D is already is a C_{m-1}-point of$S$, we can use the same disk D to show that every point in D is also a C_m-point of S. Thus we have shown that if Z_0 is a C_m-point of$S$, then there exists a discus $\{|Z - Z_0| < r\}$ such that every point of this discus is also a C_m-point of$S$.

Proof (III) Assume that Z_0 is a C_m-point of S. Then by definition of C_m-point, there exists a discus $D_0(Z_0, r_1, r_2)$ =$\{||Z - Z_0|| < r, r = min(r_1, r_2)\} \subseteq \Omega$ such that each point of $D(Z_0, r_1, r_2)$ =$\{0 < ||Z - Z_0|| < r\}$ is a C_{m-1}-point of$S$. Now, consider any subsequence S' of$S$. We want to prove that Z_0 is a C_m-point ofS'. Since Z_0 is a C_m-point of$S$. Therefore, there exists a disk such that every point of which is a C_{m-1}-point of$S$. Thus the same disk works for the subsequence S' as well. Clearly Z_0 is a C_{m-1}-point ofS'. Now, Z_0 is a C_{m-1}-point ofS', by the definition of C_m-point, Z_0 is also a C_m-point S'. Since S' is an arbitrary subsequence$S$. Therefore, it holds for all subsequences of S. Thus we have proved that if Z_0 is a C_m-point of$S$, then Z_0 is a C_m-point of every subsequence of S.

Lemma 3.7: For some integers m≥ 2 , if S is a C_m-sequence inΩ. Then S is C_{m+1} sequence in Ω and every subsequence of S is a C_m-sequence inΩ.

Proof: Suppose S is a C_m-sequence inΩ. This means that for every C_m-point $Z_0$of S, there exists a discus $D(Z_0, r_1, r_2) \subseteq \Omega$ such that each point of $D(Z_0, r_1, r_2)$ =$\{0 < ||Z - Z_0|| < r\}$ is a C_{m-1}-point of S Now, let's show that S is a C_{m+1}-sequence in Ω. Let Z_0 be a C_m-point of S, and let $D(Z_0, r_1, r_2)$ be the corresponding discus as per the definition. Since each point of $D_0(Z_0, r_1, r_2)$ is a C_{m-1}-point of S, by the induction hypothesis, S is a C_m-sequence in $D_0(Z_0, r_1, r_2)$. Now, consider the compact subset $D'(Z_0, r_1, r_2) = D(Z_0, r_1, r_2) \cup D_0(Z_0, r_1, r_2)$ of Ω. By the continuity of$S$, S is bounded on $D'(Z_0, r_1, r_2)$. Now, let's consider an arbitrary compact subset K of Ω. Since K is compact, there exists a finite number of C_m-points of S contained in$K$. Let $Z_1, Z_2, \dots, Z_n$ be these C_m-points. For each Z_i, we can find a corresponding disc $D(Z_i, r_{1i}, r_{2i})$ such that S is a C_m-sequence in $D(Z_i, r_{1i}, r_{2i})$. Now, consider the union of all these discs and K: $D' = \cup_{i=1}^{n} D(Z_i, r_{1i}, r_{2i}) \cup K$. This is a compact subset of Ω. Since S is bounded on $D'$and $K \subseteq D'$, S is bounded on K. Hence, S is a C_{m+1}-sequence in Ω.

Now, to prove that every subsequence of S is a C_m-sequence in Ω. Let S' be a subsequence of$S$. Since S' is a subset of$S$, it inherits the property of being a C_m-sequence inΩ.
Thus we have shown that if S is a C_m-sequence in Ω, then S is a C_{m+1}-sequence in Ω, and every subsequence of S is also a C_m-sequence in Ω.

Definition 3.8: A point $Z_0 \in \Omega$ is said to be non-C_m-point of S if Z_0 is not C_m-point of$S$. The set of all non-C_m-points is denoted by E.

W_m- property: If E_{Ω}^{j}, j=0,1, ... m $\neq \phi$, we say that the set E has W_m- property w.r.t Ω.

Theorem 3.9: Let $S = \{f_n(Z)\}$ be a sequence of bicomplex meromorphic functions in a domainΩ. Let $Z_0 \subseteq \Omega$. Then Z_0 is a non C_m-point of S if and only if $Z_0 \in E_{\Omega}^{m}$, where E is the set of non

C_0 -points of S in Ω.

Proof : Let E denote the set of non-C_0-points of S in Ω. Then, E contains all points where the functions in S have singularities.

To prove the theorem, we shall establish the equivalence between a point Z_0 being a non-C_m-point of S and Z_0 belonging to E_{Ω}^{m}, where E is the set of non-C_0-points of S in Ω.

For this, first assume that Z_0 is a non-C_m-point of S. This means that there does not exist a discus $D(Z_0, r_1, r_2)$ such that every point in $D_0(Z_0, r_1, r_2)$ is a C_{m-1}-point of S. Since Z_0 is not a C_m-point, it implies that $E_{\Omega}^{m} \neq \phi$, because there exists at least one point, namely Z_0, in E_{Ω}^{m}.

Conversely assume that $Z_0 \in E_{\Omega}^{m}$. we shall prove that Z_0 is a non-C_m-point of S. Since $Z_0 \in E_{\Omega}^{m}$. This means that there exists a discus $D_0(Z_0, r_1, r_2)$ such that every point in $D_0(Z_0, r_1, r_2)$ is a non-C_{m-1}-point of S. If Z_0 is a C_m-point, then every point in $D_0(Z_0, r_1, r_2)$ would be a C_{m-1}-point by definition. Since this is not the case, Z_0 cannot be a C_m-point. Therefore, Z_0 is a non-C_m-point of S. Which completes the proof of the theorem.

Definition 3.10 : Let F be a family of bicomplex meromorphic functions in a domain Ω and $m \geq 0 \in Z$, we say that F is Q_m- Normal family in Ω if from every sequence of functions of the family F, we can extract a subsequence which is C_m-sequence in Ω. That is F is Q_m- Normal at point $Z_0 \in \Omega$ if there exists a discus $D(Z_0, r_1, r_2) \subseteq \Omega$ such that F is Q_m- Normal in $D(Z_0, r_1, r_2)$. In particular, Q_0- Normal family is normal family and Q_1- Normal family is quasi-normal family in Ω.

Theorem 3.11: Let F be a family of bicomplex meromorphic functions in a domain Ω and $m \geq 0 \in Z$ if F is Q_m- Normal at each point of Ω, then F is Q_m- Normal in Ω.

Proof: First of all , we consider a sequence $\{Z_j\}$, j=1,2,3,....... in Ω such that each point of Ω is a limiting point of the sequence $\{Z_j\}$. Now by hypothesis a discus$D\left(Z_j, r_{1j}, r_{2j}\right) \subseteq \Omega$ such that F is Q_m-normal in $D\left(Z_j, r_{1j}, r_{2j}\right)$. Let R_j be the least upper bound of the set of members $\{r_{1j}, r_{2j}\}$ having this property. Define discuss$D_j = D\left(Z_j, R_j/2, R_j/2\right)$ for each Z_j. If $R_j = \infty$, then D_j covers Ω, otherwise, D_j is contained in Ω and the family is Q_m. Normal in D_j.Now let S_1=$\{f_n(z)\}$, n=1,2,.... be sequence of functions of the family F from S be just a subsequence S_1 = $f_{\alpha_1}(z)$,$f_{\alpha_2}(z)$, ..., which is a C_m -sequence in D_1. From S_1 we can get a sequenceS_g : $f_{\beta_1}(z)$,$f_{\beta_2}(z)$, ..., which is a C_m -sequence in D_2. In this way we get successively a sequenceS_p, $p =$ 1,2,3,... such that for each $p \geq 1$, $S_p$is a C_m -sequence in D_p and S_{p+1} is a subsequence of S_p. Consider the diagonal sequence

S'=$f_{\alpha_1}(z)$, $f_{\beta_2}(z)$, $f_{\gamma_3}(z)$,.......$f_{\lambda_k}(z)$....... Now S' is a subsequence of $\{f_{n_k}(z)\}$,(k=1,2,3,...) of S since for each k the terms $f_{n_k}(z)$, $f_{n_{k+1}}(z)$, ... all belongs the sequence $\{S_k\}$. Hence S' is a C_m-sequence in each of the D_j, j=1,2,3,....

Consider the point Z' of Ω. Since each D_j coversΩ, there exists a discus $D(Z', \rho_1, \rho_2)$ contained in some D_j. By construction, F is Q_m-normal in $D(Z', \rho_1, \rho_2)$. If $R_j < \infty$, then $\rho_1 < R_j/2$ and $\rho_2 < R_j/2$ for some j. Hence, F is Q_m-normal in $D(Z', 2\rho_1, 2\rho_2) \subset D_j$. If $R_j < \infty$, then since $\rho_1, \rho_2 < 1$, $D(Z', \rho_1, \rho_2)$ is contained in D_j. Thus, F is Q_m-normal in $D(Z', \rho_1, \rho_2)$. Since Z' is arbitrary, S' is a C_m-sequence in Ω. Hence, F is Q_m-normal in Ω

Which completes the proof of the theorem.

Lemma 3.12 : For an integer $m \geq 0$, If the family F of bicomplex meromorphic functions defined in a domain Ω is Q_m-Normal in Ω, then F is Q_{m+1}- Normal in Ω.

Proof: We shall prove this result by the Principle of Mathematical Induction. First, we prove the base case when $m = 0$. Suppose F is Q_0-normal inΩ. By definition, this means that for every compact subset KofΩ, there exists a constant M_K such that for every $f \in F$, we have $|f(z)| \leq M_K$ for all $Z \in K$. Now, let K be a compact subset of Ω. Since F is Q_0-normal, there exists M_K such that $|||f(Z)||| \leq M_K$ for all $f \in F$and $Z \in K$. Now, consider the set $Q_1(K) = \{Z \in \Omega: |||Z||| \leq M_K\}$. This set is compact since it is closed and bounded. Therefore, F is Q_1-normal in Ω. Now, assume that for some integer $m \geq 0$, if F is Q_m-normal in Ω, then F is Q_{m+1}-normal in Ω. Now, let's prove that if F is Q_{m+1}-normal in Ω, then F is Q_{m+2}-normal in Ω. Suppose F is Q_{m+1}-normal in Ω. By definition, this means that for every compact subset K of Ω, there exists a constant M_K such that for every $f \in F$, we have $||f(Z)||| \leq M_K(1 + |Z|)^{m+1}$ for all $Z \in K$. Now, let K be a compact subset of Ω. Since F is Q_{m+1}-normal, there exists M_K such that $|||f(Z)||| \leq M_K(1 + |Z|)^{m+1}$ for all $f \in F$ and $Z \in K$. Now, consider the set$Q_{m+2}(K) = \{Z \in \Omega: |z| \leq M_K\}$. This set is compact since it is closed and bounded. Therefore, F is Q_{m+2}-normal in Ω.

By induction, the result follows for all non-negative integers$m$. Therefore, if F is Q_m-normal inΩ, then F is Q_{m+1}-normal in Ω.

Theorem 3.13 : A family F of bicomplex meromorphic functions defined on a domain $\Omega \subseteq T$ is Q_m- normal with respect to the bicomplex chordal metric if and only if the family of meromorphic functions on F_{e_i}= $P_i(F)$ is Q_m- normal in $P_i(\Omega)$ for i=1,2 with respect to the chordal metric.

Proof: Assume that F is Q_m- normal with respect to the bicomplex chordal metric on Ω. We want to show that F_{e_i}= $P_i(F)$ is Q_m-normal in $P_i(\Omega)$ for i=1,2. For this, let $\{(f_n)_1\}$ be a sequence in F_{e_1}= $P_1(F)$. We want to show, without loss of generality that from the sequence of meromorphic functions $\{(f_n)_1$, we can extract a subsequence which is C_m-sequence in$P_1(\Omega)$.

]Since F is normal in Ω , we can find a sequence $\{f_n\}$ in F such that $\{P_1(f_n)\}$=$\{(f_n)_1\}$. Moreover for any $z_0 \in P_1(\Omega)$, we can find a $w_0 \in \Omega$ such that $P_1(w_0) = z_0$. Now consider a closed discus $\bar{D}(w_0, r, r)$ in Ω. By hypothesis, the sequence $\{f_n\}$ contains a subsequence $\{f_{n_k}\}$ which is C_m-sequence in $\bar{D}(w_0, r, r)$. Thus there exists a closed ball $\bar{B}(z_0, r) \subseteq P_1(\Omega)$ such that $\{P_1(f_{n_k})\}$=$\{(f_{n_k})_1\}$ is a C_m-sequence in $\bar{D}(w_0, r, r)$. Similarly, we can prove it for the sequence $\{(f_n)_2\}$ in F_{e_2}= $P_2(F)$ Therefore, F_{e_i}= $P_i(F)$ is Q_m- normal in $P_i(\Omega)$ for i=1,2 with respect to the chordal metric. Conversely, assume that F_{e_i}= $P_i(F)$ is Q_m-normal in $P_i(\Omega)$ =Ω_i for i=1, 2 with respect to the chordal metric. we need to show that F is Q_m- normal in Ω with respect to the bicomplex chordal metric. Let $\{f_n\}$ be any sequence in F and K be any compact subset of Ω. Then $\{P_1(f_n)\}$=$\{(f_n)_1\}$ and $\{P_2(f_n)\}$=$\{(f_n)_2\}$ are sequences in F_{e_1}= $P_1(F)$ and F_{e_2}= $P_2(F)$ respectively. Since F_{e_1}= $P_1(F)$ is Q_m- normal in $P_1(\Omega)$, $\{(f_n)_1\}$ has a subsequence $\{(f_{n_k})_1\}$ which is a C_m- sequence on compact subsets of $P_1(K)$ =K_1. Similarly, for the sequence $\{P_2(f_n)\}$=$\{(f_n)_2\}$ in F_{e_2}= $P_2(F)$ and by assumption that F_{e_2}= $P_2(F)$ is Q_m- normal in $P_2(K)$. Therefore, $\{(f_n)_2\}$ has a subsequence $\{(f_{n_k})_2\}$ which is a C_0- sequence on compact subsets of $P_2(K)$. This implies that $\{(f_{n_k})_1 e_1$+$(f_{n_k})_2 e_2\}$ is a subsequence of $\{f_n\}$ which is a C_m-sequence in $P_1(K) \times P_2(K) \supseteq K$. Thus F is Q_m- normal with respect to the bicomplex chordal metric.

Definition 3.13 : Let $S = \{f_n\}$ be a sequence of bicomplex meromorphic functions in a domain Ω, and Z_0 a point of Ω. We say that Z_0 is a μ_1-point of S if for each closed neighborhood

$\bar{D}(Z_0, r_1, r_2)$ ={$||Z - Z_0|| \leq r, r = min(r_1, r_2)$} $\subseteq \Omega$, we have $\lim_{n\to+\infty} \max_{Z\in\bar{D}} \partial(Z, f_n) = +\infty$,

Z_0 is called a μ_2-point of S, if for each open discus $D(Z_0, r_1, r_2)$ ={$||Z - Z_0|| < r$} $\subseteq \Omega$, S has a μ_1-point z' in the domain {$0 < | \; |Z - Z_0| \; | < r$}. In general, $m \geq 2$, Z_0 is called a μ_m-point of S, if for each domain $D(Z_0, r_1, r_2)$ ={$||Z - Z_0|| < r$} $\subseteq \Omega$, S has a μ_{m-1}-point in the domain{$0 < ||Z - Z_0|| < r, r = min(r_1, r_2)$}. Let S' be a subsequence of S. Then clearly, if Z_0 is a μ_1-point of S, then Z_0 is a μ_1-point of S'. By mathematical induction, we see that in general, $m \geq 1$ being an integer, if Z_0 is μ_m-point of$S$, then Z_0 is μ_m-point of $S^{'}$

Consider the sequence $S = \{f_n(Z)\}$ defined as follows:
$f_n(Z) = \frac{1}{(Z-n)^2}$, where $Z = (z_1, z_2)$ is a point in the bicomplex plane C^2 and $n \in N$.
Then $Z_0 = (0,0)$ is a μ_1-point of S, $Z_0 = (1,1)$ is a μ_2-point of S and $Z_0 = (m, m)$ is a μ_m-point of S
1.μ_1-point: Suppose$Z_0 = (0,0)$, and consider a closed discus$\bar{\Delta} = \{Z \in \Omega: ||Z - Z_0|| \leq r\}$. For any$r > 0$, as$n \to \infty$, the function $f_n(Z)$ grows unbounded as $||Z - Z_0|| \to r$. Therefore, Z_0 is a μ_1-point of S.

2. μ_2-point: Now consider$Z_0 = (1,1)$, and a discus$\Delta = \{Z \in \Omega: ||Z - Z_0|| < r\}$. In this case, for any$r > 0$, there exists a μ_1-point Z' such that $0 < ||Z' - Z_0|| < r$, because $f_n(Z)$ grows unbounded as $Z \to Z_0$. Therefore, Z_0 is a μ_2-point of S.

3. μ_m-point: For $m \geq 3$, consider $Z_0 = (m, m)$. For any discus $\Delta = \{Z \in \Omega: ||Z - Z_0|| < r\}$, there exists a μ_{m-1}-point Z' such that $0 < ||Z' - Z_0|| < r$. Hence, Z_0 is a μ_m-point of S.

Definition 3.14 : Let $m \geq 1$ be an integer. Let F be a family of of bicomplex meromorphic functions defined on a domain Ω and $\nu \geq 0$ an integer. We say that F is Q_m- normal of order ν if from every sequence of functions of the family F. We can extract a subsequence which is a C_m-sequence in D and has atmost ν non C_{m-1}-point in D. Thus for $\nu = 0$, $F$is Q_{m-1} -normal in D

Definition 3.15: *Q_m- Normal family of infinite order.*
A family F is said to be Q_m- Normal of infinite order if F is Q_m- Normal but not a Q_m- Normal family of order at most$\nu \geq 1$. on this definition we have the theorem:

Theorem 3.16: A family F of bicomplex meromorphic functions is Q_m- Normal in Ω if and only if every sequence of functions $\{f_n\}$ of F has no μ_{m+1}-point in Ω.

Proof: Suppose F is Q_m-normal in Ω. then, by definition, from every sequence of functions $\{f_n\}$ of F, we can extract a subsequence which is a C_m-sequence in Ω.
We shall prove it by contradiction. Suppose that there exists a μ_{m+1}-point Z_0 of some sequence $\{f_n\}$ in F, i.e., for every open discus $D(Z_0, r_1, r_2)$ contained in Ω, there exists a μ_m-point z' in the domain $\{0 < ||Z - Z_0|| < r\}$. This implies that we can construct a subsequence of $\{f_n\}$ that converges to $+\infty$ atZ_0, which contradicts the assumption that F is Q_m-normal. Therefore, Fcannot have a μ_{m+1}-point in Ω. Suppose F is Q_m-normal in Ω.

By definition, this means that from every sequence of functions $\{f_n\}$ of$F$, we can extract a subsequence which is a C_m-sequence in Ω. To prove this statement, we shall use proof by contradiction. We assume that there exists a -point Z_0 of some sequence $\{f_n\}$ in F. This implies that for every open discus $D(Z_0, r_1, r_2)$ contained in Ω, there exists a μ_m-point z' in the domain $\{0 < ||Z - Z_0|| < r\}$. Now, consider a sequence $\{f_n\}$ in F and let Z_0 be a μ_{m+1}-point of this sequence. For any open discus $D(Z_0, r_1, r_2)$ contained in Ω, there exists a μ_m-point z' in the domain $\{0 < ||Z - Z_0|| < r\}$. This implies that we can construct a subsequence of $\{f_n\}$ that converges to $+\infty$ at Z_0, which contradicts the assumption that F is Q_m-normal. If F were Q_m-normal, there should be no such μ_{m+1}-point. Therefore, we have shown that F cannot have a μ_{m+1}-point inΩ. In summary, we have demonstrated that if F is Q_m-normal inΩ, then it cannot have a μ_{m+1}-point in Ω. To prove the converse part, let's assume that F does not have any μ_{m+1}-point inΩ. We aim to show that F is Q_m-normal inΩ. Let $\{f_n\}$ be any sequence of functions in F. Since $\{f_n\}$ has no μ_{m+1}-point inΩ, it means that for every sequence $\{Z_k\}$ in Ω, there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $lim_{\;k\to\infty} f_{n_k}(Z_k) = +\infty$. This implies that for any closed disc $\bar{D}(Z_0, r_1, r_2)$ contained in Ω, we can find a subsequence of $\{f_n\}$, denoted by $\{f_{n_k}\}$, such that $f_{n_k}(Z_k)$ diverges to $+\infty$ as k approaches infinity, where Z_k is a sequence of points in $D(Z_0, r_1, r_2)$. Therefore, $\{f_n\}$ satisfies the condition of being a C_m-sequence. Since this holds for any sequence $\{f_n\}$ in F, it follows that F is Q_m-normal in Ω. Therefore, we have shown that a family F of bicomplex meromorphic functions is Q_m-normal in Ω if and only if every sequence of functions $\{f_n\}$ of F has no μ_{m+1}-point in Ω.

4. Conclusion

This paper extends the theory of Q_{m-} normal families of meromorphic functions using the concept of C_m-sequences from the complex variable setting to the bicomplex variable setting. It begins with a discussion of the properties of bicomplex meromorphic functions and establishes its key results in the bicomplex plane. Using C_m-sequences, the concept of Q_{m-} normal families is generalized to bicomplex meromorphic functions, offering new insights and a broader scope for understanding normality in this richer setting.This study highlights Q_{m-} normal families of bicomplex meromorphic functions as a specialized and emerging area in bicomplex analysis. It provides a foundation for future research into deeper theoretical aspects, including their properties, characterizations, and connections with other mathematical structures. Furthermore, this research encourages exploration into how the behaviors of Q_{m-} normal families in the bicomplex domain differ from those in the complex domain, necessitating the development of new theoretical frameworks. Additionally, such families have the potential to model physical phenomena in multi-dimensional spaces, making them relevant for applications in fields like quantum mechanics and relativity theory. The results presented in this paper pave the way for further studies, contributing to both the theoretical and applied dimensions of bicomplex analysis.

Acknowledgements

Authors are thankful to the Department of Mathematics, Bhagwant University for giving us this opportunity. Authors are also thankful to Dr. Muzafar Ahmad Bhat, Lecturer, Department of Sericulture, Government Degree College Poonch, University of Jammu for his constant technical support.

Author Contributions

All authors contributed in the concept and design of the present study. Material preparation, data collection and analysis performed by [Tehseen Abas Khan], [Jyoti Gupta] and [Ravinder Kumar]. The first draft of the manuscript was written by [Tehseen Abas Khan] and all authors read and approved the final manuscript. As the field progressed, researchers like Giovanni Battista Rizza and Marcel Riesz made substantial contributions to the study of bicomplex functions and their analytical properties in the mid-20th century. Their work paved the way for deeper explorations into the analytical aspects of bicomplex analysis.

References

[1] G.B. Price, *An Introduction to Multicomplex Spaces and Functions*, 1st ed., CRC Press, pp. 1-424, 1991. [CrossRef] [Google Scholar] [Publisher Link]

[2] Joel L. Schiff, *Normal Families*, Springer New York, pp. 1-236, 1993. [Google Scholar] [Publisher Link]

[3] Lawrence Zalcman, "Normal Families, New Perspectives," *Bulletin of the American Mathematical Society*, vol. 35, no. 3, pp. 215-230, 1998. [Google Scholar] [Publisher Link]

[4] Qitai Zhuang, *Normal Families of Meromorphic Functions*, World Scientific Publishing Company, pp. 1-488, 1993. [Google Scholar] [Publisher Link]

[5] John B. Conway, *Functions of One Complex Variable*, Springer-New York, pp. 1-317, 1978. [Publisher Link]

[6] James D. Riley, "Contributions to the Theory of Functions of a Bicomplex Variable," *Tohoku Mathematical Journal, Second Series*, vol. 5, no. 2, pp. 132-165, 1953. [CrossRef] [Google Scholar] [Publisher Link]

[7] Kuldeep Singh Charak, Dominic Rochon, and Narinder Sharma, "Normal Families of Bicomplex Holomorphic Functions," *Fractals*, vol. 17, no. 2, pp. 257-268, 2009. [CrossRef] [Google Scholar] [Publisher Link]

[8] Kuldeep Singh Charak, Dominic Rochon, and Narinder Sharma, "Normal Families of Bicomplex Meromorphic Functions," *Annals of Polish Mathematicians*, vol. 103, no. 3, pp. 303-317, 2011. [Google Scholar] [Publisher Link]

[9] Paul Montel, *On Quasi-Normal Families of Holomorphic Functions*, M. Lamertin, pp. 1-41, 1922. [Google Scholar] [Publisher Link]

[10] Paul Montel, "On Normal Families of Analytic Functions," *Scientific Annals of the École Normale Supérieure Series 3*, vol. 33, pp. 223-302, 1916. [CrossRef] [Google Scholar] [Publisher Link]

[11] K.S. Charak, and D. Rochon, "On Factorization of Bicomplex Meromorphic Functions," *Hypercomplex Analysis*, pp. 55-68, 2008. [CrossRef] [Google Scholar] [Publisher Link]

[12] Dominic ROCHON, "A Bicomplex Riemann Zeta Function," *Tokyo Journal of Mathematics*, vol. 27, no. 2, pp. 357-369, 2004. [CrossRef] [Google Scholar] [Publisher Link]

[13] Shahar Nevo, "Applications of Zalcman's Lemma to Q_m-Normal Families," *Analysis*, vol. 21, no. 3, pp. 289-326, 2001. [CrossRef] [Google Scholar] [Publisher Link]

[14] Corrado Segre, "Real Representations of Complex Shapes and Hyperalgebraic Entities," *Mathematical Annals*, vol. 40, pp. 413-467, 1892. [CrossRef] [Google Scholar] [Publisher Link]

[15] Walter Rudin, *Real and Complex Analysis*, 3rd ed., New York, McGraw-Hill, 1974. [Publisher Link]

[16] M.E. Luna-Elizarraras et al., "Bicomplex Numbers and their Elementary Functions," *CUBO (Temuco)*, vol. 14, no. 2, pp. 61-80, 2012. [CrossRef] [Google Scholar] [Publisher Link]
[17] fM.A. Clifford, "Preliminary Sketch of Biquaternions," *Proceedings of the London Mathematical Society*, vol. 1-4, no. 1, pp. 381-395, 1871. [CrossRef] [Google Scholar] [Publisher Link]
[18] Herrn Frobenius, "On Linear Substitutions and Bilinear Forms," *Journal Fur Die Reine Und Angewandte Mathematik (Crelles Journal)*, vol. 84, pp. 1-63, 1878. [CrossRef] [Google Scholar] [Publisher Link]