International Journal of Mathematics Trends and Technology
ISSN: 2231-5373/ https://doi.org/10.14445/22315373/IJMTT-V70I11P105
Volume 70 Issue 11, 32-37, November 2024

Original Article

Q*-Normal Spaces in General Topology

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Received: 20 August 2024 Revised: 30 September 2024 Accepted: 17 November 2024 Published: 30 November 2024

***Abstract** - In this paper, using Q^*-closed sets, we introduce a new version of normality called Q^*-normality, which is a weak form of normality. Further utilizing Q^*g-closed sets, we obtain some characterizations of Q^*-normal and normal spaces and also obtain some preservation theorems for Q^*-normal spaces.*

***Keywords** - Q^*-closed, g-closed, Q^*g-closed sets, Q^*-continuous and almost Q^*-continuous functions, Normal, Q^*-normal spaces.*

1. Introduction

In 1968, Zaitsev introduced the notion of quasi normal space [5]. In 1970, Levine [3] initiated the study of closed sets called generalized closed (briefly g-closed) sets to extend many of the most important properties of closed sets to a large family. In 1973, Singal and Singal [4] introduced the concept of mildly normal spaces and obtained their characterizations. 1986 Munshi introduced and studied notions of g-normal and g-regular spaces. In 1990, Lal and Rahman have further studied notions of quasi normal and mildly normal spaces. In 2000, Dontchev and Noiri [1] introduced the notion of πg-closed sets and, using πg-closed sets, obtained a new characterization of quasi normal spaces. In 2010, M. Murugalingam and N. Lalitha introduced the concept of Q^*-closed sets and obtained some properties of Q^*-closed sets. In 2015, P. Padma and S. Udaya kumar introduced the concept of Q^*g-closed sets and obtained some basic properties of Q^*g-closed sets. In 2018, H. Kumar [2] introduced and studied various forms of normal spaces in topological spaces in his Ph. D. Thesis. In 2024, H. Kumar et al. introduced Fg-closed sets and obtained some properties of F-normal spaces in topological spaces in terms of Fg-closed sets. In 2024, H. Kumar further studied Q^*g-closed sets and obtained a result.

2. Preliminaries

Throughout this paper, spaces (X, τ), (Y, σ), and (Z, γ) (or simply X, Y and Z) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a space X. The closure of A and interior of A are denoted by cl(A) and int(A), respectively. A subset A of a topological space (X, τ) is said to be regular open (resp. regular closed) if A = int(cl(A)) (resp. A = cl(int(A)).

2.1. Definition. A subset A of a space (X, τ) is said to be Q^*-closed if int(A) = ϕ and A is closed. The complement of a Q^*-closed set is said to be Q^*-open

2.2. Definition. A subset A of a topological space (X, τ) is said to be
1. Generalized closed (briefly g-closed) if $cl(A) \subset U$ whenever $A \subset U$ and $U \in \tau$.
2. Q^*g-closed if $cl(A) \subset U$ whenever $A \subset U$ and U is Q^*-open in X.

The complement of A is g-closed (resp. Q^*g-closed) set is said to be g-open (resp. Q^*g-open). The family of all Q^*-closed (resp. Q^*-open, Q^*g-closed, Q^*g-open) sets of a space X is denoted by Q^*-C(X) (resp. Q^*-O(X), Q^*g-C(X), Q^*g-O(X)).

2.3. Remark. We have the following implications for the properties of subsets:

$$\begin{array}{ccccccc} & & \text{regular closed} & & & & \\ & & \Downarrow & & & & \\ Q^*\text{-closed} & \Rightarrow & \text{closed} & \Rightarrow & \text{g-closed} & \Rightarrow & Q^*\text{g-closed} \end{array}$$

Where none of the implications is reversible, as can be seen from the following examples:

2.4. Example. Let X = {a, b, c} and τ = {ϕ, X, {a}, {b}, {a, b}}. Then
1. closed sets are ϕ, X, {c}, {a, c}, {b, c}.
2. Q^*-closed sets are ϕ, {c}.
3. g-closed sets are ϕ, X, {c}, {a, c}, {b, c}.
4. Q^*g-closed sets are ϕ, X, {b}, {c}, {a, c}, {b, c}.
5. regular closed sets are ϕ, X, {c}, {a, c}, {b, c}.

In the above example, every g-closed set is Q^*g-closed, but the converse is not true. Then the set A = {b} is Q^*g-closed but not g-closed.

2.5. Example. In R with the usual metric, finite sets are Q^*-closed but not regularly closed. [0, 1] is regular closed but not Q^*-closed. Hence, regular closed and Q^*-closed sets are independent of each other.

2.6. Theorem. For Q^*g-closed sets of a space X, the following properties hold:
(a) Every finite union of Q^*g-closed sets is always a Q^*g-closed.
(a) Every finite intersection of Q^*g-closed sets is always a Q^*g-closed.

2.7. Lemma. If A be a subset of X, then
(a) Q^*-cl(X – A) = X – Q^*-int(A).
(b) Q^*-int(X – A) = X – Q^*-cl(A).

2.8. Theorem. A subset A of a space X is Q^*g-open iff F $\subset$ int(A) whenever F is Q^*-closed and F $\subset$ A.

Proof. Let F be Q^*-closed set such that F $\subset$ A. Since X – A is Q^*g-closed and X – A $\subset$ X – F where F $\subset$ int(A). Conversely, Let F $\subset$ int(A) where F is Q^*-closed and F $\subset$ A. Since F $\subset$ A and X – F is Q^*-open, cl(X – A) = X – int(A) $\subset$ X – F. Therefore, A is Q^*g-open.

3. Q*- Normal Spaces

3.1. Definition. A space X is said to be **Q^*-normal** if for every pair of disjoint Q^*-closed subsets A, B of X, there exist disjoint open sets U, V of X such that A $\subset$ U and B $\subset$ V.

3.2. Definition. A space X is said to be **g-normal** [**5**] if, for every pair of disjoint g-closed subsets A, B of X, there exist disjoint open sets U, V of X such that A $\subset$ U and B $\subset$ V.

3.3. Definition. A space X is said to be **quasi normal** [**11**] if, for every pair of disjoint π-closed subsets A, B of X, there exist disjoint open sets U, V of X such that A $\subset$ U and B $\subset$ V.

3.4. Definition. A space X is said to be **mildly normal [10]** if, for every pair of disjoint regular closed subsets A, B of X, there exist disjoint open sets U, V of X such that A $\subset$ U and B $\subset$ V.

By the definitions stated above, we have the following diagram:

g-normal $\Rightarrow$ normal $\Rightarrow$ Q^*-normal
$\Downarrow$
quasi normal $\Rightarrow$ mildly normal

Where none of the implications is reversible, as can be seen from the following examples:

3.5. Example. Let X = {a, b, c, d} and $\Im$ = {ϕ, {a}, {c}, {a, c}, {a, b, d}, {b, c, d}, X}. Then the space X is normal as well as Q^*-normal.

3.6. Example. In R with the usual metric, finite sets are Q^*-closed but not regularly closed. [0, 1] is regular closed but not Q^*-closed. Hence, regular closed and Q^*-closed sets are independent of each other.

From the above example, we can say that mildly normal and Q^*-normal spaces are independent of each other.

3.7. Theorem. For a space topological X, the following properties are equivalent:

a) X is Q^*-normal.

b) For every pair of Q^*-open subsets U and V of X whose union is X, closed subsets G and H of X exist such that $G \subset U$, $H \subset V$ and $G \cup H = X$.

c) For any Q^*-closed set A and every Q^*-open set B in X such that $A \subset B$, there exists an open subset U of X such that $A \subset U \subset cl(U) \subset B$.

d) For every pair of disjoint Q^*-closed subsets A and B of X, there exist open subsets U and V of X such that $A \subset U$, $B \subset V$ and $cl(U) \cap cl(V) = \phi$.

Proof. (a) $\Rightarrow$ (b), (b) $\Rightarrow$ (c), (c) $\Rightarrow$ (d) and (d) $\Rightarrow$ (a).

(a) $\Rightarrow$ (b). Let U and V be any Q^*-open subsets of a Q^*-normal space X such that $U \cup V = X$. Then, $X - U$ and $X - V$ are disjoint Q^*-closed subsets of X. By Q^*-normality of X, there exist disjoint open subsets U_1 and V_1 of X such that $X - U \subset U_1$ and $X - V \subset V_1$. Let $G = X - U_1$ and $H = X - V_1$. Then, G and H are closed subsets of X such that $G \subset U$, $H \subset V$, and $G \cup H = X$.

(b) $\Rightarrow$ (c). Let A be a Q^*-closed and B is a Q^*-open subset of X such that $A \subset B$. Then, $X - A$ and B are Q^*-open subsets of X such that $(X - A) \cup B = X$. Then, by part (b), there exist closed subsets G and H of X such that $G \subset (X - A)$, $H \subset B$ and $G \cup H = X$. Then, $A \subset (X - G)$, $(X - B) \subset (X - H)$ and $(X - G) \cap (X - H) = \phi$. Let $U = X - G$ and $V = X - H$. Then U and V are disjoint open sets such that $A \subset U \subset X - V \subset B$. Since $X - V$ is closed, then we have $cl(U) \subset (X - V)$.
Thus, $A \subset U \subset cl(U) \subset B$.

(c) $\Rightarrow$ (d). Let A and B be any disjoint Q^*-closed subset of X. Then $A \subset X - B$, where $X - B$ is Q^*-open. By the part (c), there exists an open subset U of X such that $A \subset U \subset cl(U) \subset X - B$. Let $V = X - cl(U)$. Then, V is an open subset of X. Thus, we obtain $A \subset U$, $B \subset V$ and $cl(U) \cap cl(V) = \phi$.

(d) $\Rightarrow$ (a). It is obvious.

3.8. Theorem. For a topological space X, the following properties are equivalent:

(a) X is Q^*-normal.

(b) for any disjoint H, K $\in Q^*$-C(X), there exist disjoint Q^*g-open sets U, V such that $H \subset U$ and $K \subset V$.

(c) for any H $\in Q^*$-C(X) and any V $\in Q^*$-O(X) containing H, there exists a Q^*g-open set U of X such that, $H \subset U \subset Q^*g\text{-}cl(U) \subset V$.

(d) for any H $\in Q^*$-C(X) and any V $\in Q^*$-O(X) containing H, there exists an open set U of X such that $H \subset U \subset cl(U) \subset V$.

(e) for any disjoint H, K $\in Q^*$-C(X), there exist disjoint regular open sets U, V such that $H \subset U$ and $K \subset V$.

Proof. (a) $\Rightarrow$ (b): Since every open set is Q^*g-open, the proof is obvious.

(b) $\Rightarrow$ (c): Let H $\in Q^*$-C(X) and V be any Q^*-open set containing H. Then H, $X - V \in Q^*$-C(X) and $H \cap (X - V) = \phi$. By (b), there exist Q^*g-open sets U, G such that $H \subset U$, $X - V \subset G$ and $U \cap G = \phi$. Therefore, we have $H \subset U \subset (X - G) \subset V$. Since U is Q^*g-open and $X - G$ is Q^*g-closed, we obtain $H \subset U \subset Q^*g\text{-}cl(U) \subset (X - G) \subset V$.

(c) $\Rightarrow$ (d): Let H $\in Q^*$-C(X) and $H \subset V \in Q^*$-O(X). By (c), there exists a Q^*g-open set U_0 of X such that, $H \subset U_0 \subset Q^*g\text{-}cl(U_0) \subset V$. Since $Q^*g\text{-}cl(U_0)$ is Q^*g-closed and V $\in Q^*$-O(X), $cl(Q^*g\text{-}cl(U_0)) \subset V$. Put $int(U_0) = U$, then U is open and $H \subset U \subset cl(U) \subset V$.

(d) $\Rightarrow$ (e): Let H, K be disjoint Q^*-closed sets of X. Then $H \subset (X - K) \in Q^*$-O(X) and by (d) there exists an open set U_0 such that $H \subset U_0 \subset cl(U_0) \subset (X - K)$. Therefore, $V_0 = (X - cl(U_0))$ is an open set such that $H \subset U_0$, $K \subset V_0$ and $U_0 \cap V_0 = \phi$. Moreover, put $U = int(cl(U_0))$ and $V = int(cl(V_0))$, then U, V are regular open sets such that $H \subset U$, $K \subset V$ and $U \cap V = \phi$.

(e) $\Rightarrow$ (a): This is obvious.

3.9. Theorem. For a topological space X, the following properties are equivalent:

(a) X is normal.

(b) for any disjoint closed sets A and B, there exist disjoint Q^*g-open sets U and V such that $A \subset U$ and $B \subset V$.

(c) for any closed set A and any open set V containing A, there exists a Q^*g-open set U of X such that $A \subset U \subset cl(U) \subset V$.

Proof. (a) $\Rightarrow$ (b): This is obvious since every open set is Q^*g-open.

(b) $\Rightarrow$ (c): Let A be a closed set and V be any open set containing A. Then, A and (X – V) are disjoint closed sets. There exist disjoint Q^*g-open sets U and W such that $A \subset U$ and $(X - V) \subset W$. Since X – V is closed, we have $(X - V) \subset int(W)$ and $U \cap int(W) = \phi$. Therefore, we obtain $cl(U) \cap int(W) = \phi$ and hence $A \subset U \subset cl(U) \subset (X - int(W)) \subset V$.

(c) $\Rightarrow$ (a): Let A, B be disjoint closed sets of X. Then $A \subset (X - B)$ and (X – B) is open. By (c), there exists a Q^*g-open set G of X such that $A \subset G \subset cl(G) \subset (X - B)$. Since A is closed, we have $A \subset int(G)$. Put U = int(G) and V = (X – cl(G)). Then U and V are disjoint open sets of X such that $A \subset U$ and $B \subset V$. Therefore, X is normal.

3.10. Proposition. Let f: $X \to Y$ be a function, then:

(a) The image of open subset under an open continuous function is open.

(b) The inverse image of Q^*-open (resp. Q^*-closed) subset under an open continuous function is Q^*-open (resp. Q^*-closed) subset.

(c) The image of a closed subset under an open and a closed continuous surjective function is open.

3.11. Theorem. The image of a Q^*-normal space under an open continuous injective function is a Q^*-normal.

Proof. Let X be a Q^*-normal space, and let f: $X \to Y$ be an open continuous injective function. We need to show that f(X) is a Q^*-normal. Let A and B be any two disjoint Q^*-closed sets in f(X). Since the inverse image of a Q^*-closed set under an open continuous function is a Q^*-closed. Then, $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint Q^*-closed sets in X. By Q^*-normality of X, there exist open subsets U and V of X such that $f^{-1}(A) \subset U$, $f^{-1}(B) \subset V$ and $U \cap V = \phi$. Since f is an open continuous injective function, we have $A \subset f(U)$, $B \subset f(V)$ and $f(U) \cap f(V) = \phi$. By Proposition 3.10, we obtain f(U) and f(V) are disjoint open sets in f(X) such that $A \subset f(U)$ and $B \subset f(V)$. Hence, f(X) is Q^*-normal.

3.12. Theorem. For a topological space X, the following properties are equivalent:

(a) X is Q^*-normal.

(b) For any disjoint Q^*-closed sets H and K, there exist disjoint g-open sets U and V such that $H \subset U$ and $K \subset V$.

(c) For any disjoint Q^*-closed sets H and K, there exist disjoint Q^*g-open sets U and V such that $H \subset U$ and $K \subset V$.

(d) For any Q^*-closed set H and any Q^*-open set V containing H, there exists an g-open set U of X such that, $H \subset U \subset cl(U) \subset V$.

(e) For any Q^*-closed set H and any Q^*-open set V containing H, there exists a Q^*g-open set U of X such that, $H \subset U \subset cl(U) \subset V$.

Proof. (a) $\Rightarrow$ (b), (b) $\Rightarrow$ (c), (c) $\Rightarrow$ (d), (d) $\Rightarrow$ (e) and (e) $\Rightarrow$ (a).

(a) (b). Let X be Q^*-normal space. Let H, K be disjoint Q^*-closed sets of X. By assumption, there exist disjoint open sets U, V such that $H \subset U$ and $K \subset V$. Since every open set is g-open, so U and V are g-open sets such that $H \subset U$ and $K \subset V$.

(b) (c). Let H and K be two disjoint Q^*-closed sets. By assumption, there exist disjoint g-open sets U and V such that $H \subset U$ and $K \subset V$. Since every g-open set is Q^*g-open, so U and V are Q^*g-open sets such that $H \subset U$ and $K \subset V$.

(c) (d). Let H be any Q^*-closed set, and V be any Q^*-open set containing H. By assumption, there exist disjoint Q^*g-open sets U and W such that $H \subset U$ and $X - V \subset W$. By Theorem 2.8, we get $X - V \subset int(W)$ and $cl(U) \cap int(W) = \phi$. Hence $H \subset U \subset cl(U) \subset X - int(W) \subset V$.

(d) (e). Let H be any Q^*-closed set, and V be any Q^*-open set containing H. By assumption, there exists g-open set U of X such that $H \subset U \subset cl(U) \subset V$. Since every g-open set is Q^*g-open, there exists Q^*g-open set U of X such that, $H \subset U \subset cl(U) \subset V$.

(e) (a). Let H, K be any two disjoint Q^*-closed sets of X. Then $H \subset X - K$ and $X - K$ is Q^*-open. By assumption, there exists Q^*g-open set G of X such that $H \subset G \subset cl(G) \subset X - K$. Put $U = int(G)$, $V = X - cl(G)$. Then U and V are disjoint open sets of X such that $H \subset U$ and $K \subset V$.

4. Some Related Functions with Q*- Normal Spaces

4.1. Definition. A function $f: X \to Y$ is said to be
(1) almost Q^*g-continuous if for any regular open set V of Y, $f^{-1}(V) \in Q^*$g-O(X).
(2) almost Q^*g-closed if for any regular closed set F of X, $f(F) \in Q^*$g-C(Y).

4.2. Definition. A function $f: X \to Y$ is said to be
(1) Q^*-irresolute (resp. Q^*- continuous) if for any Q^*-open (resp. open) set V of Y, $f^{-1}(V)$ is Q^*-open in X.
(2) pre-Q^*-closed (resp. Q^*- closed) if for any Q^*-closed (resp. closed) set F of X, f(F) is Q^*-closed in Y.

4.3. Theorem. A function $f: X \to Y$ is an almost Q^*g-closed surjection if and only if for each subset S of Y and each regular open set U containing $f^{-1}(S)$, there exists a Q^*g-open set V such that $S \subset V$ and $f^{-1}(V) \subset U$.

Proof. Necessity. Suppose that f is almost Q^*g-closed. Let S be a subset of Y and U be a regular open set of X containing $f^{-1}(S)$. Put $V = Y - f(X - U)$, then V is a Q^*g-open set of Y such that $S \subset V$ and $f^{-1}(V) \subset U$.

Sufficiency. Let F be any regular closed set of X. Then $f^{-1}(Y - f(F)) \subset (X - F)$ and $X - F$ is regular open. There exists a Q^*g-open set V of Y such that $(Y - f(F)) \subset V$ and $f^{-1}(V) \subset (X - F)$. Therefore, we have $f(F) \supset (Y - V)$ and $F \subset f^{-1}(Y - V)$. Hence, we obtain $f(F) = Y - V$ and f(F) are Q^*g-closed in Y. This shows that f is almost Q^*g-closed.

4.4. Theorem: If $f: X \to Y$ is an almost Q^*g-closed Q^*-irresolute (resp. Q^*-continuous) surjection and X is Q^*-normal, then Y is Q^*-normal (resp. normal).

Proof. Let A and B be any disjoint Q^*-closed (resp. closed) sets of Y. Then $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint Q^*-closed sets of X. Since X is Q^*-normal, there exist disjoint open sets U and V of X such that $f^{-1}(A) \subset U$ and $f^{-1}(B) \subset V$. Put $G = int(cl(U))$ and $H = int(cl(V))$, then G and H are disjoint regular open sets of X such that $f^{-1}(A) \subset G$ and $f^{-1}(B) \subset H$.

By Theorem 4.3, there exist Q^*g-open sets K and L of Y such that $A \subset K$, $B \subset L$. $f^{-1}(K) \subset G$ and $f^{-1}(L) \subset H$. Since G and H are disjoint, so K and L are also disjoint. It follows from Theorem 3.8 (resp. Theorem 3.9) that Y is Q^*-normal (resp. normal).

4.5. Theorem. If $f: X \to Y$ is a continuous almost Q^*g-closed surjection, and X is a normal space, then Y is normal.

Proof. The proof is similar to that of Theorem 4.4.

4.6. Theorem. If $f: X \to Y$ is an almost Q^*g-continuous pre-Q^*-closed (resp. Q^*-closed) injection, Y is Q*-normal, and X is Q^*-normal (resp. normal).

Proof. Let H and K be disjoint Q^*-closed (resp. closed) sets of X. Since f is a pre-Q^*-closed (resp. Q^*-closed) injection, f(H) and f(K) are disjoint Q^*-closed sets of Y. Since Y is Q^*-normal, there exist disjoint open sets P and Q such that $f(H) \subset P$ and $f(K) \subset Q$. Now, put $U = int(cl(P))$ and $V = int(cl(Q))$, then U and V are disjoint regular open sets such that $f(H) \subset U$ and $f(K) \subset V$.

Since f is almost Q^*g-continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint Q^*g-open sets such that $H \subset f^{-1}(U)$ and, $K \subset f^{-1}(V)$. It follows from Theorem 3.8 (resp. Theorem 3.9) that X is Q^*-normal (resp. normal).

5. Conclusion

In this paper, we introduce and study a new class of spaces, namely Q*- normal spaces, using Q*- open sets. The relationship among normality, Q*- normality, which is a weaker form of normality. Also, we obtained some characterization of Q*- normal spaces and properties of the forms of Q*g- closed functions. Of course, the entire content will be a successful tool for the researchers to find a way to obtain the results in the context of such types of normal spaces. This idea can be extended to topologically ordered, bitopological ordered, fuzzy topological spaces, etc.

Conflicts of Interest

We certify that this work is original, has never been published before, and is not being considered for publication elsewhere. This publication is free from any conflicts of interest. As the Corresponding Author, I certify that each listed author has read the paper and given their approval for submission.

References

[1] J. Dontchev, and T. Noiri, "Quasi-Normal Spaces and πg-Closed Sets," *Acta Mathematica Hungarica*, vol. 89, no. 3, pp. 211-219, 2000. [CrossRef] [Google Scholar] [Publisher Link]

[2] Hemant Kumar, "*Some Weaker forms of Normal Spaces in Topological Spaces*," Ph.D. Thesis, C. C. S. University, Meerut, 2018. [Google Scholar] [Publisher Link]

[3] Norman Levine, "Generalized Closed Sets in Topology," *Rendiconti del Circolo Matematico di Palermo*, vol. 19, pp. 89-96, 1970. [CrossRef] [Google Scholar] [Publisher Link]

[4] M.K. Singal, and Asha Rani Singal, "Mildly Normal Spaces," *Kyungpook Mathematical Journal*, vol. 13, no. 1, pp. 27-31, 1973. [Google Scholar] [Publisher Link]

[5] V. Zaitsev, "On Certain Classes of Topological Spaces and Their Biocompactifications," *Doklady Akademii Nauk: SSSR*, vol. 178, no. 4, pp. 778-779, 1968. [Google Scholar]