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Original Article

Complete Strong φ-Metric Spaces

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Abstract *- In this paper, after presenting the concept of the strong φ-metric space, we establish some properties for complete spaces, dense or bounded subsets, and even uniformly continuous functions defined in such spaces. Our results boost (generalize) plenty of pre-existing results in the literature.*

Keywords *- Strong φ-metric, Strong φ-metric space, Completeness, Dense space, Uniformly continuous function.*

1. Introduction

The idea of strong φ-metric was first presented in, inspired by a huge supply of generalizations about the metric function like in [1,2,3,4]. Also, building properties about the strong φ-metric space has been a goal for the authors since then. [5, 6] In this article, we try to list some properties and definitions of the strong φ-metric space in the completeness framework and hope to extend our work in fixed point theory.

Firstly, we introduce some basic definitions of our new metric space.

Definition1.1. [1] The function $d_s: X \times X \to \mathbb{R} \geq 0$, is called a *strong φ-metric* and satisfies the following conditions:

a) $d_s(x, y) = 0 \Leftrightarrow x = y$;

b) $d_s(x, y) = d_s(y, x)$;

c) $d_s(x, z) \leq K d_s(x, y) + d_s(y, z) + \varphi(x, y, z)$, $\forall x, y, z \in X$, and $K \geq 1$, with $\varphi: X \times X \times X \to \mathbb{R} \geq 0$ a function fulfilling:

   i) $\varphi(x, y, z) = 0$ if $x = z$ or $y = z$;

   ii) $\varphi(x, y, z) = \varphi(y, x, z)$;

   iii) $\forall \varepsilon > 0, \exists \delta > 0$ such that $\varphi(x, y, z) < \varepsilon$, whenever $d_s(x, y) < \delta$ or $d_s(y, z) < \delta$, $\forall x, y, z \in X$. The ordered pair (X, d_s) is called a *strong φ-metric space*.

Definition 1.2. Let (X, d_s) be a *strong φ-metric space*. A sequence (x_n) in X is called a *Cauchy sequence* if, for any $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that $d_s(x_m, x_n) < \varepsilon$ for any $m > N, n > N$.

Definition 1.3. A strong φ-metric space (X, d_s) is said to be *complete* if every Cauchy sequence in X converges to a point in X.

Definition 1.4. Let (X, d_s) be a strong φ-metric space and A, a nonempty subset of it. The *diameter of A* is defined as below:

$$diam(A) = sup\{d_s(x, y) | x, y \in A\}.$$

We say that A is bounded if *diam(A)* is finite.

Definition 1.5. Let f be a function from a strong φ-metric space (X, d_s) into a strong φ-metric (Y, p_s). Then the function f is *uniformly continuous* if, for a given $\varepsilon > 0$, there $\exists \delta > 0$ such that for any $x, y \in X$, $d_s(x, y) < \delta$ implies $p(f(x), f(y)) < \varepsilon$.

Definition 1.6. Let f be a function from a strong φ-metric space (X, d_s) into a strong φ-metric (Y, p_s). The function f is said to be an isometry if $d_s(a, b) = p(f(a), f(b))$ for any $a, b \in X$.

Definition 1.7. Completion of a strong φ-metric space (X, d_s) is a pair consisting of a completely strong φ-metric space (X', d'_s) and an isometry $\psi: X \to X'$ such that $\psi[X]$ is dense in X'.

2. Main Results

This paper aims to introduce a new couple of these intermediate properties in the context of the strong φ-metric space.

Theorem 2.1. In a strong φ-metric space, any convergent sequence is a Cauchy sequence.

Proof. Suppose that $(\mathbf{x_n})$ is a sequence that converges to x. For an $\boldsymbol{\varepsilon > 0}$ then there is an $\mathbf{N} \in \mathbb{N}$ such that:

$$\boldsymbol{d_s(x_n, x) < \frac{\varepsilon}{3}, d_s(x_m, x) < \frac{\varepsilon}{3K}} \text{ and } \boldsymbol{\varphi(x_m, x_n, x) < \frac{\varepsilon}{3}} \text{ for all } \boldsymbol{n \geq N}$$

Let $\boldsymbol{m, n} \in \mathbb{N}$ such that $\boldsymbol{n \geq N}$ and $\boldsymbol{m \geq N}$. Then:

$$d_s(x_m, x_n) \leq K d_s(x_m, x) + d_s(x_n, x) + \varphi(x_m, x_n, x) < K \frac{\varepsilon}{3K} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Hence (x_n) is a Cauchy sequence.

Theorem 2.2. A closed subset of a complete strong φ-metric space is a complete subspace.

Proof. Let S, be a closed subspace of a completely strong φ-metric space X. Let (x_n) be a Cauchy sequence in S. Then (x_n) is a Cauchy sequence in X, and hence, it must converge to a point x in X. But then $x \in \bar{S} = S$. Thus, S is complete.

Theorem 2.3. A complete subspace of a strong φ-metric space is a closed subset.

Proof. Let S, be a complete subspace of a strong φ-metric space X. Let $x \in \bar{S}$. Then, there is a sequence (x_n) in S, which converges to $x \in X$. Hence (x_n) is a Cauchy sequence in S. Since S is complete, (x_n) must convergent to some point $y \in S$. By the uniqueness of the limit, we must have $x = y \in S$. Hence $\bar{S} = S$, so S is closed.

Theorem 2.4. A uniformly continuous function maps Cauchy sequences into Cauchy sequences.

Proof. Let $f: (X, d_s) \to (Y, p_s)$ be a uniformly continuous function on strong φ-metric space. Let (x_n) be a Cauchy sequence in X. To see that $\left(f(x_n)\right)$ is a Cauchy sequence, let $\varepsilon > 0$. Then there is a $\delta > 0$ such that:

$$\forall x, y \in X, d_s(x, y) < \delta \Rightarrow p_s\left(f(x), f(y)\right) < \varepsilon.$$

Thus, there exists an $N \in \mathbb{N}$ such that $d_s(x_m, x_n) < \delta$ for any $m, n \geq N$. It follows that $p_s\left(f(x_m), f(x_n)\right) < \varepsilon$ for any $m, n \geq N$. Hence $\left(f(x_m)\right)$ is a Cauchy sequence in Y.

Theorem 2.5. Let $f: (X, d_s) \to (Y, p_s)$ be an isometry. Then, it is injective and uniformly continuous. Furthermore, its inverse $f^{-1}: (f[X], p_s) \to (X, d_s)$ is also an isometry.

Theorem 2.6. Let A be a dense subset of a strong φ-metric space (X, d_s). Let f be a uniformly continuous function (isometry) from A to a strong φ-metric (Y, p_s). Then, there is a unique uniformly continuous function (isometry) $g: X \to Y$, which extends f.

Proof. Let (X, d_s) be a strong φ-metric space and (Y, p_s) a complete strong φ-metric space. Let A be a dense subset of X, and let f be a uniformly continuous $f: A \to Y$.

- Define a function $g: X \to Y$.

For each $x \in X = \bar{A}$, there is a sequence (x_n) in *A*, which converges to *x*. Then (x_n) is a Cauchy sequence in *X*. Thus $\big(f(x_n)\big)$ is a Cauchy sequence in *Y*. Since *Y* is complete, $\big(f(x_n)\big)$ is a convergent sequence. Define:

$$g(x) = \lim_{n\to\propto} f(x_n)$$

for any $x \in X$, where (x_n) is a sequence in *A* that converges to *x*.

- The function g is well-defined, independent of the choice of (x_n).

Let (x_n) and (y_n) be any sequence in *A* which converges to $x \in \bar{A} = X$. Then the sequence $(x_1, y_1, x_2, y_2, \dots, x_n, y_n, \dots)$ must converge to *x*. Hence, the sequence $(f(x_1), f(y_1), f(x_2), f(y_2), \dots, f(x_n), f(y_n), \dots)$converges to some point $z \in Y$. Since $(f(x_1), f(x_2), \dots)$ and $(f(y_1), f(y_2), \dots)$ are its subsequences, they must also converge to *z*. Hence $z = g(x)$ does not depend on the choice of the sequences.

- The function g is an extension of f.

Let $a \in A$ and let a_n= a for each $n \in \mathbb{N}$. Then $(a_n)$is a sequence in *A* which converges to a.

Hence $g(a) = \lim_{n\to\infty} f(a_n) = f(a)$. This shows that g is an extension of f.

- The function g is uniformly continuous on *X*.

Let $\varepsilon > 0$ then there is a $\delta > 0$ such that:

$$\forall a, b \in A, d_s(a, b) < \delta \Rightarrow p_s\big(f(a), f(b)\big) < \tfrac{\varepsilon}{3}.$$

Let $x, y \in X$ be such that $d_s(x, y) < \delta$. Then, there are sequences $(x_n) \to x$ and $(y_n) \to y$. Hence $f(x_n) \to g(x)$ and $f(y_n) \to g(y)$. Choose $N \in \mathbb{N}$ such that:

$$d_s(x_N, x) < \tfrac{\delta - 2d_s(x,y)}{4K} \text{ and } d_s(y_N, y) < \tfrac{\delta - 2d_s(x,y)}{4K} \text{ with } \varphi(x_N, x, y) < \tfrac{\delta}{4} \text{ and } \varphi(x, y, x_N) < \tfrac{\delta}{4}.$$

Also,

$$p_s(f(x_N), g(x)) < \tfrac{\varepsilon}{2K} \text{ and } p_s(f(y_N), g(y)) < \tfrac{\varepsilon}{2K}.$$

Then:

$$d_s(x_N, y_N) \le K d_s(x_N, x) + d_s(y_N, y) + \varphi(x_N, x, y) + \varphi(x_N, x, y) < \frac{\delta}{2} + \frac{\delta}{4} + \frac{\delta}{4} = \delta$$

This leads to: $p_s\big(f(x_N), f(y_N)\big) < \frac{\varepsilon}{3}$, by the uniform continuity of f on *A*.

Hence,

$$p_s\big(g(x), g(y)\big) \le K p_s\big(g(x), f(x_N)\big) + p_s\big(f(x_N), f(y_N)\big) + K p_s\big(f(y_N), g(y)\big) + \varphi\big(g(x), f(x_N), f(y_N)\big) + \varphi\big(f(x_N), f(y_N), g(y)\big) \le K\tfrac{\varepsilon}{2K} + \tfrac{\varepsilon}{3} + K\tfrac{\varepsilon}{2K} + \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} = \varepsilon.$$

With $\varphi\big(g(x), f(x_N), f(y_N)\big) < \frac{\varepsilon}{3}$ and $\varphi\big(f(x_N), f(y_N), g(y)\big) < \frac{\varepsilon}{3}$. This shows that g is uniformly continuous on *X*.

- The function g is unique.

Let g and h be (uniformly) continuous functions on *X*, which extends *f* on a dense subset *A*. To see that $g = h$, let $x \in X$. Then, there is a sequence (x_n) in *A*, which converges to *x*. By continuity of g and h,

$$g(x) = \lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} h(x_n) = h(x).$$

Hence $g = h$.

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