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Original Article

Fixed Point Theorems On 5- Dimensional Ball Metric Spaces

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Abstract - *The paper introduces the 5-dimensional ball metric space, an advanced extension of metric spaces, b-metric spaces, S-metric spaces, and B_4-metric spaces. It develops unique fixed-point theorems for self-mappings on complete 5-dimensional ball metric spaces under specific contractive conditions, accompanied by illustrative applications and examples where applicable.*

Keywords - *5-dimensional ball metric spaces, B_4-metric spaces, Fixed-point theorems and S-metric spaces.*

1. Introduction

The Banach contraction mapping principle is a very useful theorem. It is a very popular tool for solving problems in many mathematical analysis branches. Banach fixed point theorem has many applications in and beyond mathematics. Banach fixed point theorem has been generalized and extended by many authors in various ways. In 2023, Sarma et al. [1] introduced the B_4-metric space, which emerged as a natural extension of the S-metric space initially proposed by Sedghi et al. [3]. These spaces have spurred significant interest and exploration within the mathematical community, establishing fixed-point theorems in Adewale and Iluno [4] and Van et al. [5]. Rao et al. [2] introduced 4-dimensional ball metric spaces. This paper introduces 5-dimensional ball metric spaces, which are natural extensions of metric spaces, b-metric spaces, S-metric spaces, B4-metric spaces and 4-dimension ball metric spaces. We establish unique fixed-point theorems on 5-dimensional ball metric space with their applications and supporting examples.

2. Results

Firstly, we define 5-dimensional ball metric space with examples. It is further followed by proving and using a few lemmas in the main theorem.

Definition 2.1. Let $\$$ be a non- empty set and $B_5: \$^5 \rightarrow R+$ satisfy the following conditions: for all $ꝓ_1, ꝓ_2, ꝓ_3, ꝓ_4, ꝓ_5, \alpha \in \$$.

1. $B_5(ꝓ_1, ꝓ_2, ꝓ_3, ꝓ_4, ꝓ_5) = 0$ if and only if $ꝓ_1= ꝓ_2= ꝓ_3= ꝓ_4= ꝓ_5$
2. $B_5(ꝓ_1, ꝓ_2, ꝓ_3, ꝓ_4, ꝓ_5) \leq B_5(ꝓ_1, ꝓ_1, ꝓ_1, ꝓ_1, \alpha) + B_5(ꝓ_2, ꝓ_2, ꝓ_2, ꝓ_2,\alpha) + B_5(ꝓ_3, ꝓ_3, ꝓ_3, ꝓ_3, \alpha) + B_5(ꝓ_4, ꝓ_4, ꝓ_4, ꝓ_4, \alpha) + B_5(ꝓ_5, ꝓ_5, ꝓ_5, ꝓ_5, \alpha)$.

We then define B_5 as a 5-dimensional ball metric on $\$$ and the pair $(\$, B_5)$ is a 5-dimensional ball metric space.

Example 2.2. Suppose, $\$ = N \cup \{0\}$ and define $B_5: \$^5 \rightarrow R^+ \cup \{0\}$ by

$$B_5(ꝓ_1, ꝓ_2, ꝓ_3, ꝓ_4, ꝓ_5) = 0, \text{ if } ꝓ_1 = ꝓ_2 = ꝓ_3 = ꝓ_4 = ꝓ_5 = ꝓ_1^2 + ꝓ_2^2 + ꝓ_3^2 + ꝓ_4^2 + ꝓ_5^2, \text{ otherwise, where } ꝓ_1, ꝓ_2, ꝓ_3, ꝓ_4, ꝓ_5 \in \$.$$

Then $(\$, B_5)$ is a 5- dimensional ball metric space.

Example 2.3. Suppose, $\$ = N \cup \{0\}$ and define $B_5: \$^5 \rightarrow R^+ \cup \{0\}$ by

$$B_5(ꝓ_1, ꝓ_2, ꝓ_3, ꝓ_4, ꝓ_5) = 0, \text{ if } ꝓ_1 = ꝓ_2 = ꝓ_3 = ꝓ_4 = ꝓ_5 = ꝓ_1 + ꝓ_2 + ꝓ_3 + ꝓ_4 + ꝓ_5, \text{ otherwise, where } ꝓ_1, ꝓ_2, ꝓ_3, ꝓ_4, ꝓ_5 \in \$.,$$

Then $(\$, B_5)$ is a 5- dimensional ball metric space.

Example 2.4. Suppose, $\$ = N \cup \{0\}$, $\lambda > 0$ and define B_5: $\$^5 \rightarrow R^+ \cup \{0\}$ by

$$B_5(p_1, p_2, p_3, p_4, p_5) = 0, \text{ if } p_1 = p_2 = p_3 = p_4 = p_5 = \lambda, \text{ otherwise, where } p_1, p_2, p_3, p_4, p_5 \in \$.$$

Then ($, B5) is a 5- dimensional ball metric space.

Definition 2.5. 1. A sequence $\{p_n\}$ in $ converges to p if $B_5(p_n, p_n, p_n, p_n, p) \rightarrow 0$, as $n \rightarrow \infty$.
i.e. given $\varepsilon > 0$, there exists $n_0 \in N$ such that for all $n \geq n_0$, $B_5(p_n, p_n, p_n, p_n, p) < \varepsilon$, we denote this by $\lim_{n\rightarrow\infty} p_n = p$ or $\lim_{n\rightarrow\infty} B_5(p_n, p_n, p_n, p_n, p) = 0$.

2. A sequence $\{p_n\}$ in $ Cauchy sequence if $B_5(p_n, p_n, p_n, p_n, p_m) \rightarrow 0$, as $n, m \rightarrow \infty$.
   i.e. given $\varepsilon > 0$, there exists $n_0 \in N$ such that for all $n, m \geq n_0$, $B_5(p_n, p_n, p_n, p_n, p_m) < \varepsilon$.

3. A 5- dimensional ball metric space ($, B_5) is called complete if every Cauchy sequence in $ is convergent.
   We now state a few lemmas we use in further development.

Lemma 2.6. Let ($, B_5) is a 5-dimensional ball metric space. Then, $B_5(p_1, p_2, p_2, p_2, p_2) = B_5(p_2, p_2, p_2, p_2, p_1)$, for all $p_2, p_2 \in \$$.

Lemma 2.7. $p_m \rightarrow p$ if and only if $B_5(p, p, p, p, p_m) \rightarrow -0$, as $n \rightarrow \infty$.

Lemma 2.8. If $p_m \rightarrow p_1$ and $p_m \rightarrow p_2$ implies $p_1 = p_2$.

Lemma 2.9. If $p_m \rightarrow p$ implies $\{p_m\}$ is a Cauchy Sequence.

Lemma 2.10. Let $\$ \neq \varphi$ and B_5: $\$^5 \rightarrow R^+ \cup \{0\}$ ba a 5- dimensional ball metric space on $. Then, $B_5(\alpha, \beta, \beta, \beta, \beta) \leq B_5(\alpha, \delta, \delta, \delta, \delta) + 4 B_5(\delta, \beta, \beta, \beta, \beta)$, for all $\alpha, \beta, \delta \in \$$.

Proof. Let ($, B_5) is a 5- dimensional ball metric space. Replacing p_1 by α , p_2, p_3, p_4, p_5 by β and α by δ in definition 2.1 (2), we get

$$B_5(\alpha, \beta, \beta, \beta, \beta) \leq B_5(\alpha, \alpha, \alpha, \alpha, \delta) + B_5(\beta, \beta, \beta\beta, \delta) + B_5(\beta, \beta, \beta\beta, \delta) + B_5(\beta, \beta, \beta\beta, \delta) + B_5(\beta, \beta, \beta, \beta, \delta).$$

Therefore

$$B_5(\alpha, \beta, \beta, \beta, \beta) \leq B_5(\alpha, \alpha, \alpha, \alpha, \delta) + 4 B_5(\beta, \beta, \beta, \beta, \delta).$$

Therefore

$$B_5(\alpha, \beta, \beta, \beta, \beta) \leq B_5(\alpha, \delta, \delta, \delta, \delta) + 4 B_5(\delta, \beta, \beta, \beta, \beta) \text{ (by lemma 2.6).}$$

Now, we state and prove our main theorem on 5-dimensional ball metric spaces.

Theorem 2.11. Let ($, B_5) be a complete 5-dimensional ball metric space and H: $ → $ is a mapping. Let $0 \leq K < 1/4$ is such that for all $p_1, p_2, p_3, p_4, p_5 \in \$$.

$$B_5(Hp_1, Hp_2, Hp_3, Hp_4, Hp_5) \leq k\, B_5(p_1, p_2, p_3, p_4, p_5). \qquad (2.1.1)$$

Then, H has a unique fixed point.

Proof. For, $p_1, p_2 \in \$$, from 3.1.1, taking $p_2 = p_3 = p_4 = p_5$, we have $B_5(Hp_1, Hp_2, Hp_2, Hp_2, Hp_2) \leq k\, B_5(p_1, p_2, p_2, p_2, p_2)$. Let $p_0 \in \$$. Define the sequence $\{p_n\}$ by $p_{n+1} = H^n p_0$, for n = 1, 2, 3, …

Then, $p_{n+1} = Hp_n$.

$$B_5(p_n, p_n, p_n, p_n, p_{n+1}) = B_5(Hp_{n-1}, Hp_{n-1}, Hp_{n-1}, Hp_{n-1}, Hp_n) \leq k\, B_5(p_{n-1}, p_{n-1}, p_{n-1}, p_{n-1}, p_n). \qquad (2.1.2)$$

Let

$$S_n = B_5(p_n, p_n, p_n, p_n, p_{n+1}) \quad \text{We have } S_n \leq k\, S_{n-1} \leq k^2\, S_{n-2}$$

Therefore,

$$S_n \leq k^n S_0, \text{ for all } n \in N. \ldots \qquad (2.1.3)$$

This shows that $S_n \rightarrow 0$, as $n \rightarrow \infty$. Suppose, $m > n$.
By using Definition 2.1, for $p_{n+1}, p_{n+2}, p_{n+3} \ldots p_{m-1}$, we have $B_5(p_n, p_m, p_m, p_m, p_m)$

$$\leq B_5(p_n, p_n, p_n, p_n, p_{n+1}) + B_5(p_m, p_m, p_m, p_m, p_{n+1}) + B_5(p_m, p_m, p_m, p_m, p_{n+1}) + B_5(p_m, p_m, p_m, p_m, p_{n+1}) + B_5(p_m, p_m, p_m, p_m, p_{n+1})$$
$$= B_5(p_n, p_n, p_n, p_n, p_{n+1}) + 4\, B_5(p_m, p_m, p_m, p_m, p_{n+1})$$

$$\le S_n + 4\,B_5(p_m, p_m, p_m, p_m, p_{n+1})$$
$$\le S_n + 4\,S_{n+1} + 4^2\,B_5\,(p_m, p_m, p_m, p_m, p_{n+2})$$
$$\le S_n + 4\,S_{n+1} + 4^2\,S_{n+2} + 4^3\,B_5(p_m, p_m, p_m, p_m, p_{n+3})$$
$$\le S_n + 4\,S_{n+1} + 4^2\,S_{n+2} + 4^3\,S_{n+3} + 4^4\,S_{n+4} + \ldots + 4^{m-n-1}\,S_{m-1} \text{ (lemma 2.10)} \ldots \qquad (2.1.4)$$

From 2.1.3 and 2.1.4, we have

$$B_5(p_n, p_m, p_m, p_m, p_m) \le S_n + 4\,k\,S_{n+1} + 4^2\,k^2\,S_{n+2} + 4^3\,k^3\,S_{n+3} + 4^4\,k^4\,S_{n+4} + \ldots\, 4^{m-n-1}\,k^{m-n-1}\,S_{m-1}$$
$$\le S_n\,(1 + 4\,k + 4^2\,k^2 + 4^3\,k^3 + 4^4\,k^4 + \ldots\, 4^{m-n-1}\,k^{m-n-1})$$
$$= S_n\,(1/\,1 - 4k) \text{ (since } 4k < 1 \text{, by hypothesis)}$$
$$\le k^n\,S_0\,(1/\,1 - 4k) \to 0, \text{ as } n \to \infty, \text{ since } k < 1/4.$$

Hence, $\{p_n\}$ is a Cauchy sequence.
Since \$ is complete, there exists $p^* \,\varepsilon\, \$$ such that

$$p_n \to p^* \ \ldots \quad (2.1.5)$$

Now,

$$B_5(p_{n+1}, Hp^*, Hp^*, Hp^*, Hp^*)$$
$$= B_5(Hp_n, Hp^*, Hp^*, Hp^*, Hp^*)$$
$$\le k\,B_5(p_n, p^*, p^*, p^*, p^*) \to 0, \text{ as } n \to \infty \text{ (by 2.1.5)}$$

Therefore, $p_{n+1} \to Hp^*$
Therefore, $Hp^* = p^*$ (by lemma 2.7)
Therefore, p^* is a fixed point of H.

Uniqueness of fixed point

Suppose, p^{**} is a fixed point of H.
Then, $B_5(Hp^*, Hp^{**}, Hp^{**}, Hp^{**}, Hp^{**}) \le k\,B_5(p^*, p^{**}, p^{**}, p^{**}, p^{**})$.
Therefore, $B_5(p^*, p^{**}, p^{**}, p^{**}, p^{**}) \le k\,B_5(p^*, p^{**}, p^{**}, p^{**}, p^{**})$.
Therfore, $B_5(p^*, p^{**}, p^{**}, p^{**}, p^{**}) = 0$.
Therefore , $p^{**} = p^*$. (by Definition 2.1)
Thus, H has a unique fixed point.

3. Discussion of Results and its Applications

In this section, we obtain applications of the Theorem 2.11.

Theorem 3.1. Let (\$, B_5) be a complete 5-dimensional ball metric space and $H: \$ \to \$$ is a mapping. Let $0 \le K < 1/17$ is such that for all $p_1, p_2, p_3, p_4, p_5 \,\varepsilon\, \$$.

$$B_5(Hp_1, Hp_2, Hp_3, Hp_4, Hp_5) \le k\,\{ B_5(p_1, Hp_1, Hp_1, Hp_1, Hp_1) + B_5(p_2, Hp_2, Hp_2, Hp_2, Hp_2) + B_5(p_3, Hp_3, Hp_3, Hp_3, Hp_3) + B_5(p_4, Hp_4, Hp_4, Hp_4, Hp_4) + B_5(p_5, Hp_5, Hp_5, Hp_5, Hp_5)\} \ldots \qquad (3.1.1)$$

Then, H has a unique fixed point.

Proof. For, $p_1, p_2 \,\varepsilon\, \$$, from 3.1.1 , taking $p_2 = p_3 = p_4 = p_5$, we have $B_5(Hp_1, Hp_2, Hp_2, Hp_2, Hp_2) \le k\,\{ B_5(p_1, Hp_1, Hp_1, Hp_1, Hp_1) + 4B_5(p_2, Hp_2, Hp_2, Hp_2, Hp_2)\}$.

Let Let $p_0 \,\varepsilon\, \$$. Define the sequence $\{p_n\}$ by $p_{n+1} = H^n p_0$, for n = 0, 1, 2, 3,…

We have

$$B_5(p_n, p_n, p_n, p_n, p_{n+1}) \le k\,\{ B_5(p_{n-1}, p_{n-1}, p_{n-1}, p_{1-n-1}, p_n) + B_5(p_{n-1}, p_{n-1}, p_{n-1}, p_{1-n-1}, p_n) + B_5(p_{n-1}, p_{n-1}, p_{n-1}, p_{1-n-1}, p_n) + B_5(p_{n-1}, p_{n-1}, p_{n-1}, p_{1-n-1}, p_n) + B_5(p_n, p_n, p_n, p_n, p_{n+1}).$$

So that, $B_5(p_n, p_n, p_n, p_n, p_{n+1}) \le 4k/1\text{-}k\ \ B_5(p_{n-1}, p_{n-1}, p_{n-1}, p_{1-n-1}, p_n)$.
Since, $4k/1\text{-}k < ¼$, by Theorem 2.11, the result follows.
Write $S_n = B_5(p_n, p_n, p_n, p_n, p_{n+1})$, we have $S_n \le k\,S_{n-1} \le k^2\,S_{n-2}$.
Therefore, $S_n \le k^n\,S_0$, for all $n \,\varepsilon\, N$, we have for all $n, m \,\varepsilon\, N$ with $n \ne m$, $p_n \ne p_m$.
By repeated use of (2) in Definition 2.1,

$$B_5(p_n, p_m, p_m, p_m, p_m)$$
$$\le B_5(p_n, p_n, p_n, p_n, p_{n+1}) + B_5(p_m, p_m, p_m, p_m, p_{n+1}) + B_5(p_m, p_m, p_m, p_m, p_{n+1}) + B_5(p_m, p_m, p_m, p_m, p_{n+1}) + B_5(p_m, p_m, p_m, p_m, p_{n+1})$$
$$= B_5(p_n, p_n, p_n, p_n, p_{n+1}) + 4\,B_5(p_m, p_m, p_m, p_m, p_{n+1})$$
$$\le S_n + 4\ B_5(p_m, p_m, p_m, p_m, p_{n+1})$$

$$\leq S_n + 4\,S_{n+1} + 4^2\,B_5(ꝑ_m, ꝑ_m, ꝑ_m, ꝑ_m, ꝑ_{n+2})$$
$$\leq S_n + 4\,S_{n+1} + 4^2\,S_{n+2} + 4^3\,B_5(ꝑ_m, ꝑ_m, ꝑ_m, ꝑ_m, ꝑ_{n+3}) \leq S_n + 4\,S_{n+1} + 4^2\,S_{n+2} + 4^3\,S_{n+3} + 4^4\,S_{n+4} + \ldots 4^{m-n-1}\,S_{m-1}$$

We have,

$$B_5(ꝑ_n, ꝑ_m, ꝑ_m, ꝑ_m, ꝑ_m) \leq S_n + 4\,k\,S_{n+1} + 4^2\,k^2\,S_{n+2} + 4^3\,k^3\,S_{n+3} + 4^4\,k^4\,S_{n+4} + \ldots 4^{m-n-1}\,k^{m-n-1}\,S_{m-1}$$
$$\leq S_n\,(1 + 4\,k + 4^2\,k^2 + 4^3\,k^3 + 4^4\,k^4 + \ldots 4^{m-n-1}\,k^{m-n-1})$$
$$= S_n\,(1/1-4k) \text{ (since } 4k < 1 \text{, by hypothesis)}$$
$$\leq k^n\,S_0\,(1/1-4k) \to 0, \text{as } n \to \infty, \text{ since } k < 1/4.$$

Hence, { $ꝑ_n$} is a Cauchy sequence.
Since \$ is complete, there exists $ꝑ^*$ ε \$ such that $ꝑ_n \to ꝑ^*$
Now,

$$B_5(ꝑ_{n+1}, Ꜧꝑ^*, Ꜧꝑ^*, Ꜧꝑ^*, Ꜧꝑ^*) = B_5(Ꜧꝑ_n, Ꜧꝑ^*, Ꜧꝑ^*, Ꜧꝑ^*, Ꜧꝑ^*) \leq k\,B_5(ꝑ_n, ꝑ^*, ꝑ^*, ꝑ^*, ꝑ^*) \to 0, \text{ as } n \to \infty$$

Therefore, $ꝑ_{n+1} \to Ꜧꝑ^*$
Therefore, $Ꜧꝑ^* = ꝑ^*$.
Therefore, $ꝑ^*$ is a fixed point of Ꜧ.

Uniqueness of Fixed Point

Suppose**,** $ꝑ^{**}$ is a fixed point of Ꜧ. Then,

$$B_5(Ꜧꝑ^*, Ꜧꝑ^{**}, Ꜧꝑ^{**}, Ꜧꝑ^{**}, Ꜧꝑ^{**}) \leq k\,\{B_5(ꝑ^*, Ꜧꝑ^*, Ꜧꝑ^*, Ꜧꝑ^*, Ꜧꝑ^*) + B_5(ꝑ^{**}, Ꜧꝑ^{**}, Ꜧꝑ^{**}, Ꜧꝑ^{**}, Ꜧꝑ^{**}) + B_5(ꝑ^{**}, Ꜧꝑ^{**}, Ꜧꝑ^{**}, Ꜧꝑ^{**}, Ꜧꝑ^{**}) +$$
$$B_5(ꝑ^{**}, Ꜧꝑ^{**}, Ꜧꝑ^{**}, Ꜧꝑ^{**}, Ꜧꝑ^{**}) + B_5(ꝑ^{**}, Ꜧꝑ^{**}, Ꜧꝑ^{**}, Ꜧꝑ^{**}, Ꜧꝑ^{**})\}$$
$$\leq k\,\{B_5(ꝑ^*, ꝑ^*, ꝑ^*, ꝑ^*, ꝑ^*) + B_5(ꝑ^{**}, ꝑ^{**}, ꝑ^{**}, ꝑ^{**}, ꝑ^{**}) + B_5(ꝑ^{**}, ꝑ^{**}, ꝑ^{**}, ꝑ^{**}, ꝑ^{**}) + B_5(ꝑ^{**}, ꝑ^{**}, ꝑ^{**}, ꝑ^{**}, ꝑ^{**}) + B_5(ꝑ^{**}, ꝑ^{**}, ꝑ^{**}, ꝑ^{**}, ꝑ^{**})\}$$
$$\text{Therefore, } B_5(ꝑ^*, ꝑ^{**}, ꝑ^{**}, ꝑ^{**}, ꝑ^{**}) \leq k\,B_5(ꝑ^*, ꝑ^{**}, ꝑ^{**}, ꝑ^{**}, ꝑ^{**}).$$

Therfore, $B_5(ꝑ^*, ꝑ^{**}, ꝑ^{**}, ꝑ^{**}, ꝑ^{**}) = 0$.
Therefore , $ꝑ^{**} = ꝑ^*$.
Thus, Ꜧ has a unique fixed point.

Theorem 3.2. Let (\$, B_5) be a complete 5-dimensional ball metric space and Ꜧ: \$ → \$ is a mapping. Suppose, the real numbers d_1, d_2, d_3, d_4, d_5, such that

$$0 \leq d_1 < 1/4, \quad 0 \leq d_2 < 1/5, \quad 0 \leq d_3 < 1/5, \quad 0 \leq d_4 < 1/5, \quad 0 \leq d_5 < 1/5.$$

Write $\lambda = \max\,\{ d_1, d_2/1-d_2, d_3/1-d_3, d_4/1-d_4, d_5/1-d_5\}$.
Assume that for all $ꝑ_1, ꝑ_2, ꝑ_3, ꝑ_4, ꝑ_5$ ε \$, $B_5(Ꜧꝑ_1, Ꜧꝑ_2, Ꜧꝑ_3, Ꜧꝑ_4, Ꜧꝑ_5)$

$$\leq \lambda/5\,\{ B_5(ꝑ_1, ꝑ_2, ꝑ_3, ꝑ_4, ꝑ_5) + 4\,\lambda/5 B_5(ꝑ_1, ꝑ_1, ꝑ_1, ꝑ_1, Ꜧꝑ_1) \quad (3.2.1)$$

Then , Ꜧ has a unique fixed point.

Proof. From 3.2.1 , taking For, $ꝑ_1, ꝑ_2$ ε \$, $ꝑ_2 = ꝑ_3 = ꝑ_4 = ꝑ_5$, we have $B_5(Ꜧꝑ_1, Ꜧꝑ_2, Ꜧꝑ_2, Ꜧꝑ_2, Ꜧꝑ_2) \leq \lambda/5$ { $B_5(ꝑ_1, ꝑ_2, ꝑ_2, ꝑ_2, ꝑ_2) + 4\,\lambda/5\,B_5(ꝑ_1, ꝑ_1, ꝑ_1, ꝑ_1, Ꜧꝑ_1)$ }.

Let Let $ꝑ_0$ ε \$. Define the sequence {$ꝑ_n$} by $ꝑ_{n+1} = Ꜧ^n ꝑ_0$ for n = 0, 1, 2, 3,
Then , $B_5(ꝑ_{n+1}, ꝑ_n, ꝑ_n, ꝑ_n, ꝑ_n) = B_5(Ꜧꝑ_n, Ꜧꝑ_{n-1}, Ꜧꝑ_{n-1}, Ꜧꝑ_{n-1}, Ꜧꝑ_{n-1}) \leq \lambda/5\{ B_5(ꝑ_n, ꝑ_{n-1}, ꝑ_{n-1}, ꝑ_{1-n-1}, ꝑ_n) + 4\lambda/5 B_5(ꝑ_n, ꝑ_{n-1}, ꝑ_{n-1}, ꝑ_{n-1}, ꝑ_{n-1}) = \lambda\,B_5(ꝑ_n, ꝑ_{n-1}, ꝑ_{n-1}, ꝑ_{1-n-1}, ꝑ_n)$.

Write $S_n = B_5(ꝑ_n, ꝑ_n, ꝑ_n, ꝑ_n, ꝑ_{n+1})$, we have $S_n \leq k\,S_{n-1} \leq k^2\,S_{n-2}$.
Therefore, $S_n \leq k^n\,S_0$, for all n ε N, we have for all n,m ε N with $n \neq m$, $ꝑ_n \neq ꝑ_m$.
By repeated use of (2) in Definition 2.1,

$$B_5(ꝑ_n, ꝑ_m, ꝑ_m, ꝑ_m, ꝑ_m)$$
$$\leq B_5(ꝑ_n, ꝑ_n, ꝑ_n, ꝑ_n, ꝑ_{n+1}) + B_5(ꝑ_m, ꝑ_m, ꝑ_m, ꝑ_m, ꝑ_{n+1}) + B_5(ꝑ_m, ꝑ_m, ꝑ_m, ꝑ_m, ꝑ_{n+1}) + B_5(ꝑ_m, ꝑ_m, ꝑ_m, ꝑ_m, ꝑ_{n+1}) + B_5(ꝑ_m, ꝑ_m, ꝑ_m, ꝑ_m, ꝑ_{n+1})$$
$$= B_5(ꝑ_n, ꝑ_n, ꝑ_n, ꝑ_n, ꝑ_{n+1}) + 4\,B_5(ꝑ_m, ꝑ_m, ꝑ_m, ꝑ_m, ꝑ_{n+1})$$
$$\leq S_n + 4\,B_5(ꝑ_m, ꝑ_m, ꝑ_m, ꝑ_m, ꝑ_{n+1}) \leq S_n + 4\,S_{n+1} + 4^2\,B_5(ꝑ_m, ꝑ_m, ꝑ_m, ꝑ_m, ꝑ_{n+2})$$
$$\leq S_n + 4\,S_{n+1} + 4^2\,S_{n+2} + 4^3\,B_5(ꝑ_m, ꝑ_m, ꝑ_m, ꝑ_m, ꝑ_{n+3})$$
$$\leq S_n + 4\,S_{n+1} + 4^2\,S_{n+2} + 4^3\,B_5(ꝑ_m, ꝑ_m, ꝑ_m, ꝑ_m, ꝑ_{n+3}) \leq S_n + 4\,S_{n+1} + 4^2\,S_{n+2} + 4^3\,S_{n+3} + 4^4\,S_{n+4} + \ldots 4^{m-n-1}\,S_{m-1}$$

We have,

$$B_5(ꝑ_n, ꝑ_m, ꝑ_m, ꝑ_m, ꝑ_m) \leq S_n + 4\,k\,S_{n+1} + 4^2\,k^2\,S_{n+2} + 4^3\,k^3\,S_{n+3} + 4^4\,k^4\,S_{n+4} + \ldots 4^{m-n-1}\,k^{m-n-1}\,S_{m-1}$$
$$\leq S_n\,(1 + 4\,k + 4^2\,k^2 + 4^3\,k^3 + 4^4\,k^4 + \ldots 4^{m-n-1}\,k^{m-n-1})$$
$$= S_n\,(1/1-4k) \text{ (since } 4k < 1 \text{, by hypothesis)}$$
$$\leq k^n\,S_0\,(1/1-4k) \to 0, \text{as } n \to \infty, \text{ since } k < 1/4.$$

Hence, $\{ ꝑ_n \}$ is a Cauchy sequence.
Since $\$$ is complete, there exists $ꝑ^* \varepsilon \$$ such that $ꝑ_n \to ꝑ^*$
Now,

$$B_5(ꝑ_{n+1}, Ꜧꝑ^*, Ꜧꝑ^*, Ꜧꝑ^*, Ꜧꝑ^*) = B_5(Ꜧꝑ_n, Ꜧꝑ^*, Ꜧꝑ^*, Ꜧꝑ^*, Ꜧꝑ^*) \leq kB_5(ꝑ_n, ꝑ^*, ꝑ^*, ꝑ^*, ꝑ^*) \to 0, \text{ as } n \to \infty$$

Therefore, $ꝑ_{n+1} \to Ꜧꝑ^*$
Therefore, $Ꜧꝑ^* = ꝑ^*$.
Therefore, $ꝑ^*$ is a fixed point of $Ꜧ$.

Uniqueness of fixed point

Suppose**,** $ꝑ^{**}$ is a fixed point of $Ꜧ$.
Then**,** $B_5(Ꜧꝑ^*, Ꜧꝑ^{**}, Ꜧꝑ^{**}, Ꜧꝑ^{**}, Ꜧꝑ^{**}) \leq \lambda/5 \ B_5(ꝑ^*, ꝑ^{**}, ꝑ^{**}, ꝑ^{**}, ꝑ^{**}) + 4\lambda/5 B_5(ꝑ^*, ꝑ^{**}, ꝑ^{**}, ꝑ^{**}, ꝑ^{**}).,ꝑ^{**})$.
Therefore**,** $B_5(ꝑ^*, ꝑ^{**}, ꝑ^{**}, ꝑ^{**}, ꝑ^{**}) \leq \lambda B_5(ꝑ^*, ꝑ^{**}, ꝑ^{**}, ꝑ^{**}, ꝑ^{**})$.
Therfore, $B_5(ꝑ^*, ꝑ^{**}, ꝑ^{**}, ꝑ^{**}, ꝑ^{**}) = 0$.
Therefore, $ꝑ^{**} = ꝑ^*$.
Thus, $Ꜧ$ has a unique fixed point.

3. Conclusion

This paper introduces the notion of the 5-dimensional ball metric spaces. We establish unique fixed-point theorems for self-mapping on complete 5-dimensional ball metric spaces and illustrate their applications with supporting examples. This work contributes to the understanding and applying 5-dimensional ball metric space in mathematical analysis and its allied areas. The possibility of extending the results of this paper to n-dimensional ball metric spaces is under active investigation.

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