International Journal of Mathematics Trends and Technology
ISSN: 2231-5373/ https://doi.org/10.14445/22315373/IJMTT-V70I11P106

Volume 70 Issue 11, 38-44, November 2024

Original Article

A Brief Study of Operators of Generalized Fractional Integration

Blessy Jayaron Jose

Department of Information Technology, University of Technology and Applied Sciences, Al Mussannah, Sultanate of Oman.

Corresponding Author : joannajayaron@gmail.com

Received: 25 September 2024 Revised: 01 November 2024 Accepted: 18 November 2024 Published: 30 November 2024

Abstract *- Fractional calculus broadens the scope of conventional calculus by introducing derivatives and integrals of non-whole number orders. This mathematical field expands the ideas of differentiation and integration beyond integer values, offering versatile methods for describing intricate processes across diverse scientific and engineering disciplines. The abstract explores the fundamental definitions, properties, and applications of fractional operators, including the Riemann-Liouville, Holmgren, and Grünwald-Letnikov approaches given by different mathematicians like the Mellin transform which have established connections, while a few of them explored the relationships of the Hankel transform. In this survey, ideas from Kiryakov. V was taken especially related to a more unusual instance of kernels that were special functions like the Gauss and generalized hypergeometric functions, including arbitrary G- and H-functions, kernels and to create a theory of the associated GFC with several applications. Additionally, five more authors brought attention to their respective contributions in this area. In this survey, the Riemann-Liouville fractional integral is simplified to the Weyl integral, and a brief study is done on the hypergeometric functions of one and more variables, such as the generalized hypergeometric function contributed by a few mathematicians.*

Keywords *- Gamma-function, Fractional-order differentiation and integration, Riemann-Liouville fractional integrals, Weyl integrals, Saigo operators.*

1. Introduction

Mathematics is well renowned for the apparent multiplicity of thick and tools required to learn about the subject matter in special functions theory [1]. Because of their importance in mathematical analysis, functional analysis, geometry, physics, and other applications, some mathematical functions have names and notations devised for them. These functions are known as fractional operators. [2]. It could be Euler who first discussed several common fractional operators in 1720. He defined the Gamma function as a continuation of the factorial and handled the Bessel and elliptic functions. Thus, ideas such as the incomplete Gamma- and Beta-functions, the Error functions, the Airy, Whittaker, and others, as well as Special Functions of Mathematical Physics" or "Named Functions", were published. Other ideas included the Bessel and cylindrical functions, the Gauss, Kummer, Tricomi, confluent and generalized hypergeometric functions, and the classical orthogonal polynomials (such as Laguerre, Jacobi, Gegenbauer, Legendre, Tchebisheff, Hermite, etc.) [3]. In this survey, we simply briefly mention a few of them. The potency of the special functions is completely unknown to 2 out of 40 mathematicians. A paper with a Bessel function or Legendre polynomial instantly goes on to the following article. Hopefully, these lectures will demonstrate the value of hypergeometric functions. There are very few known facts about them, but this few knowledge can be quite helpful in a variety of circumstances. In a study done by Mathai and Habould [24], however, the discussion of fractional integrals and derivatives indicates a growing area of interest where more research could be conducted to apply these concepts to complex systems in various scientific domains. This paper, however, discusses the various works done in fractional integration by various authors and paves the way for improvement in the research.

2. Literature Review

How fractional calculus was created?

In a study conducted by Hilfer. R [27], he thoroughly researched the initial value of Fractional Calculus and found its values for $t \to 0$While the fractional initial value problem is well defined and the solution finite at all times, its values for $t \to 0$ are divergent. Gutierrez(et al.) concluded that fractional calculus is the area of mathematics in which arbitrary order integrals and derivatives are studied and used in various applications. With the development of the classical ones came the notion of fractional operators. [4]. Fractional calculus has a plethora of applications these days. It is safe to say that fractional calculus techniques and tools are employed in almost all modern engineering and research (David et al.). Applications for this include, but are not limited to, rheology, viscoelasticity, acoustics, optics, chemical and statistical physics, robotics, control theory, electrical and mechanical engineering, bioengineering, etc. [5].

As a matter of fact, real-world occurrences are generally described by fractional order systems. Because these new fractional-order models are often more accurate than integer-order ones, the fractional-order model has more degrees of freedom than the corresponding classical one—FC applications have mainly been successful. Research by Ding and colleagues[6] found that fractional operators effectively model widespread and non-localized effects common in technical systems and natural phenomena. This is because fractional operators consider the entire historical context of the studied activity. As a result, fractional calculus provides excellent methods for describing how materials and processes retain information from past states and exhibit hereditary traits.

In a study conducted by SØRENSEN (1999) [7], research on Niels Henrik Abel's theory of equations was done in which the distinction of submitting the initial application was made, which he did in 1823. Abel applied fractional calculus techniques to resolve an integral equation from formulating the tautochrone problem. In a letter written on September 30, 1695, L'Hôpital inquired about a particular notation Leibnitz had employed in his works to represent the nth derivative of a linear function.

$f(x) = x, \frac{D^n x}{Dx^n}$ [8].What will happen if $n = \frac{1}{2}$? was the query L'Hopital's put to Leibnitz. Leibnitz's response was an apparent paradox from which sometimes insightful conclusions may be made [9]. As a result, fractional calculus was born on that day, making it appropriate to celebrate its birthday. Numerous well-known mathematicians, including Fourier, Euler, and Laplace, contributed to fractional calculus's development [10]. Many mathematicians introduced the concept of fractional calculus, covering fractional-order differentiation and integration, and developed their symbolic representations for these operations. Most fractional calculus mathematical theory was established in the 20th century. The Riemann-Liouville [27]and Grunwald-Letnikov definitions are the most renowned in fractional calculus, though they remain relatively unknown outside the field. Caputo [28] modified the traditional Riemann-Liouville fractional derivative to enable solving fractional equations using integer-order initial conditions. Later, in 1996. Kolowankar [29] modified the Riemann-Liouville fractional derivative approach to study fractal functions that are not differentiable at any point.

This groundbreaking work originated in the 19th century, involving contributions from numerous eminent mathematicians. A chronological list of mathematicians who contributed to the field in the 19th century were Euler, Laplace (1812), Fourier (1822), Abel (1823-1826), Liouville (1832-1873), Riemann (1847), Holmgren (1865-67), Grunwald (1867-1872), and Letnikov (1868-1872). However, an effort has recently been made to characterize fractional derivatives as localized operations directly connected to fractal science concepts [12]. Perhaps the calculus of the twenty-first century is fractional calculus. In this paper, fractional integral operators have been identified, and a brief discussion on GAUSS HYPERGEOMETRIC FUNCTION is included in a study conducted by Kiryakova. V [34], the research proposes a unified way to understand and work with Special Functions of Fractional Calculus (SFs of FC). These functions are increasingly important because they help solve fractional order differential/integral equations in many fields like physics, engineering, biology, economics, etc.

3. Methodology

The methodology involves a brief survey of fractional operators

3.1. Fractional Integral Operators

Fractional integration is usually generalized as an integration that is repeated several times. Studies in this area concentrate on the integral variables and the Srivastava and Daou functions. The integral

$$R_{0,y}^{\varepsilon} f = {}_0D_y^{-\varepsilon} f(x) = \frac{1}{\Gamma(\varepsilon)} \int_0^x (y-t)^{\varepsilon-1} f(t)dt \ldots\ldots\ldots\ldots (1.2.1)$$

$= \frac{y^{\varepsilon}}{\Gamma(\varepsilon)} \int_0^1 (1-v)^{\varepsilon-1} f(xv)dv, R(\varepsilon) > 0 \ldots\ldots\ldots (1.2.2)$ is called the Riemann-Liouville fractional integral of order ε. The fractional (1.2.1) is convergent for a wide class of functions $f(t)$, if $R(\varepsilon) > 0$. The upper limit y may be real and complex; in the latter case, the integration path is the segment $t = yr, 0 \leq r \leq 1$. The integral

$$W_{y,\infty}^{\varepsilon} f = {}_yW_{\infty}^{-\varepsilon} f(x) = \frac{1}{\Gamma(\varepsilon)} \int_x^{\infty} (t-y)^{\varepsilon-1} f(t)dt \ldots .1.2.3$$

$$= \frac{y^{\varepsilon}}{\Gamma(\varepsilon)} \int_{1}^{\infty} (v-1)^{\varepsilon-1} f(yv)\, dv, \qquad R(\varepsilon) > 0.....1.2.4$$

is called the Weyl fractional integral of degree ε.

In a 1949 study, Reiz [30]extended fractional integral concepts to functions with multiple complex variables. In 1950, Baker and Copson [31] then used this work to tackle partial differential equations. Subsequently, numerous scientists have investigated how fractional integration operators relate to other types of integral transforms. During the 1940s, Doetsch and Widder[32] separately explored how fractional integrals relate to the Laplace transform. Kober (1940, 1941)[13] examined the connection between fractional integrals and the Mellin transform, while Erdélyi[14] and Kober (1940)[13] focused on their link to the Hankel transform. Zygmund (1959) [15]and Herrera (1952)[16]incorporated fractional integrals into Fourier series theory. In the early 1970s, Bora, Kalla, and Saxena [17], followed by Bora and Saxena [18], investigated the relationships between Riemann-Liouville fractional integrals, Weyl integrals (1917)[19], and other mathematical concepts.

The operators are characterized through the following equations :

$$R^{m}_{\eta,\varepsilon}[f(y)] = \frac{my^{m(\eta+\varepsilon)}}{\Gamma(\varepsilon)} \int_0^y (y^m - t^m)^{\varepsilon-1} t^{m(\varepsilon+1)-1} f(t)dt, \varepsilon > 0 \dots\dots\dots (1.2.5)$$

$$= \frac{y^{1-n(\eta+\varepsilon-1)}}{\Gamma(1+\eta)} \frac{d}{dx} \int_0^y (x^p - t^p)^{\eta} t^{p(\varepsilon+1)-1} f(t)dt, -1 < \eta < 0 \dots\dots\dots (1.2.6)$$

And

$$K^{n}_{\sigma,\eta}[f(y)] = \frac{py^{p\sigma}}{\Gamma(\eta)} \int_x^{\infty} (t^p - x^p)^{\eta-1} t^{n(1-\eta-\sigma)-1} f(t)dt, \eta > 0 \dots\dots\dots (1.2.7)$$

$$= \frac{x^{m(\delta-1)+1}}{\Gamma(1+\eta)} \frac{d}{dx} \int_x^{\infty} (t^m - x^m)^{\eta} t^{m(1-\eta-\delta)-1} f(t)dt, -1 < \eta < 0 \dots\dots (1.2.8)$$ ….continue

Provided that

$$q \leq 1, q^{-1} + r^{-1} = 1, p = 0, Re(\eta) > -r^{-1}, Re(\sigma) > -q^{-1},$$

$$f(x) \in L_q(0,\infty) \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (1.2.9)$$

Conditions (1.2.9) guarantee that both $R[f(y)]$ and $K[f(y)]$ exist and also that both belong to $L_q(0,\infty)$

Operations (1.2.5) and (1.2.7) are present for positive values , while operations (1.2.6) and (1.2.8) are present for negative values η. Saigo and Maeda [1996] [20] proposed that generalized fractional integration operators of arbitrary order can be expressed using a kernel that includes the Appell function F_3 in the given kernel.

$$I_{0,y}^{\varepsilon,\, \varepsilon',\, \eta,\, \eta',\, \gamma} f = \frac{x^{-\varepsilon}}{\Gamma(\gamma)} \int_0^y (y-t)^{\gamma-1} t^{-\varepsilon'} F_3(\varepsilon, \varepsilon', \eta, \eta'; \gamma; 1 - \frac{t}{y}, 1 - \frac{y}{t}) f(t)dt,$$

$$= \frac{d^k}{dx^k} I_{0,x}^{\varepsilon,\varepsilon',\eta+k,\eta',\gamma+k} f, Re(\gamma) \leqq 0; k = [-Re(\gamma)] + 1.$$

And

$$J_{y,\infty}^{\eta,\eta'',\varepsilon,\varepsilon',\gamma} f = \frac{x^{-\varepsilon'}}{\Gamma(\gamma)} \int_y^{\infty} (t-y)^{\gamma-1} t^{-\varepsilon} F_3(\varepsilon, \varepsilon', \eta, \eta'; \gamma; 1 - \frac{y}{t}, 1 - \frac{t}{y}) f(t)dt,$$

$$Re(\gamma) > 0; \dots\dots\dots\dots (1.2.12)$$

$$= (-1)^k \frac{d^k}{dx^k} J_{y,\infty}^{\eta,\eta'',\varepsilon,\varepsilon'+k,\gamma+k} f, Re(\gamma) \leqq 0; k = [-Re(\gamma)] + 1. \dots (1.2.13)$$

For $\varepsilon' = 0$, When certain conditions are met, the operators described earlier can be reduced to the Saigo operators [1978] [21], which are defined as:

$$I_{0,y}^{\varepsilon,\eta,\varpi} f = \frac{x^{-\varepsilon-\eta}}{\Gamma(\varepsilon)} \int_0^x (x-t)^{\varepsilon-1} 2F_1(\varepsilon+\eta, -\varpi; \varepsilon; 1-\frac{t}{y}) f(t)dt,$$
$$Re(\varepsilon) > 0 \ldots\ldots\ldots\ldots (1.2.14)$$

$$= \frac{d^k}{dy^k} I_{0,y}^{\eta+k,\varepsilon-k,\omega-k} f, Re(\varepsilon) \leqq 0; k = [Re(-\varepsilon')] + 1. \ldots\ldots.(1.2.15)$$

$$J_{z,\infty}^{\eta,\varepsilon,\omega} f = \frac{1}{\Gamma(\eta)} \int_x^\infty (t-y)^{\eta-1} t^{-\eta-\varepsilon} 2F_1(\eta+\varepsilon, -\alpha'; \eta; 1-\frac{y}{t}) f(t)dt,$$
$$Re(\varepsilon) > 0; \ldots\ldots\ldots (1.2.16)$$

$$= (-1)^k \frac{d^k}{dx^k} J_{,\infty}^{\eta+k,\varepsilon-k,\omega} f, Re(\varepsilon) \leqq 0; k = [Re(-\varepsilon)] + 1 \ldots\ldots\ldots (1.2.17)$$

For specific parameter values, when $\alpha = -\beta$, equations (1.2.14) and (1.2.15) simplify to the Riemann-Liouville and Weyl fractional operators, respectively, as shown:

$$R_{0,x}^{\varepsilon} f = I_{0,y}^{\varepsilon,-\varepsilon,\omega} f = \frac{1}{\Gamma(\varepsilon)} \int_0^x (y-t)^{\varepsilon-1} f(t)dt \ldots\ldots\ldots\ldots\ldots(1.2.18)$$

and

$$W_{y,\infty}^{\varepsilon} f = J_{y,\infty}^{\varepsilon,-\mu,\omega} f = \frac{1}{\Gamma(\varepsilon)} \int_x^\infty (t-y)^{\varepsilon-1} f(t)dt$$

When $\alpha = 0$, (1.2.16) and (1.2.17) reduce to Erdélyi - Kober operators as given below:

$$E_{0,y}^{\varepsilon,\omega} f = I_{0,x}^{\varepsilon,0,\omega} f = \frac{x^{-\varepsilon-\omega}}{\Gamma(\varepsilon)} \int_0^x (y-t)^{\varepsilon-1} t^{\omega} f(t)dt \ldots\ldots\ldots (1.2.20)$$

and

$$K_{y,\infty}^{\varepsilon,\omega} f = J_{y,\infty}^{\varepsilon,0,\omega} f = \frac{x^{\omega}}{\Gamma(\varepsilon)} \int_x^\infty (t-x)^{\varepsilon-1} t^{-\varepsilon-\omega} f(t)dt \ldots\ldots\ldots (1.2.21)$$

3.2. Fractional Integration Operators Involving the Gauss Hypergeometric Function

Fractional integration operators given by Erdélyi and Kober were generalized by Saxena [22] in the year 1967 in the following form:

$$I[f(x)] = I[\eta, \varepsilon, \gamma, p: f(x)] = \frac{y^{-\gamma-1}}{\Gamma(1-\eta)} \int_0^x 2F_1(\eta, \varepsilon+y; \varepsilon; \frac{t}{y})\, t^{\gamma} f(t)dt \ldots\ldots\ldots\ldots\ldots(1.3.1)$$

$$R[f(x)] = I[\eta, \varepsilon, \sigma, p: f(x)] = \frac{y^{\delta}}{\Gamma(1-\beta)} \int_x^\infty 2F_1(\eta, \varepsilon+p; \varepsilon; \frac{y}{t})\, t^{-\sigma-1} f(t)dt \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(1.3.2)$$

where $F(\eta, \varepsilon; \gamma; y)$ denotes the ordinary hypergeometric function and $\eta, \varepsilon, \gamma$ are complex parameters, the operators in (1.3.1) and (1.3.2) exist, provided that

$$f(y) \in L_q(0,\infty), \quad Re(1-\eta) > p, Re(\gamma) > -\frac{1}{r}, Re(\delta) > -\frac{1}{q}; \frac{1}{q} + \frac{1}{r} = 1,$$

$$\mathrm{B} \neq 0, -1, -2, \ldots, q \geq 1, p \in \mathbb{N}_0$$

Kalla and Saxena [23] further developed the concept with the operators defined in equations (1.3.1) and (1.3.2) by introducing more generalized versions through integral equations in their work published in 1969 and 1979.

$$I[f(y)] = I[\eta, \varepsilon, \gamma; p, \nu, \omega, b; f(y)]$$

$$= \frac{\nu y^{-\omega-1}}{\Gamma(1-\eta)} \int_0^x 2F_1(\eta, \varepsilon + p; \gamma; b\frac{t^\nu}{y^\nu})\, t^\omega f(t)dt \qquad (1.3.3)$$

and

$$R[f(x)] = I[\eta, \varepsilon, \gamma; p, \nu, \delta, b; f(y)]$$
$$= \frac{\nu y^{\sigma}}{\Gamma(1-\eta)} \int_x^\infty 2F_1(\eta, \varepsilon + p; \gamma; \frac{by^\nu}{t^\nu})\, t^{-\sigma-1} f(t)dt \qquad (1.3.4)$$

The operators defined in equations (1.3.3) and (1.3.4) are valid under the following constraints:

(i) $1 < q, r < \infty, \frac{1}{q} + \frac{1}{r} = 1, \nu > 0, |arg(1 - b)| < \pi$

(ii) $Re(\eta) > 0, Re(\omega) > -\frac{1}{q}, Re(\sigma) > -\frac{1}{p}, Re(1 + \gamma - \eta - \varepsilon - p) > 0,$
$p \in \mathbb{N}_0, \ \gamma \neq 0, -1, -2, \ldots;$

(iii) $f(x) \in L_q(0, \infty)$

However, the final requirement guarantees the existence and confirms their membership. Additionally, Kalla and Saxena's 1969 work establishes specific characteristics related to the Mellin transforms of these operators.

4. Discussion

A thorough study has been done on the survey, which would then delve into generalized fractional integration that extends classical integral operators to fractional orders. Key topics, however, included Riemann-Liouville fractional integrals.

5. Improvements

The survey might conclude by touching on recent advancements in the field, such as:

1. New classes of special functions related to fractional calculus
2. Extensions of classical fractional operators
3. Numerical methods for evaluating generalized fractional integrals
4. Emerging applications in areas like anomalous diffusion, viscoelasticity, and control theory.

6. Conclusion

This paper has presented an extensive review of the principal operators within the domain of generalized fractional integration, elucidating their fundamental definitions, characteristics, and practical uses. Since its emergence, the area of generalized fractional integration has exhibited significant expansion and diversification, extending the foundational concepts of traditional fractional calculus.

References

[1] Daniel Isaacson, *The Reality of Mathematics and the Case of Set Theory*, Truth, Reference and Realism, edited by Zsolt Novak and Andras Simonyi, Budapest, Hungary: Central European University Press, pp. 1-76, 2011. [CrossRef] [Google Scholar] [Publisher Link]

[2] Ricardo Almeida, Shakoor Pooseh, and Delfim F.M. Torres, *Computational Methods in The Fractional Calculus of Variations*, World Scientific Publishing Company, pp. 29-50, 2015. [CrossRef] [Google Scholar] [Publisher Link]

[3] Virginia Kiryakova, "A Guide to Special Functions in Fractional Calculus," *Mathematics*, vol. 9, no. 1, 2021. [CrossRef] [Google Scholar] [Publisher Link]

[4] Ricardo Enrique Gutiérrez, João Maurício Rosário, and José Tenreiro Machado, "Fractional Order Calculus: Basic Concepts and Engineering Applications," *Mathematical Problems in Engineering*, vol. 2010, pp.1-19, 2010. [CrossRef] [Google Scholar] [Publisher Link]

[5] Sérgio Adriani David, Juan Lopez Linares, and Eliria Maria de Jesus Agnolon Pallone, "Fractional Order Calculus: Historical Apologia, Basic Concepts and Some Applications," *Brazilian Journal of Physics Teaching*, vol. 33, no. 4, pp.4302-4302, 2011. [CrossRef] [Google Scholar] [Publisher Link]

[6] Wei Ding et al., "Applications of Distributed-Order Fractional Operators: A Review," *Entropy*, vol. 23, no. 1, 2021. [CrossRef] [Google Scholar] [Publisher Link]

[7] Henrik Kragh Sørensen, Niels Henrik Abel and The Theory of Equations 1," *University of Aarhus, Denmark*, 1999. [Google Scholar] [Publisher Link]

[8] Dina dos Santos Tavares, "*Fractional Calculus of Variations*," University of Aveiro (Portugal), 2017. [Google Scholar] [Publisher Link]

[9] *Mathematical Methods*, Exercises in Environmental Physics, Springer, New York, NY, pp. 1-79, 2006. [CrossRef][Google Scholar] [Publisher Link]

[10] Dumitru Baleanu, *Fractional Calculus: Models and Numerical Methods*, World Scientific, 2nd ed., 2024. [CrossRef] [Google Scholar] [Publisher Link]

[11] Manuel D. Ortigueira, "A Factory of Fractional Derivatives," *Symmetry*, vol. 16, no. 7, 2024. [CrossRef] [Google Scholar] [Publisher Link]

[12] Adebayo Ridwan, "*Galerkin Method to the Analytical Solution of Generalized Fractional-Order Integro-Differential Equations via Shifted Gegen Bauer Polynomial*," Kwara State University (Nigeria), 2023. [Google Scholar] [Publisher Link]

[13] H. Kober, "On A Theorem of Schur and on Fractional Integrals of Purely Imaginary Order," *Transactions of the American Mathematical Society*, vol. 50, no. 1, pp. 160-174, 1941. [CrossRef] [Google Scholar] [Publisher Link]

[14] Bateman Manuscript Project, Harry Bateman, and Arthur Erdélyi, *Higher transcendental Functions, vol. II*, McGraw - Hill, New York, 1953. [Google Scholar] [Publisher Link]

[15] Antoni Zygmund, *Trigonometric Series*, 2nd ed., Cambridge University Press, 2002. [Google Scholar] [Publisher Link]

[16] Harrers, Felix, E., "Una note sobra la derivation de un ordon real cualquiera," Revita Mathematics, vol. 9, 1952.

[17] S. L. Bora, S. L. Kalla, and R. K. Saxena, "On Integral Transforms," *Univ Nac Tucuman Rev Ser A*, vol. 20, pp. 181-188, 1970. [Google Scholar]

[18] S.L. Bora, and R.K. Saxena, "On fractional integration," *Publications of the Mathematical Institute*, vol. 11, no. 25, pp. 19-22, 1971. [Google Scholar] [Publisher Link]

[19] Neunter Jahrgang, *Quarterly of the Society of Natural Sciences in Zurich,* Society of Natural Sciences in Zurich, vol. 61, pp. 296-306, 1917. [Google Scholar] [Publisher Link]

[20] Megumi Saigo, and Nobuyuki Maeda, "More Generalization of Fractional Calculus," *Transform Methods and Special Functions, Varna. Bulgaria*, pp. 386-400, 1996. [Google Scholar]

[21] Shantanu Das, *Introduction to Fractional Calculus*, Springer Berlin, Heidelberg, 2nd ed., pp. 1-50, 2011. [CrossRef] [Google Scholar] [Publisher Link]

[22] Megumi Saigo, "A Remark on Integral Operators Involving the Gauss Hypergeometric Functions," *Department of Mathematics, College of General Education, Kyushu University*, vol. 11, no. 2, pp. 135-143, 1978. [CrossRef] [Google Scholar] [Publisher Link]

[23] R.K. Saxena, "A Formal Solution of Certain Dual Integral Equations Involving H-Functions," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 63, no. 1, pp. 171-178, 2008. [CrossRef] [Google Scholar] [Publisher Link]

[24] S.L. Kalla, and R.K. Saxena, "Integral Operators Involving Hypergeometric Functions," *Mathematische Zeitschrift*, vol. 108, pp. 231-234, 1969. [CrossRef] [Google Scholar] [Publisher Link]

[25] Kenneth S. Miller, and Bertram Ross, "An Introduction to the Fractional Calculus and Fractional Differential Equations," John Wiley and Sons, New York, 1993. [Google Scholar] [Publisher Link]

[26] A.M. Mathai, and Hans J. Haubold, *Special Functions for Applied Scientists*, Springer New York, 1st ed., 2008. [CrossRef] [Google Scholar] [Publisher Link]

[27] Yuriy Furgala, and Bohdan Rusyn, "Peculiarities of Melin Transform Application to Symbol Recognition," *In 2018 14th International Conference on Advanced Trends in Radioelecrtronics, Telecommunications and Computer Engineering (TCSET)*, Lviv-Slavske, Ukraine, 2018. [CrossRef] [Google Scholar] [Publisher Link]

[28] Alexander D. Poularikas, *Transforms and Applications Handbook*, CRC Press, Boca Raton, 3rd ed., 2010. [CrossRef] [Google Scholar] [Publisher Link]

[29] R. Hilfer, "Fractional Diffusion Based on Riemann-Liouville Fractional Derivatives," *The Journal of Physical Chemistry B*, vol. 104, no. 16, pp. 3914-3917, 2000. [CrossRef] [Google Scholar] [Publisher Link]

[30] M. Cristina Caputo, and Delfim F.M. Torres, "Duality for The Left and Right Fractional Derivatives," *Signal Processing*, vol. 107, pp. 265-271, 2015. [CrossRef] [Google Scholar] [Publisher Link]

[31] Adam Loverro, "Fractional Calculus: History, Definitions and Applications for The Engineer," *Rapport technique, Univeristy of Notre Dame: Department of Aerospace and Mechanical Engineering*, pp.1-28, 2004. [Google Scholar] [Publisher Link]

[32] Marcel Riesz, The Riemann-Liouville Integral and The Cauchy Problem, *Acta Mathematica*, vol. 8, pp. 1-223, 1949. [CrossRef] [Google Scholar] [Publisher Link]

[33] Bevan Braithwaite Baker, and Edward Thomas Copson, *The mathematical theory of Huggen's principle*, American Mathematical Society, 2003. [Google Scholar] [Publisher Link]

[34] Doetsch, G., Theorie and Unwinding der Laplace Transformations, Dover, New York, 1943.

[35] H. Kober, "On A Theorem of Schur and on Fractional Integrals of Purely Imaginary Order," *Transactions of the American Mathematical Society*, vol. 50, no. 1, pp. 160-174, 1941. [CrossRef] [Google Scholar] [Publisher Link]

[36] Virginia Kiryakova, "The Special Functions of Fractional Calculus as Generalized Fractional Calculus Operators of Some Basic Functions," *Computers & Mathematics with Applications*, vol. 59, no. 3, pp.1128-1141, 2010. [CrossRef] [Google Scholar] [Publisher Link]