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Original Article

Spatial Growing Stability of Shear Flows

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Abstract *- The spatial instability of an inviscid, incompressible shear flows with variable density is studied. A criterion for instability to spatially growing disturbances is derived. The range of the instability region is determined, which is found to depend on the basic velocity profile and Richardson number.*

Keywords *- Shear flows, Breadth function, Variable topography, Stability.*

2010 AMS subject classification number *- 76E05.*

1. Introduction

The extended Taylor-Goldstein problem obtained more attention in recent years because of its application in mathematics and Ocean Engineering. Recently, [1] introduced the study of shear flows with variable cross section. [2] improved the theory for such flows. [3], [4] laid out mathematical foundation. [4] proved that Richardson number must be greater than or equal to 0.25 for stability. [5] derived that (c_r, c_i) lies inside a semi-ellipse whose diameter depends on basic velocity profile. [6] obtained long wave criterion.

For temporal stability, the wave number k is real number and the frequency ω is sought which is complex. In contrast, for spatial mode is that the ω is real, wave number $k = k_r + ik_i$ is to be determined.

In this paper, we consider spatial stability, we derived a criterion for instability to spatially growing disturbances i.e., $b\left(\frac{v'}{b}\right)' - \frac{2N^2(v-c_r)}{|v-c|^2}$ must be positive at least one point in the flow domain in [0, D]. Also we proved results on the location of the complex eigen values of $k = k_r + ik_i$, the eigen value lies outside the given instability region whose range depends on the basic velocity profile and Richardson number J_0.

2. Stratified Equation

The stratified equation is given by (cf. [4])

$$\left[\frac{(bu)'}{b}\right]' + \left[\frac{N^2}{(v-c)^2} - \frac{b\left(\frac{v'}{b}\right)'}{(v-c)} - k^2\right]u = 0, \tag{1}$$

with $u(0) = 0 = u(D)$. (2)

Here $k > 0$ is the wave number, $c = c_r + ic_i$ is the phase speed, v is the basic velocity, $N^2 \geq 0$ is the stratification parameter and $b(z)$ is the breadth.

Now using $u = (v-c)^{\frac{1}{2}} h$ into Eq.(1), we have

$$\left[(v-c)\frac{(bh)'}{b}\right]' - \frac{1}{2}b\left(\frac{v'}{b}\right)' h - k^2(v-c)h - \frac{\left[\frac{(v')^2}{4} - N^2\right]}{(v-c)}h = 0\,, \tag{3}$$

with $h(0) = 0 = h(D)$. (4)

3. General Analytical Results

Theorem 1

For waves advancing in the positive direction, i.e., k_r, ω are positive and with $c_i > 0$, a necessary condition for instability to spatially growing disturbances is that

$$b\left(\frac{v'}{b}\right)' > \frac{2N^2(v-c_r)}{|v-c|^2}\,.$$

Proof:

Applying method of complex eigen function with (bu^*), we have

$$\int\left[\frac{\left|(bu)'\right|^2}{b} + k^2 b|u|^2\right]dz + \int\frac{b\left(\frac{v'}{b}\right)'}{(v-c)}b|u|^2\,dz - \int\frac{N^2}{(v-c)^2}b|u|^2\,dz = 0\,.$$

Equating imaginary part and considering the fact that $k = k_r + ik_i$, we get

$$2\int k_r k_i b|u|^2\,dz + c_i\int\frac{b\left(\frac{v'}{b}\right)'}{|v-c|^2}b|u|^2\,dz - 2c_i\int\frac{N^2(v-c_r)}{|v-c|^4}b|u|^2\,dz = 0\,. \tag{5}$$

Considering k as complex and the frequency ω is real, with definition $\omega = kc$.

i.e., $k_i = \dfrac{-c_i|k|^2}{\omega}$, (6)

$$c_r = \frac{\omega k_r}{|k|^2}, \tag{7}$$

$$c_i = \frac{-\omega k_i}{|k|^2}, \tag{8}$$

Substituting Eq.(6) into Eq.(5) and since $c_i > 0$, we get

$$\int \left[\frac{2k_r |k|^2}{\omega} - \frac{b\left(\frac{v'}{b}\right)'}{|v-c|^2} + \frac{2N^2\left(v-c_r\right)}{|v-c|^4} \right] b|u|^2\, dz = 0\,.$$

This implies that

$$\frac{2k_r |k|^2}{\omega}|v-c|^2 = b\left(\frac{v'}{b}\right)' - \frac{2N^2\left(v-c_r\right)}{|v-c|^2}\,.$$

We consider waves advancing in the positive direction, i.e., k_r, ω are positive, we get

$$b\left(\frac{v'}{b}\right)' > \frac{2N^2\left(v-c_r\right)}{|v-c|^2}\,.$$

Thus, $b\left(\frac{v'}{b}\right)' - \frac{2N^2\left(v-c_r\right)}{|v-c|^2}$ must be positive minimum one point in the domain in [0, D]

For homogeneous case, $N^2 = 0$, a condition for instability to spatially growing disturbances is that $b\left(\frac{v'}{b}\right)' > 0$.

For comprehensively understanding the stability behavior of spatially growing disturbances, the spatial instability of shear flows with variable cross section will be studied in the following.

Theorem 2

The complex eigen value $k = k_r + ik_i$ given by Eqs.(1) and (2) with $c_i > 0$ lie inside a region given by

$$k_r^2 + \left[1 + \frac{4\omega^2 - 8v_{\min}k_r\omega}{\left(v'\right)^2_{\max}\left(1-4J_0\right)}\right]k_i^2 > 0\,.$$

Proof:

Applying method of complex eigen function with $\left(bh^*\right)$, we get

$$\int(v-c)\left[\frac{\left|(bh)'\right|^2}{b}+k^2b|h|^2\right]dz+\frac{1}{2}\int b\left(\frac{v'}{b}\right)'b|h|^2\,dz+\int\frac{\left[\frac{\left(v'\right)^2}{4}-N^2\right]}{(v-c)}b|h|^2\,dz=0\,.$$

Equating imaginary part and considering the fact that $k=k_r+ik_i$, we get

$$2\int\left(v-c_r\right)k_rk_ib|h|^2\,dz-c_i\int\frac{\left|(bh)'\right|^2}{b}dz-c_i\int\left(k_r^2-k_i^2\right)b|h|^2\,dz+c_i\int\frac{\left[\frac{\left(v'\right)^2}{4}-N^2\right]}{|v-c|^2}b|h|^2\,dz=0\,.$$

Using Eq.(6), we have

$$2\int\left(v-c_r\right)\frac{k_r|k|^2}{\omega}dz+\int\frac{\left|(bh)'\right|^2}{b}dz+\int\left(k_r^2-k_i^2\right)b|h|^2\,dz-\int\frac{\left[\frac{\left(v'\right)^2}{4}-N^2\right]}{|v-c|^2}b|h|^2\,dz=0\,.$$

Dropping the second integration term $\int\frac{\left|(bh)'\right|^2}{b}dz$ being positive, we get

$$\int\left[\frac{2\left(v-c_r\right)k_r|k|^2}{\omega}+k_r^2-k_i^2-\frac{\left[\frac{\left(v'\right)^2}{4}-N^2\right]}{|v-c|^2}\right]b|h|^2\,dz\le 0\,.$$

Substituting Eq.(7) in the above equation, we get

$$\int\left[\frac{2vk_r|k|^2}{\omega}-|k|^2-\frac{\left[\frac{\left(v'\right)^2}{4}-N^2\right]}{|v-c|^2}\right]b|h|^2\,dz\le 0\,.$$

We know that $\frac{1}{|U_0 - c|^2} \le \frac{1}{c_i^2}$, substituting this in the above equation, we get

$$\frac{2v_{\min} k_r |k|^2}{\omega} - |k|^2 - \frac{(v')^2_{\max}\left[\frac{1}{4} - \left[\frac{N^2}{(v')^2}\right]_{\min}\right]}{c_i^2} \le 0;$$

i.e., $\frac{2v_{\min} k_r |k|^2}{\omega} - |k|^2 - \frac{(v')^2_{\max}[1-4J_0]}{4c_i^2} \le 0.$

Substituting Eq.(8) in the above equation, we get

$$\frac{2v_{\min} k_r}{\omega} - 1 - \frac{(v')^2_{\max}[1-4J_0]|k|^2}{4\omega^2 k_i^2} \le 0;$$

i.e., $|k|^2 - \frac{\left(\frac{2v_{\min} k_r}{\omega} - 1\right)\omega^2 k_i^2}{\frac{(v')^2_{\max}}{4}(1-4J_0)} > 0.$

This implies that

$$k_r^2 + \left[1 + \frac{4\omega^2 - 8v_{\min} k_r \omega}{(v')^2_{\max}(1-4J_0)}\right] k_i^2 > 0.$$

When $J_0 = 0$, our result reduces to [7].

4. Concluding Remarks

We studied the spatial instability of shear flows in variable cross section. First, we obtained a criterion for instability to spatially growing disturbances i.e., $b\left(\frac{v'}{b}\right)' - \frac{2N^2(v-c_r)}{|v-c|^2}$ must be positive at least one point in the flow domain in [0, D]. Next, we present the region of complex eigen values of $k = k_r + ik_i$ i.e., the eigen value lies outside the given instability region whose range depends on the basic velocity profile, Richardson number J_0.

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