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Original Article

Fixed Point Results in Complex-Valued Double Controlled Metric-like Spaces

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Abstract - *In this paper, we prove fixed point results of Kannan and Reich interpolative contraction in complex-valued double contraction metric–like spaces. Moreover, examples have also been provided to underpin and exemplify the results.*

Keywords **-** *Interpolative contraction, fixed point, double-controlled metric space, complex-valued double-controlled metric-like spaces.*

1. Introduction and Preliminaries

The Banach Contraction Principle [1] is one of the key tools to show the existence of solutions related to various mathematical problems, especially those characterizing differential equations, integral equations, and fractional differential equations. Bakhtin [2]and Czerwik[3] initiated the concept of b-metric spaces, which resulted in many fixed-point results (see [4],[12],[13]). Kamran et al. [5] generalized b-metric spaces and the triangle inequality, due to which control functions in the contractive condition do not have a role. This generalization enabled the extension of the Banach contraction from metric spaces to b-metric spaces and then to controlled metric-type spaces.(see [6],[9],[13]).

Abdeljawad et al.[7] also introduced double control metric spaces, later obtaining many fixed-point results (see [7],[15]). Harandi[22] introduced the concept of metric-like spaces in 2012 as a generalization of metric spaces. Subsequently, Mlaiki [16] generalized controlled metric spaces by introducing controlled metric-like spaces in which the self-distance does not necessarily have to be zero. After this, Azam et al.[14] gave their concept of complex-valued metric spaces, which Hosseini and Karazaki[21] more generally extended to complex-valued metric-like spaces(see[18]). Building on this, Aslam et al.[19] extended this concept into complex-valued controlled metric-type spaces and complex-valued double controlled metric-like spaces. Recently, Singh et al.[20] provided several new interpolative contractions, such as the (λ, a)-interpolative Kannan contraction, the (λ, a, b)-interpolative Kannan contraction, and the (λ, a, b, c)-interpolative Reich contraction, with all extensive fixed-point results formed in complete controlled metric spaces.

This article contains fixed-point results for the Kannan and Reich interpolating contractions in complex-valued double-controlled metric-like spaces. Examples have also been provided to underpin and exemplify the results of this study.

2. Preliminaries

Further, let us remember some definitions and results.

Let C be the set of complex numbers and w_1 and w_2 be elements of C. $w_1 \leq w_2$ iff Re $(w_1) \leq$ Re (w_2) or (Re (w_1) = Re (w_2) and Im $(w_1) \leq$ Im (w_2)). Regarding the earlier definition, we have that $w_1 \leq w_2$ if one of the other conditions is satisfied

1. Re $(w_1) <$ Re (w_2) and Im$(w_1) <$ Im (w_2),
2. Re $(w_1) <$ Re (w_2) and Im (w_1) = Im (w_2),
3. Re $(w_1) <$ Re (w_2) and Im $(w_1) >$ Im (w_2),
4. Re (w_1) = Re (w_2) and Im $(w_1) <$ Im (w_2).

Definition 2.1 [2] Let Ɦ ≠ ϕ and $\alpha \geq 1$be a given real number. Let $Ᵽ_b$: Ɦ x Ɦ → [0,+ ∞) be a function is called b- metric if 1. $Ᵽ_b(\rho, \sigma) \geq 0$,

2. $Ᵽ_b(\rho, \sigma) = 0$ if and only if $\rho = \sigma$,
3. $Ᵽ_b(\rho, \sigma) = Ᵽ_b(\sigma, \rho)$,
4. $Ᵽ_b(\rho, \sigma) \leq \alpha [Ᵽ_b(\rho, \delta) + Ᵽ_b(\delta, \sigma)]$, for all ρ, σ, δ ε Ɦ.

A pair (Ɦ, $Ᵽ_b$) is called a b-metric space. It is clear that b-metric space is an extension on usual metric space.

Definition 2.2 [5] Let Ħ ≠ ϕ and given a function α: Ħ x Ħ→ [1,+ ∞). Let $Ᵽ_{eb}$: Ħx Ħ → [0,+ ∞) be a function is called extended b- metric if 1. $Ᵽ_{eb}$(ρ, σ) ≥ 0,
2. $Ᵽ_{eb}$(ρ, σ) = 0 if and only if ρ = σ,
3. $Ᵽ_{eb}$(ρ, σ) = $Ᵽ_{eb}$(σ, ρ),
4. $Ᵽ_{eb}$(ρ, σ) ≤ α(ρ, σ) [$Ᵽ_{eb}$(ρ, δ) + $Ᵽ_{eb}$(δ, σ)], for all ρ, σ, δ ε Ħ.
A pair (Ħ, $Ᵽ_{eb}$) is called a extended b-metric space. It is clear that extension b-metric space is an extension of b-metric metric space.

Definition 2.3 [6] Let Ħ≠ ϕ and given a function α : Ħ x Ħ → [1,+ ∞). Let $Ᵽ_c$: Ħx Ħ → [0,+ ∞) be a function is called controlled metric if 1. $Ᵽ_c$(ρ, σ) ≥ 0,
2. $Ᵽ_c$(ρ, σ) = 0 if and only if ρ = σ,
3. $Ᵽ_c$(ρ, σ) = $Ᵽ_c$(σ, ρ),
4. $Ᵽ_c$(ρ, σ) ≤ α(ρ, δ) $Ᵽ_c$(ρ, δ) + α(δ,σ) $Ᵽ_c$(δ, σ), for all ρ, σ, δ ε Ħ.
A pair (Ħ, $Ᵽ_c$) is called a controlled metric space. It is clear that controlled space is an b- metric and extension b- metric space.

Definition 2.4 [7] Let Ħ ≠ ϕ and given a function α, β : Ħ x Ħ → [1,+ ∞). Let $Ᵽ_{dc}$: Ħ x Ħ → [0,+ ∞) be a function .is called double controlled metric if
1. $Ᵽ_{dc}$(ρ, σ) ≥ 0,
2. $Ᵽ_{dc}$(ρ, σ) = 0 if and only if ρ = σ,
3. $Ᵽ_{dc}$(ρ, σ) = $Ᵽ_{dc}$(σ, ρ),
4. $Ᵽ_{dc}$(ρ, σ) ≤ α(ρ, δ) $Ᵽ_{dc}$(ρ, δ) + β(δ,σ) $Ᵽ_{dc}$(δ, σ), for all ρ, σ, δ ε Ħ.
A pair (Ħ, $Ᵽ_{dc}$) is called a double controlled metric space. It is clear that double controlled space is an b- metric, extension b- metric and controlled metric space

Definition 2.5 [14] Let Ħ ≠ ϕ and α ≥ 1 be a given real number. Let $Ᵽ_{cvb}$: Ħ x Ħ → C be a function called complex valued b- metric if 1. $Ᵽ_{cvb}$(ρ, σ) ≥ 0,
2. $Ᵽ_{cvb}$(ρ, σ) = 0 if and only if ρ = σ,
3. $Ᵽ_{cvb}$(ρ, σ) = $Ᵽ_{cvb}$(σ, ρ),
4. $Ᵽ_{cvb}$(ρ, σ) ≤ α [$Ᵽ_{cvb}$(ρ, δ) + $Ᵽ_{cvb}$(δ, σ)], for all ρ, σ, δ ε Ħ.
A pair (Ħ, $Ᵽ_{cvb}$) is called complex valued b-metric space. Complex-valued b-metric space is an extension of complex-valued metric space.

Definition 2.6 [7] Let Ħ ≠ ϕ and α : Ħ x Ħ → [1,+ ∞). Let $Ᵽ_{ecvb}$: Ħ x Ħ → C be a function is called complex valued extended b- metric if 1. $Ᵽ_{ecvb}$(ρ, σ) ≥ 0,
2. $Ᵽ_{ecvb}$(ρ, σ) = 0 if and only if ρ = σ,
3. $Ᵽ_{ecvb}$(ρ, σ) = $Ᵽ_{ecvb}$(σ, ρ),
4. $Ᵽ_{ecvb}$(ρ, σ) ≤ α(ρ, σ) [$Ᵽ_{ecvb}$(ρ, δ) + $Ᵽ_{ecvb}$(δ, σ)], for all ρ, σ, δ ε Ħ.
A pair (Ħ, $Ᵽ_{ecvb}$) is called complex valued extended b-metric space. It is clear that complex vaued extended b-metric space is an extension of complex valued b- metric space.

Definition 2.7 [21] Let Ħ ≠ ϕ and α : Ħ x Ħ → [1,+ ∞). Let $Ᵽ_{cvc}$: Ħ x Ħ → C be a function is called complex valued controlled metric space if
1. $Ᵽ_{cvc}$(ρ, σ) ≥ 0,
2. $Ᵽ_{cvc}$(ρ, σ) = 0 if and only if ρ = σ,
3. $Ᵽ_{cvc}$(ρ, σ) = $Ᵽ_{cvc}$(σ, ρ),
4. $Ᵽ_{cvc}$(ρ, σ) ≤ α(ρ, σ) [$Ᵽ_{cvc}$(ρ, δ) + $Ᵽ_{cvc}$(δ, σ)], for all ρ, σ, δ ε Ħ.
A pair (Ħ, $Ᵽ_{cvc}$) is called complex valued controlled metric space. It is clear that complex valued controlled metric space is an extension of complex valued b- metric space and complex valued extended b- metric space.

Definition 2.8 [18] Let Ħ ≠ ϕ and α,β : Ħ x Ħ → [1,+ ∞). Let $Ᵽ_{cvdc}$: Ħ x Ħ → C be a function is called complex valued double controlled metric space if

1. Ᵽ$_{cvdc}$(ρ, σ) ≥ 0,
2. Ᵽ$_{cvdc}$(ρ, σ) = 0 if and only if ρ = σ,
3. Ᵽ$_{cvdc}$(ρ, σ) = Ᵽ$_{cvdc}$(σ, ρ),
4. Ᵽ$_{cvc}$(ρ, σ) ≤ α(ρ, δ) Ᵽ$_{cvdc}$(ρ, δ) + α(δ, σ) Ᵽ$_{cvdc}$(δ, σ)], for all ρ, σ, δ ε Ḩ.

A pair (Ḩ, Ᵽ$_{cvdc}$) is called complex valued double controlled metric space. It is clear that complex valued doble controlled metric space is an extension of complex valued b- metric space , complex valued extended b- metric space and complex valued controlled metric space

Definition 2.9 [16]. Let Ḩ ≠ ϕ and α : Ḩ x Ḩ → [1,+ ∞). Let Ᵽ$_{cvcl}$: Ḩ x Ḩ → [0, ∞) be a function is called complex valued controlled metric like- space if

1. Ᵽ$_{cvcl}$(ρ, σ) = 0 implies ρ = σ,
2. Ᵽ$_{cvcl}$(ρ, σ) = Ᵽ$_{cvcl}$(σ, ρ),
3. Ᵽ$_{cvcl}$(ρ, σ) ≤ α(ρ, δ) Ᵽ$_{cvcl}$(ρ, δ) + α(δ, σ) Ᵽ$_{cvcl}$(δ, σ)], for all ρ, σ, δ ε Ḩ.

A pair (Ḩ, Ᵽ$_{cvcl}$) is called complex valued controlled metric- like space.

Definition2.10 [19] Let Ḩ ≠ ϕ and α,β : Ḩ x Ḩ → [1,+ ∞). Let Ᵽ$_{cvdcl}$: Ḩ x Ḩ → [0, ∞) be a function called complex valued double controlled metric like- space if

A$_1$. Ᵽ$_{cvdcl}$ (ρ, σ) = 0 => ρ = σ,
A$_2$. Ᵽ$_{cvdcl}$ (ρ, σ) = Ᵽ$_{cvdcl}$ (σ, ρ),
A$_3$. Ᵽ$_{cvdcl}$ (ρ, σ) ≤ α(ρ, δ) Ᵽ$_{cvdcl}$ (ρ, δ) + β(δ, σ) Ᵽ$_{cvdcl}$ (δ, σ), for all ρ, σ, δ ε Ḩ.

A pair (Ḩ, Ᵽ$_{cvdcl}$) is a complex-valued double controlled metric like space. A complex-valued double-controlled metric-like space is an extension of a complex-valued controlled metric-like space.

A complex-valued double controlled metric- type space is also a complex-valued double controlled metric –space in general. The converse is not true in general. Further, this is also a more generalized form than complex-valued extended b–metric–type space.

Example 2.1 [19] Let Ḩ = {1,2,3}. Consider the complex-valued double controlled metric- like Ᵽ$_{cvdcl}$, defined by

Ᵽ$_{cvdcl}$ **(ρ, σ)**	**1**	**2**	**3**
1	0	2+4i	1-i
2	2+4i	0	1
3	1-i	1	i/2

Take α, β : Ḩ x Ḩ → [1, ∞) to be symmetric and defined by

α (ρ, σ)	**1**	**2**	**3**
1	1	6/5	151/100
2	6/5	1	8/5
3	151/100	8/5	1

β (ρ, σ)	**1**	**2**	**3**
1	1	6/5	8/3
2	6/5	1	33/20
3	8/3	33/20	1

One can easily show that (Ḩ, Ᵽ$_{cvdcl}$) is double controlled metric–like space rather than a controlled metric-type space. when ρ = 2, δ = 3, σ = 1,

I Ᵽ$_{cvdcl}$ (ρ,σ) I = I Ᵽ$_{cvdcl}$ (2,1) I = $\sqrt{20}$ ≥ 6$[1+\sqrt{2}]/5$ = α (2,1)[I Ᵽ$_{cvdcl}$ (2,3) I + I Ᵽ$_{cvdcl}$ (3,1) I] = α (ρ,σ) I Ᵽ$_{cvdcl}$ (ρ,δ) I + I Ᵽ$_{cvdcl}$ (δ,σ) I]. Thus, Ᵽ$_{cvdcl}$ is not a complex-valued extended b- metric space.

Definition 2.11 [19] Let (Ħ, Ᵽ$_{cvdcl}$) be a complex-valued double controlled metric like space by one or two functions.

1. The sequence {σ$_n$} is convergent to some σ in Ḩ if for each positive ε, there is some integer Z$_ε$ such that Ᵽ$_{cvdcl}$ (σ$_n$,σ) < ε for each n ≥ Z$_ε$.

It is written as $\lim_{n\to\infty}$ σ$_n$ = σ.

2. The sequence $\{\sigma_n\}$ is said Cauchy, if for every $\varepsilon > 0$, $Ꝑ_{cvdcl}(\sigma_n,\sigma_m) < \varepsilon$ for all $m,n \geq Z_\varepsilon$, where Z_ε is some integer.

3. (Ή, $Ꝑ_{cvdcl}$) is complete if every Cauchy sequence converges.

Definition 2.12 [19 **]** Let (Ή, $Ꝑ_{cvdcl}$) be a complex-valued double controlled metric like space by one or two functions for σ ε Ӈ and $l > 0$.

1. $B(\sigma, l) = [\, y$ εӇ, $Ꝑ_{cvdcl}(\sigma, y) < l \,\}$.

2. The self-mapping Ţ on Ӈ is said to be continuous at σ in Ӈ if, for all $\delta > 0$, there exists $l > 0$ such that Ţ$(B(\sigma, l)) \subseteq B($Ţ$\sigma,\delta)$.

If Ţ is continuous at σ in (Ӈ, $Ꝑ_{cvdcl}$), then $\sigma_n \to \sigma$ implies that Ţ$\sigma_n \to$Ţσ when $n \to\infty$.

Lemma 2.1. [19] Let (Ή, $Ꝑ_{cvdcl}$) be a complex-valued double controlled metric like space and assume a sequence $\{\sigma_n\}$ in Ӈ. Then $\{\sigma_n\}$ is Cauchy sequence $\Leftrightarrow Ꝑ_{cvdcl}(\sigma_n,\sigma_m) \to 0$ as $n, m \to\infty$ where n,m ε N.

Lemma 2.2 [19] Let (Ή, $Ꝑ_{cvdcl}$) be a complex-valued double controlled metric like space . Then a sequence $\{\sigma_n\}$ in Ӈ is a Cauchy sequence, such that $\sigma_n \neq \sigma_m$, whenever $m \neq n$. Then $\{\sigma_n\}$ converges to at most one point.

Lemma 2.3 [19] Let (Ή, $Ꝑ_{cvdcl}$) be a complex-valued double controlled metric like space and assume a sequence $\{\sigma_n\}$ in Ӈ. Then $\{\sigma_n\}$ is converges to σ $\Leftrightarrow Ꝑ_{cvdcl}(\sigma_n,\sigma) \to 0$ as $n\to\infty$.

Lemma2.3[19] For a given complex-valued controlled space (Ӈ, $Ꝑ_{cvdcl}$), the complex-valued double controlled metric like function Ꝑ: Ӈ x Ӈ → C is continuous, with respect to the partial order " ≤ ".

Lemma 2.4[19] Let (Ή, $Ꝑ_{cvdcl}$) be a complex-valued double controlled metric like space. The limit of every convergent sequence in Ӈ is unique if the functional $Ꝑ_{cvdcl}$: Ӈ x Ӈ →Ӈ is continuous.

3. Result

This section provides several new interpolative contractions, such as the (λ, a)-interpolative Kannan contraction, the (λ, a, b)-interpolative Kannan contraction, and the (λ, a, b, c)-interpolative Reich contraction in complex-valued double controlled metric –like space, with a given theorem on the Kannan contraction and Reich contraction with examples to support the theorem.

Definition 3.1 Let (Ή, $Ꝑ_{cvdcl}$) be a complex, double-controlled metric-like space. Let Ţ: Ӈ → Ӈ be self mapping. We shall Ţ a (k, a) – interpolative Kannan contraction if exists k ε [0,1). A ε (0,1) such that

$$Ꝑ_{cvdcl}(\text{Ţ}\sigma, \text{Ţ}\rho) \leq k\,(Ꝑ_{cvdcl}(\sigma, \text{Ţ}\sigma))^a\,(Ꝑ_{cvdcl}(\rho, \text{Ţ}\rho))^{1-a} \qquad 3.1$$

for all σ, ρ ε Ӈ , with $\sigma \neq \rho$.

Definition 3.2 Let (Ή, $Ꝑ_{cvdcl}$) be a complex, double-controlled metric-like space. Let Ţ: Ӈ → Ӈ be self mapping. We shall Ţ a (k, a, b) – interpolative Kannan contraction if there exists k ε [0,1), a, b ε (0,1) , $a + b < 1$ such that

$$Ꝑ_{cvdcl}(\text{Ţ}\sigma, \text{Ţ}\rho) \leq k\,(Ꝑ_{cvdcl}(\sigma, \text{Ţ}\sigma))^a\,(Ꝑ_{cvdcl}(\rho, \text{Ţ}\rho))^{b} \qquad 3.2$$

for all σ, ρ ε Ӈ , with $\sigma \neq \rho$.

Definition 3.3 Let (Ή, $Ꝑ_{cvdcl}$) be a complex-valued double controlled metric like space. Let Ţ: Ӈ → Ӈ be self mapping. We shall Ţ a (k, a, b, c) – interpolative Riech contraction if there exists k ε [0,1), a, b, c ε (0,1) , $a + b + c < 1$ such that

$$Ꝑ_{cvdcl}(\text{Ţ}\sigma, \text{Ţ}\rho) \leq k\,(Ꝑ_{cvdcl}(\sigma, \rho))^a\,(Ꝑ_{cvdcl}(\sigma, \text{Ţ}\sigma))^b\,(Ꝑ_{cvdcl}(\rho, \text{Ţ}\rho))^{c} \qquad 3.3$$

for all σ, ρ ε Ӈ , with $\sigma \neq \rho$.

Theorem 3.1 Let (Ή, $Ꝑ_{cvdcl}$) be a complete complex valued double controlled metric like space. Let Ţ: Ӈ → Ӈ be self mapping. We shall Ţ a (k, a) – interpolative Kannan contraction. For σ_0 ε Ӈ, take $\sigma_n = \text{Ţ}^n\sigma_0$. Assume that

$$\sup_{m\geq 1}\lim_{i\to\infty} \alpha(\sigma_{i+1}, \sigma_{i+2})\, \beta(\sigma_{i+1}, \sigma_m)/\, \alpha(\sigma_i, \sigma_{i+1}) < 1/k \qquad 3.4$$

Then Ţ has a fixed point.

Proof. Let σ_0 ε Ӈ be an initial point. Define a sequence { σ_n} as σ_{n+1} = Ţσ_n for all n ε N. Obviously, if there exists n_0 ε N for which $\sigma_{n0+1} = \sigma_{n0}$, then Ţ$\sigma_{n0} = \sigma_{n0}$, and the proof is complete. Thus, we suppose that $\sigma_{n+1} \neq \sigma_n$ for each n ε N. Thus, by 3.1, we have

$$Ꝑ_{cvdcl}(\sigma_n, \sigma_{n+1}) = Ꝑ_{cvdcl}(Ţ\sigma_{n-1}, Ţ\sigma_n) \leq k\ (Ꝑ_{cvdcl}(\sigma_{n-1}, Ţ\sigma_{n-1}))^a\ (Ꝑ_{cvdcl}(\sigma_n, Ţ\sigma_n))^{1-a}$$
$$= k\ (Ꝑ_{cvdcl}(\sigma_{n-1}, \sigma_n))^a\ (Ꝑ_{cvdcl}(\sigma_n, \sigma_{n+1}))^{1-a}$$

$$(Ꝑ_{cvdcl}(\sigma_n, \sigma_{n+1}))^a \leq k(Ꝑ_{cvdcl}(\sigma_{n-1}, \sigma_n))^a \qquad 3.5$$

Since a < 1, we have

$$Ꝑ_{cvdcl}(\sigma_n, \sigma_{n+1}) \leq k^{1/a}(Ꝑ_{cvdcl}(\sigma_{n-1}, \sigma_n)) \leq k\ Ꝑ_{cvdcl}(\sigma_{n-1}, \sigma_n)$$

$$Ꝑ_{cvdcl}(\sigma_n, \sigma_{n+1}) \leq k\ Ꝑ_{cvdcl}(\sigma_{n-1}, \sigma_n) \leq k^2\ Ꝑ_{cvdcl}(\sigma_{n-2}, \sigma_{n-1}) \leq k^3\ Ꝑ_{cvdcl}(\sigma_{n-3}, \sigma_{n-2}) \ldots \leq k^n\ Ꝑ_{cvdcl}(\sigma_0, \sigma_1) \qquad 3.6$$

For all n,m ε N and n < m, we have

$$Ꝑ_{cvdcl}(\sigma_n, \sigma_m) \leq \alpha(\sigma_n, \sigma_{n+1})\ Ꝑ_{cvdcl}(\sigma_n, \sigma_{n+1}) + \beta(\sigma_{n+1}, \sigma_m)\ Ꝑ_{cvdcl}(\sigma_{n+1}, \sigma_m)$$

$$\leq \alpha(\sigma_n, \sigma_{n+1})\ Ꝑ_{cvdcl}(\sigma_n, \sigma_{n+1}) + \beta(\sigma_{n+1}, \sigma_m)\ \{\ \alpha(\sigma_{n+1}, \sigma_{n+2})\ Ꝑ_{cvdcl}(\sigma_{n+1}, \sigma_{n+2}) + \beta(\sigma_{n+2}, \sigma_m)\ Ꝑ_{cvdcl}(\sigma_{n+2}, \sigma_m)\ \}$$

$$= \alpha(\sigma_n, \sigma_{n+1})\ Ꝑ_{cvdcl}(\sigma_n, \sigma_{n+1}) + \beta(\sigma_{n+1}, \sigma_m)\ \alpha(\sigma_{n+1}, \sigma_{n+2})\ Ꝑ_{cvdcl}(\sigma_{n+1}, \sigma_{n+2})$$

$$+ \beta(\sigma_{n+1}, \sigma_m)\ \beta(\sigma_{n+2}, \sigma_m)\ Ꝑ_{cvdcl}(\sigma_{n+2}, \sigma_m)$$

$$\leq \alpha(\sigma_n, \sigma_{n+1})\ Ꝑ_{cvdcl}(\sigma_n, \sigma_{n+1}) + \beta(\sigma_{n+1}, \sigma_m)\ \alpha(\sigma_{n+1}, \sigma_{n+2})\ Ꝑ_{cvdcl}(\sigma_{n+1}, \sigma_{n+2})$$

$$+ \beta(\sigma_{n+1}, \sigma_m)\ \beta(\sigma_{n+2}, \sigma_m)\ \alpha(\sigma_{n+2}, \sigma_{n+3})\ Ꝑ_{cvdcl}(\sigma_{n+2}, \sigma_{n+3})$$

$$+ \beta(\sigma_{n+1}, \sigma_m)\ \beta(\sigma_{n+2}, \sigma_m)\ \beta(\sigma_{n+3}, \sigma_m)\ Ꝑ_{cvdcl}(\sigma_{n+3}, \sigma_m)$$

$$\leq \alpha(\sigma_n, \sigma_{n+1})\ Ꝑ_{cvdcl}(\sigma_n, \sigma_{n+1}) + \sum_{i=n+1}^{m-2}\Big(\prod_{j=n+1}^{i} \beta(\sigma_j, \sigma_m)\Big)\ \alpha(\sigma_i, \sigma_{i+1})\ Ꝑ_{cvdcl}(\sigma_i, \sigma_{i+1})$$

$$+ \prod_{k=n+1}^{m-1} \beta(\sigma_k, \sigma_m)\ Ꝑ_{cvdcl}(\sigma_{m-1}, \sigma_m)$$

$$\leq \alpha(\sigma_n, \sigma_{n+1})\ k^n\ Ꝑ_{cvdcl}(\sigma_0, \sigma_1) + \sum_{i=n+1}^{m-2}\Big(\prod_{j=n+1}^{i} \beta(\sigma_j, \sigma_m)\Big)\ \alpha(\sigma_i, \sigma_{i+1})\ k^i\ Ꝑ_{cvdcl}(\sigma_o, \sigma_1)$$

$$+ \prod_{k=n+1}^{m-1} \beta(\sigma_k, \sigma_m)\ k^{m-1}\ Ꝑ_{cvdcl}(\sigma_o, \sigma_1)$$

$$\leq \alpha(\sigma_n, \sigma_{n+1})\ k^n\ Ꝑ_{cvdcl}(\sigma_0, \sigma_1) + \sum_{i=n+1}^{m-1}\Big(\prod_{j=n+1}^{i} \beta(\sigma_j, \sigma_m)\Big)\ \alpha(\sigma_i, \sigma_{i+1})\ k^i\ Ꝑ_{cvdcl}(\sigma_o, \sigma_1) \ \ldots \qquad 3.7$$

$$\text{Let } S_i = \sum_{i=0}^{i}\Big(\prod_{j=0}^{i} \beta(\sigma_j, \sigma_m)\Big)\ \alpha(\sigma_i, \sigma_{i+1})\ k^i\ Ꝑ_{cvdcl}(\sigma_o, \sigma_1). \ \ldots \qquad 3.8$$

$$\text{Consider } V_i = \prod_{j=0}^{i} \beta(\sigma_j, \sigma_m)\ \alpha(\sigma_i, \sigma_{i+1}) k^i\ Ꝑ(\sigma_o, \sigma_1). \ \ldots \qquad 3.9$$

We have $V_{i+1} / V_i = \beta(\sigma_{i+1}, \sigma_m)\ \alpha(\sigma_{i+1}, \sigma_{i+2})k / \alpha(\sigma_i, \sigma_{i+1})$.

In view of condition (3.1) and the ratio test, we ensure that the series. $\sum_i V_i$ converges. Thus

$\lim_{n\to\infty} S_n$ exists. Hence , the real sequence { S_n} is Cauchy. Now, using (3.7), we have

$$Ꝑ_{cvdcl}(\sigma_n, \sigma_m) \leq Ꝑ_{cvdcl}(\sigma_o, \sigma_1)\ \ [k^n\ \alpha(\sigma_n, \sigma_{n+1}) + S_{m-1} - S_n\ ]. \ \ldots \qquad 3.10$$

Above, we used $\alpha(\sigma, \rho) \geq 1$. Letting n,m →∞ in (3.10) we obtain

$$\text{Lim}_{m,n\to\infty}\ Ꝑ_{cvdcl}(\sigma_n, \sigma_m) = 0. \ \ldots \qquad 3.11$$

Thus, the sequence { σ_n} ia a Cauchy in the complete double metric space (Ӈ,$Ꝑ_{cvdcl}$), so there is some σ^*ε Ӈ so the

$$\lim_{n\to\infty} Ᵽ_{cvdcl}(\sigma_n, \sigma^*) = 0. \quad \ldots \qquad 3.12$$

That is $\sigma_m \to \sigma^*$ as $n\to\infty$. Now we prove that σ^* is a fixed point of Ӈ. By (3.1) and condition (4) in def[2.10], we get

$Ᵽ_{cvdcl}(\sigma^*, Ţ\sigma^*) \le \alpha(\sigma^*, \sigma_{n+1})\ Ᵽ_{cvdcl}(\sigma^*, \sigma_{n+1}) + \beta(\sigma_{n+1}, Ţ\sigma^*)\ Ᵽ_{cvdcl}(\sigma_{n+1}, Ţ\sigma^*)$

$= \alpha(\sigma^*, \sigma_{n+1})\ Ᵽ_{cvdcl}(\sigma^*, \sigma_{n+1}) + \beta(\sigma_{n+1}, Ţ\sigma^*)\ Ᵽ_{cvdcl}(Ţ\sigma_n, Ţ\sigma^*)$

$\le \alpha(\sigma^*, \sigma_{n+1})\ Ᵽ_{cvdcl}(\sigma^*, \sigma_{n+1}) + \beta(\sigma_{n+1}, Ţ\sigma^*)\ [k(Ᵽ_{cvdcl}(\sigma_n, Ţ\sigma_n))^a (Ᵽ_{cvdcl}(\rho^*, Ţ\sigma^*))^{1-a}]$

$\le \alpha(\sigma^*, \sigma_{n+1})\ Ᵽ_{cvdcl}(\sigma^*, \sigma_{n+1}) + \beta(\sigma_{n+1}, Ţ\sigma^*)\ [k(Ᵽ_{cvdcl}(\sigma_n, \sigma_{n+1}))^a (Ᵽ_{cvdcl}(\sigma^*, Ţ\sigma^*))^{1-a}]$.

Taking $n\to\infty$ and using (3.11), (3.12) we obtain that

$$Ᵽ_{cvdcl}(\sigma^*, Ţ\sigma^*) = 0. \quad \ldots \qquad 3.13$$

Implies, $\sigma^* = Ţ\sigma^*$.

Now, we prove the uniqueness of σ^*. Let ρ^* be another fixed point of Ţ inӇ, then

by 3.1, we have

$Ᵽ_{cvdcl}(\sigma^*, \rho^*) = Ᵽ_{cvdcl}(Ţ\sigma^*, Ţ\rho^*) \le k\,[Ᵽ_{cvdcl}(\sigma^*, Ţ\sigma^*)]^a [Ᵽ_{cvdcl}(\rho^*, Ţ\rho^*)]^{1-a} = 0.$

Implies , $\sigma^* = \rho^*$. Complete the proof.

Example 2.1 Let Ӈ = {1,2,3}. Consider the complex-valued double controlled metric- like $Ᵽ_{cvdcl}$ defined by

$Ᵽ_{cvdcl}$ **(ρ, σ)**	**1**	**2**	**3**
1	0	2-4i	1+i
2	2-4i	0	1
3	1+i	1	i/2

Take α, β: Ӈ x Ӈ → [1, ∞) to be symmetric and defined by

α (ρ, σ)	**1**	**2**	**3**
1	1	6/5	151/100
2	6/5	1	8/5
3	151/100	8/5	1

β (ρ, σ)	**1**	**2**	**3**
1	1	6/5	8/3
2	6/5	1	33/20
3	8/3	33/20	1

Now we define the self mapping Ţ on Ӈ as follows Ţ1 = Ţ2 = Ţ3 = 2.

Now we verify the first condition of theorem 3.1

Case 1. σ = 1, ρ = 2
$I\,Ᵽ_{cvdcl}(Ţ\sigma, Ţ\rho)I = I\,Ᵽ_{cvdcl}(Ţ1, Ţ2)I = I\,Ᵽ_{cvdcl}(2,2)I = 0 \le k\, I(Ᵽ_{cvdcl}(1, Ţ1))I^a\, I(Ᵽ_{cvdcl}(2, Ţ2))I^{1-a} = 0$
$= k\, I(Ᵽ_{cvdcl}(\sigma, Ţ\sigma))I^a\, I(Ᵽ_{cvdcl}(\rho, Ţ\rho))I^{1-a}$

Case 2. σ = 1, ρ = 3
$I\,Ᵽ_{cvdcl}(Ţ\sigma, Ţ\rho)I = I\,Ᵽ_{cvdcl}(Ţ1, Ţ3)I = I\,Ᵽ_{cvdcl}(2,2)I = 0 \le k\, I(Ᵽ_{cvdcl}(1, Ţ1))I^a\, I(Ᵽ_{cvdcl}(3, Ţ3))I^{1-a} = k\, I(Ᵽ_{cvdcl}(1, 2))I^a$
$I(Ᵽ_{cvdcl}(3, 2))I^{1-a} = k\,(\sqrt{20})^a \cdot 1 = k\, I(Ᵽ_{cvdcl}(\sigma, Ţ\sigma))I^a\, I(Ᵽ_{cvdcl}(\rho, Ţ\rho))I^{1-a}$

Case 3. $\sigma = 2, \rho = 1$

$I\, \text{Ꝑ}_{cvdcl}(\text{Ţ}\sigma, \text{Ţ}\rho)I = I\, \text{Ꝑ}_{cvdcl}(\text{Ţ}2, \text{Ţ}1)I = I\, \text{Ꝑ}_{cvdcl}(2,2)I = 0 \leq k\, I(\text{Ꝑ}_{cvdcl}(2, \text{Ţ}2))I^{a}\, I(\text{Ꝑ}_{cvdcl}(1, \text{Ţ}1))I^{1-a} = k\, I(\text{Ꝑ}_{cvdcl}(2, 2))I^{a}\, I(\text{Ꝑ}_{cvdcl}(1, 2))I^{1-a} = 0 = k\, I(\text{Ꝑ}_{cvdcl}(\sigma, \text{Ţ}\sigma))I^{a}\, I(\text{Ꝑ}_{cvdcl}(\rho, \text{Ţ}\rho))I^{1-a}$

Case 4. $\sigma = 2, \rho = 3$

$I\, \text{Ꝑ}_{cvdcl}(\text{Ţ}\sigma, \text{Ţ}\rho)I = I\, \text{Ꝑ}_{cvdcl}(\text{Ţ}2, \text{Ţ}3)I = I\, \text{Ꝑ}_{cvdcl}(2,2)I = 0 \leq k\, I(\text{Ꝑ}_{cvdcl}(2, \text{Ţ}2))I^{a}\, I(\text{Ꝑ}_{cvdcl}(3, \text{Ţ}3))I^{1-a} = k\, I(\text{Ꝑ}_{cvdcl}(2, 2))I^{a}\, I(\text{Ꝑ}_{cvdcl}(3, 2))I^{1-a} = 0 = k\, I(\text{Ꝑ}_{cvdcl}(\sigma, \text{Ţ}\sigma))I^{a}\, I(\text{Ꝑ}_{cvdcl}(\rho, \text{Ţ}\rho))I^{1-a}$

Case 5. $\sigma = 3, \rho = 1$

$I\, \text{Ꝑ}_{cvdcl}(\text{Ţ}\sigma, \text{Ţ}\rho)I = I\, \text{Ꝑ}_{cvdcl}(\text{Ţ}3, \text{Ţ}1)I = I\, \text{Ꝑ}_{cvdcl}(2,2)I = 0 \leq k\, I(\text{Ꝑ}_{cvdcl}(3, \text{Ţ}3))I^{a}\, I(\text{Ꝑ}_{cvdcl}(1, \text{Ţ}1))I^{1-a} = k\, I(\text{Ꝑ}_{cvdcl}(3, 2))I^{a}\, I(\text{Ꝑ}_{cvdcl}(1, 2))I^{1-a} = k\, 1.\, (\sqrt{20})^{1-a} = k\, I(\text{Ꝑ}_{cvdcl}(\sigma, \text{Ţ}\sigma))I^{a}\, I(\text{Ꝑ}_{cvdcl}(\rho, \text{Ţ}\rho))I^{1-a}$

Case 6. $\sigma = 3, \rho = 2$

$I\, \text{Ꝑ}_{cvdcl}(\text{Ţ}\sigma, \text{Ţ}\rho)I = I\, \text{Ꝑ}_{cvdcl}(\text{Ţ}3, \text{Ţ}2)I = I\, \text{Ꝑ}_{cvdcl}(2,2)I = 0 \leq k\, I(\text{Ꝑ}_{cvdcl}(3, \text{Ţ}3))I^{a}\, I(\text{Ꝑ}_{cvdcl}(2, \text{Ţ}2))I^{1-a} = k\, I(\text{Ꝑ}_{cvdcl}(3, 2))I^{a}\, I(\text{Ꝑ}_{cvdcl}(2, 2))I^{1-a} = 0 = k\, I(\text{Ꝑ}_{cvdcl}(\sigma, \text{Ţ}\sigma))I^{a}\, I(\text{Ꝑ}_{cvdcl}(\rho, \text{Ţ}\rho))I^{1-a}$

Case 7. $\sigma = 1, \rho = 1$

$I\, \text{Ꝑ}_{cvdcl}(\text{Ţ}\sigma, \text{Ţ}\rho)I = I\, \text{Ꝑ}_{cvdcl}(\text{Ţ}1, \text{Ţ}1)I = I\, \text{Ꝑ}_{cvdcl}(2,2)I = 0 \leq k\, I(\text{Ꝑ}_{cvdcl}(1, \text{Ţ}1))I^{a}\, I(\text{Ꝑ}_{cvdcl}(1, \text{Ţ}1))I^{1-a} = k\, I(\text{Ꝑ}_{cvdcl}(1, 2))I^{a}\, I(\text{Ꝑ}_{cvdcl}(1, 2))I^{1-a} = k\, (\sqrt{20})^{1} = k\, I(\text{Ꝑ}_{cvdcl}(\sigma, \text{Ţ}\sigma))I^{a}\, I(\text{Ꝑ}_{cvdcl}(\rho, \text{Ţ}\rho))I^{1-a}$

Case 8. $\sigma = 2, \rho = 2$

$I\, \text{Ꝑ}_{cvdcl}(\text{Ţ}\sigma, \text{Ţ}\rho)I = I\, \text{Ꝑ}_{cvdcl}(\text{Ţ}2, \text{Ţ}2)I = I\, \text{Ꝑ}_{cvdcl}(2,2)I = 0 \leq k\, I(\text{Ꝑ}_{cvdcl}(2, \text{Ţ}2))I^{a}\, I(\text{Ꝑ}_{cvdcl}(2, \text{Ţ}2))I^{1-a} = k\, I(\text{Ꝑ}_{cvdcl}(2, 2))I^{a}\, I(\text{Ꝑ}_{cvdcl}(2, 2))I^{1-a} = 0 = k\, I(\text{Ꝑ}_{cvdcl}(\sigma, \text{Ţ}\sigma))I^{a}\, I(\text{Ꝑ}_{cvdcl}(\rho, \text{Ţ}\rho))I^{1-a}$

Case 9. $\sigma = 3, \rho = 3$

$I\, \text{Ꝑ}_{cvdcl}(\text{Ţ}\sigma, \text{Ţ}\rho)I = I\, \text{Ꝑ}_{cvdcl}(\text{Ţ}3, \text{Ţ}3)I = I\, \text{Ꝑ}_{cvdcl}(2,2)I = 0 \leq k\, I(\text{Ꝑ}_{cvdcl}(3, \text{Ţ}3))I^{a}\, I(\text{Ꝑ}_{cvdcl}(3, \text{Ţ}3))I^{1-a} = k\, I(\text{Ꝑ}_{cvdcl}(3, 2))I^{a}\, I(\text{Ꝑ}_{cvdcl}(3, 2))I^{1-a} = 1 = k\, I(\text{Ꝑ}_{cvdcl}(\sigma, \text{Ţ}\sigma))I^{a}\, I(\text{Ꝑ}_{cvdcl}(\rho, \text{Ţ}\rho))I^{1-a}$

For all an $\varepsilon\ (0,1)$, hence, all the above conditions are satisfied, and these conditions are also satisfied for $\text{Ţ}1 = \text{Ţ}2 = \text{Ţ}3 = 1$. For any $\sigma_0\ \varepsilon\ \text{Ꜧ}$, the second condition of theorem 3.1 holds. Therefore, a fixed point exists at 1.

Theorem3.2 Let (Ħ, Ꝑ_{cvdcl}) be a complete complex valued double controlled metric like space. Let $\text{Ţ}: \text{Ꜧ} \to \text{Ꜧ}$ be self mapping. We shall Ţ a (k, a,b) – interpolative Kannan contraction. For $\sigma_0\ \varepsilon\ \text{Ꜧ}$, take $\sigma_n = \text{Ţ}^n\sigma_0$. Assume that

$$\sup_{m\geq 1}\lim_{i\to\infty} \alpha(\sigma_{i+1}, \sigma_{i+2})\, \beta(\sigma_{i+1}, \sigma_m)/\, \alpha(\sigma_i, \sigma_{i+1}) < 1/k \qquad 3.15$$

Then Ţ has a fixed point.

Proof. Let $\sigma_0\ \varepsilon\ \text{Ꜧ}$ be initial point. Define a sequence $\{\sigma_n\}$ as $\sigma_{n+1} = \text{Ţ}\sigma_n$ for all n ε N. Obviously, if there exists n_0 ε N for which $\sigma_{n0+1} = \sigma_{n0}$, then $\text{Ţ}\sigma_{n0} = \sigma_{n0}$, and the proof is complete. Thus, we suppose that $\sigma_{n+1} \neq \sigma_n$ for each n ε N. Thus ,by 3.2, we have

$$\text{Ꝑ}_{cvdcl}(\sigma_n, \sigma_{n+1}) = \text{Ꝑ}_{cvdcl}(\text{Ţ}\sigma_{n-1}, \text{Ţ}\sigma_n) \leq k\, (\text{Ꝑ}_{cvdcl}(\sigma_{n-1}, \text{Ţ}\sigma_{n-1}))^{a}\, (\text{Ꝑ}_{cvdcl}(\sigma_n, \text{Ţ}\sigma_n))^{b} = k\, (\text{Ꝑ}_{cvdcl}(\sigma_{n-1}, \sigma_n))^{a}\, (\text{Ꝑ}_{cvdcl}(\sigma_n, \sigma_{n+1}))^{b}$$
$$(\text{Ꝑ}_{cvdcl}(\sigma_n, \sigma_{n+1}))^{1-b} \leq k(\text{Ꝑ}_{cvdcl}(\sigma_{n-1}, \sigma_n)) \qquad 3.16$$

When,a+b < 1

$$\text{Ꝑ}_{cvdcl}(\sigma_n, \sigma_{n+1}) \leq k^{1/1-b}(\text{Ꝑ}_{cvdcl}(\sigma_{n-1}, \sigma_n)) \leq k\, \text{Ꝑ}_{cvdcl}(\sigma_{n-1}, \sigma_n)$$

and then

$$\text{Ꝑ}_{cvdcl}(\sigma_n, \sigma_{n+1}) \leq k\, \text{Ꝑ}_{cvdcl}(\sigma_{n-1}, \sigma_n) \leq k^2\, \text{Ꝑ}_{cvdcl}(\sigma_{n-2}, \sigma_{n-1}) \leq k^3\, \text{Ꝑ}_{cvdcl}(\sigma_{n-3}, \sigma_{n-2}) \ldots \leq k^n\, \text{Ꝑ}_{cvdcl}(\sigma_0, \sigma_1) \qquad 3.17$$

As already elaborated in the proof of theorem 3.1, the classical procedure leads to the existence of a fixed point $\sigma^*\ \varepsilon\ \text{Ꜧ}$.

Theorem 3.3 Let (Ħ, Ꝑ_{cvdcl}) be a complete complex valued double controlled metric like space. Let $\text{Ţ}: \text{Ꜧ} \to \text{Ꜧ}$ be self mapping. We shall Ţ a (k, a,b,c) – interpolative Riech contraction. For $\sigma_0\ \varepsilon\ \text{Ꜧ}$, take $\sigma_n = \text{Ţ}^n\sigma_0$. Assume that

$$\sup_{m\geq 1}\lim_{i\to\infty} \alpha(\sigma_{i+1}, \sigma_{i+2})\, \beta(\sigma_{i+1}, \sigma_m)/\, \alpha(\sigma_i, \sigma_{i+1}) < 1/k \qquad 3.18$$

Then Ţ has a fixed point.

Proof. Let σ_0 ε Ꜧ be an initial point. Define a sequence $\{\sigma_n\}$ as $\sigma_{n+1} = Ţ\sigma_n$ for all n ε N. Obviously, if there exists n_0 ε N for which $\sigma_{n0+1} = \sigma_{n0}$, then $Ţ\sigma_{n0} = \sigma_{n0}$, and the proof is complete. Thus, we suppose that $\sigma_{n+1} \neq \sigma_n$ for each n ε N. Thus, by 3.3, we have

$Ꝑ_{cvdcl}(\sigma_n, \sigma_{n+1}) = Ꝑ_{cvdcl}(Ţ\sigma_{n-1}, Ţ\sigma_n) \leq k\, Ꝑ_{cvdcl}(\sigma_{n-1}, \sigma_n)^a \; (Ꝑ_{cvdcl}(\sigma_{n-1}, Ţ\sigma_{n-1}))^b \; (Ꝑ_{cvdcl}(\sigma_n, Ţ\sigma_n))^c$
$= k\, Ꝑ_{cvdcl}(\sigma_{n-1}, \sigma_n)^a \,(Ꝑ_{cvdcl}(\sigma_{n-1}, \sigma_n))^b \;(Ꝑ_{cvdcl}(\sigma_n, \sigma_{n+1}))^c = k\, Ꝑ_{cvdcl}(\sigma_{n-1}, \sigma_n)^{a+b}\,(Ꝑ_{cvdcl}(\sigma_n, \sigma_{n+1}))^c$

Since a+b < 1-c, the last inequalities gives,

$$(Ꝑ_{cvdcl}(\sigma_n, \sigma_{n+1}))^{1-c} \leq k(Ꝑ_{cvdcl}(\sigma_{n-1}, \sigma_n))^{a+b} \leq k(Ꝑ_{cvdcl}(\sigma_{n-1}, \sigma_n))^{1-c} \quad 3.19$$

$Ꝑ_{cvdcl}(\sigma_n, \sigma_{n+1}) \leq k^{1/1-c}(Ꝑ_{cvdcl}(\sigma_{n-1}, \sigma_n)) \leq k\, Ꝑ_{cvdcl}(\sigma_{n-1}, \sigma_n)$

and then

$Ꝑ_{cvdcl}(\sigma_n, \sigma_{n+1}) \leq k\, Ꝑ_{cvdcl}(\sigma_{n-1}, \sigma_n) \leq k^2\, Ꝑ_{cvdcl}(\sigma_{n-2}, \sigma_{n-1}) \leq k^3\, Ꝑ_{cvdcl}(\sigma_{n-3}, \sigma_{n-2}) \ldots \leq k^n\, Ꝑ_{cvdcl}(\sigma_0, \sigma_1)$ 3.21

As already elobrated in the proof of Theorem 3.1, the classical procedure leads to the existence of a fixed point σ^* ε Ꜧ.

Example 3.2 Let Ꜧ = {1,2,3}. Consider the complex- valued double controlled metric- like $Ꝑ_{cvdcl}$ defined by

$Ꝑ_{cvdcl}$ **(ρ, σ)**	**1**	**2**	**3**
1	0	2+i	1-i
2	2+i	0	i
3	1-i	i	i/2

Take α, β: Ꜧ x Ꜧ → [1, ∞) to be symmetric and defined by

α (ρ, σ)	**1**	**2**	**3**
1	1	1	3/2
2	1	1	8/7
3	3/2	8/7	1

β (ρ, σ)	**1**	**2**	**3**
1	1	7/6	1
2	7/6	1	9/2
3	1	9/2	1

Now we define Ţ: Ꜧ→Ꜧ as follows Ţ1 = Ţ2 = Ţ3 = 2.

Now, we verify the first condition of theorem 3.3.

Case 1. σ = 1, ρ = 2
$I\, Ꝑ_{cvdcl}(Ţ\sigma, Ţ\rho) I = I\, Ꝑ_{cvdcl}(Ţ1, Ţ2) I = I\, Ꝑ_{cvdcl}(2,2) I = 0 \leq k\; I\, Ꝑ_{cvdcl}(1, 2) I^a\, I(Ꝑ_{cvdcl}(1, Ţ1)) I^b\, I(Ꝑ_{cvdcl}(2, Ţ2)) I^c = k\; I\, Ꝑ_{cvdcl}(1, 2) I^a\, I(Ꝑ_{cvdcl}(1, 2)) I^b\, I(Ꝑ_{cvdcl}(2, 2)) I^c = 0 = k\, I(Ꝑ_{cvdcl}(\sigma, \rho)) I^a\, I(Ꝑ_{cvdcl}(\sigma, Ţ\sigma)) I^b\, I(Ꝑ_{cvdcl}(\rho, Ţ\rho)) I^c$

Case 2. σ = 1, ρ = 3
$I\, Ꝑ_{cvdcl}(Ţ\sigma, Ţ\rho) I = I\, Ꝑ_{cvdcl}(Ţ1, Ţ3) I = I\, Ꝑ_{cvdcl}(2,2) I = 0 \leq k\, I(Ꝑ_{cvdcl}(1,3)) I^a\, I(Ꝑ_{cvdcl}(1, Ţ1)) I^b\, I(Ꝑ_{cvdcl}(3, Ţ3)) I^c = k\, I(Ꝑ_{cvdcl}(1,3)) I^a\, I(Ꝑ_{cvdcl}(1, 2)) I^b\, I(Ꝑ_{cvdcl}(3, 2)) I^c = k.\, (\sqrt{2})^a (\sqrt{5})^b.1 = k\, I(Ꝑ_{cvdcl}(\sigma, \rho)) I^a\, I(Ꝑ_{cvdcl}(\sigma, Ţ\sigma)) I^b\, I(Ꝑ_{cvdcl}(\rho, Ţ\rho)) I^c$

Case 3. σ = 2, ρ = 1
$I\, Ꝑ_{cvdcl}(Ţ\sigma, Ţ\rho) I = I\, Ꝑ_{cvdcl}(Ţ2, Ţ1) I = I\, Ꝑ_{cvdcl}(2,2) I = 0 \leq k\, I(Ꝑ_{cvdcl}(2,1)) I^a\, I(Ꝑ_{cvdcl}(2, Ţ2)) I^b\, I(Ꝑ_{cvdcl}(1, Ţ1)) I^c = k\, I(Ꝑ_{cvdcl}(2,1)) I^a\, I(Ꝑ_{cvdcl}(2, 2)) I^b\, I(Ꝑ_{cvdcl}(1, 2)) I^c = 0 = k\, I(Ꝑ_{cvdcl}(\sigma, \rho)) I^a\, I(Ꝑ_{cvdcl}(\sigma, Ţ\sigma)) I^b\, I(Ꝑ_{cvdcl}(\rho, Ţ\rho)) I^c$

Case 4. σ = 2, ρ = 3
$I\, Ꝑ_{cvdcl}(Ţ\sigma, Ţ\rho) I = I\, Ꝑ_{cvdcl}(Ţ2, Ţ3) I = I\, Ꝑ_{cvdcl}(2,2) I = 0 \leq k\, I(Ꝑ_{cvdcl}(2,3)) I^a\, I(Ꝑ_{cvdcl}(2, Ţ2)) I^b\, I(Ꝑ_{cvdcl}(3, Ţ3)) I^c = k\, I(Ꝑ_{cvdcl}(2,3)) I^a\, I(Ꝑ_{cvdcl}(2, 2)) I^b\, I(Ꝑ_{cvdcl}(3, 2)) I^c = 0 = k\, I(Ꝑ_{cvdcl}(\sigma, \rho)) I^a\, I(Ꝑ_{cvdcl}(\sigma, Ţ\sigma)) I^b\, I(Ꝑ_{cvdcl}(\rho, Ţ\rho)) I^c$

Case 5. $\sigma = 3, \rho = 1$
$I\, Ᵽ_{cvdcl} (Ţ\sigma, Ţ\rho)I = I\, Ᵽ_{cvdcl} (Ţ3, Ţ1)I = I\, Ᵽ_{cvdcl} (2,2)I = 0 \leq k\, I(Ᵽ_{cvdcl} (3,1))I^{a}\, I(Ᵽ_{cvdcl} (3, Ţ3))I^{b}\, I(Ᵽ_{cvdcl} (1, Ţ1))I^{c}$
$= k\, I(Ᵽ_{cvdcl} (3,1))I^{a}\, I(Ᵽ_{cvdcl} (3, 2))I^{b}\, I(Ᵽ_{cvdcl} (1, 2))I^{c} = k. (\sqrt{2})^{a} . 1. (\sqrt{5})^{c} = k\, I(Ᵽ_{cvdcl} (\sigma, \rho))I^{a}\, I(Ᵽ_{cvdcl} (\sigma, Ţ\sigma))I^{b}\, I(Ᵽ_{cvdcl} (\rho, Ţ\rho))I^{c}$

Case 6. $\sigma = 3, \rho = 2$
$I\, Ᵽ_{cvdcl} (Ţ\sigma, Ţ\rho)I = I\, Ᵽ_{cvdcl} (Ţ3, Ţ2)I = I\, Ᵽ_{cvdcl} (2,2)I = 0 \leq k\, I(Ᵽ_{cvdcl} (3,2))I^{a}\, I(Ᵽ_{cvdcl} (3, Ţ3))I^{b}\, I(Ᵽ_{cvdcl} (2, Ţ2))I^{c} = k$
$I(Ᵽ_{cvdcl} (3,2))I^{a}\, I(Ᵽ_{cvdcl} (3, 2))I^{b}\, I(Ᵽ_{cvdcl} (2, 2))I^{c} = 0 = k\, I(Ᵽ_{cvdcl} (\sigma, \rho))I^{a}\, I(Ᵽ_{cvdcl} (\sigma, Ţ\sigma))I^{b}\, I(Ᵽ_{cvdcl} (\rho, Ţ\rho))I^{c}$

Case 7. $\sigma = 1, \rho = 1$
$I\, Ᵽ_{cvdcl} (Ţ\sigma, Ţ\rho)I = I\, Ᵽ_{cvdcl} (Ţ1, Ţ1)I = I\, Ᵽ_{cvdcl} (2,2)I = 0 \leq k\, I(Ᵽ_{cvdcl} (1,1))I^{a}\, I(Ᵽ_{cvdcl} (1, Ţ1))I^{b}\, I(Ᵽ_{cvdcl} (1, Ţ1))I^{c}$
$= k\, I(Ᵽ_{cvdcl} (1,1))I^{a}\, I(Ᵽ_{cvdcl} (1, 2))I^{b}\, I(Ᵽ_{cvdcl} (1, 2))I^{c} = 0 = k\, I(Ᵽ_{cvdcl} (\sigma, \rho))I^{a}\, I(Ᵽ_{cvdcl} (\sigma, Ţ\sigma))I^{b}\, I(Ᵽ_{cvdcl} (\rho, Ţ\rho))I^{c}$

Case 8. $\sigma = 2, \rho = 2$
$I\, Ᵽ_{cvdcl} (Ţ\sigma, Ţ\rho)I = I\, Ᵽ_{cvdcl} (Ţ2, Ţ2)I = I\, Ᵽ_{cvdcl} (2,2)I = 0 \leq k\, I(Ᵽ_{cvdcl} (2,2))I^{a}\, I(Ᵽ_{cvdcl} (2, Ţ2))I^{b}\, I(Ᵽ_{cvdcl} (2, Ţ2))I^{c}$
$= k\, I(Ᵽ_{cvdcl} (2,2))I^{a}\, I(Ᵽ_{cvdcl} (2, 2))I^{b}\, I(Ᵽ(2, 2))I^{c} = 0 = k\, I(Ᵽ_{cvdcl} (\sigma, \rho))I^{a}\, I(Ᵽ_{cvdcl} (\sigma, Ţ\sigma))I^{b}\, I(Ᵽ_{cvdcl} (\rho, Ţ\rho))I^{c}$

Case 9. $\sigma = 3, \rho = 3$
$I\, Ᵽ_{cvdcl} (Ţ\sigma, Ţ\rho)I = I\, Ᵽ_{cvdcl} (Ţ3, Ţ3)I = I\, Ᵽ_{cvdcl} (2,2)I = 0 \leq k\, I(Ᵽ_{cvdcl} (3,3))I^{a}\, I(Ᵽ_{cvdcl} (3, Ţ3))I^{b}\, I(Ᵽ_{cvdcl} (3, Ţ3))I^{c} = k$
$I(Ᵽ_{cvdcl} (3,3))I^{a}\, I(Ᵽ_{cvdcl} (3, 2))I^{b}\, I(Ᵽ_{cvdcl} (3, 2))I^{c} = k.1/2 .1 = k\, I(Ᵽ_{cvdcl} (\sigma, \rho))I^{a}\, I(Ᵽ_{cvdcl} (\sigma, Ţ\sigma))I^{b}\, I(Ᵽ_{cvdcl} (\rho, Ţ\rho))I^{c}$

For all a, b,c ε (0,1) with a+ b+ c < 1, it is clear that the above conditions are satisfied; these conditions are also satisfied for Ţ1 = Ţ2 = Ţ3 = 1. For any $\sigma_0\, \varepsilon$ Ḩ, the second condition of theorem 3.3 holds. Therefore, a fixed point exists at 1.

4. Conclusion

Considering the results [19], this paper has some fixed point results on complex-valued double-controlled metric-like spaces and supporting examples in this setting. The related Kannan Type and Reich type fixed point results are presented. This result is more generalized than [19] and others. This work contributes to understanding complex valued double controlled metrics like space in mathematical analysis and its allied areas.

Authors' Contributions

Both authors contributed equally and significantly to writing this article. Both authors read and approved the final manuscript.

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